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The fractional moments of shifted power law distribution by Caputo definition

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Abstract

In this paper, the shifted power low distribution is studied in the direction of Fractioned moment. This type of distribution is a generalization for standard power low distributions. In this study, the fractional definition of Caputo is used to generalize the fractional moments of the maintained type of distribution to get a closed useful form.

Keywords: Fractional moment, Caputo definition, Şhifted Power law distribution functions, Fourier transform.
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1. Introduction

We do not reveal a secret if we say that fractional calculus plays nowadays a remarkable role in the different areas of science and engineering. Since fractional integral and fractional derivative have been introduced in several ways by different operators which all share some specific properties and varies among them by their uses and conformity to phenomena. These phenomena were described in powerful models, for instance, viscoelasticity and non-Newtonian fluid characteristics. [6]

In this work, the moments of integer order are related to the derivative of fractional orders of the characteristic function of distribution functions. In a general sense, the fractional calculus could be used to evaluate the fractional moments, which is proposed for finding in a closed-form in this paper to be applicable and useful for applied manner.

The term of fractional calculus is considered an ancient term since the era of Leibniz [2] who presents the core question of the topic of non-integer derivative and integral to create a very useful tool in modelling the wide range of applications in different areas of science, for instance, but not

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limited to, mathematics, social sciences, physics and engineering [3]. There are a lot of operators that represent the definition of derivative and integral of arbitrary real number order or even complex order of derivative which combines derivative and integration in one operator depending on the order of such operator which indicates for the derivative or integration, for example, Riemann-Liovelle definition, Caputo and Grünwald–Letnikov definition. There are some obstacles that appear when dealing with most definitions in fractional calculus, for seeking to overcome these difficulties there are some definitions were designed, such as, a conformable fractional derivative which designed by Khalil et al. in [4] who share most properties of the original Newton's derivative which make the fractional derivative easy to achieve. The conformable and some other definitions of fractional derivatives are stated in the next section. In this work, the definition of Caputo for the fractional derivative and integration is utilized to propose a useful closed form for the fractional moment of non-central type with the handle of Fourier transform FT which was originally presented for a central moment by Singh [3]. Also, this work is considered as a generalization for previous work in [1].

The goal functions under consideration in this work are the power-law distribution functions which appear in many applications in different areas of science, for instance, the motion of humans or animals in searching for their food could be modelled in some power-law distribution functions [2] as well as the motion of carried charges in semiconductors [3]. These applications are quite specific for such applications, so the non-central moments may have thereby much specific meaning, which may study in future work. There are many studies related to the power-law functions maintained in [5] where the function $h(x) \sim |x|^{-\alpha}, \alpha > 0$ and $h(x)$ is the probability function. The non-central moment of fractional order may have rare applications that represent some phenomena, however, it is convenient to study these concepts as applications of fractional derivative in the statistical spaces.

Historically, since the 17th century when the idea of fractional calculus (FC) was formulated, there are many mathematical contributions to the development of FC, for example, Newton, Euler, Riesz and Caputo [2]. Generally, there are many applications in modern mathematics that appear as a new fruitful area of research. One of these new domains is probability and statistics. Specifically, in this paper we study the shifted power distribution functions which occur in some wonderful applications, for example, the number of files in some computers, words number of some scientific or literary article, number and distribution of flares in the sun and earthquakes [5].

In the rest of this work, the main definitions and concepts needed in this work are cited briefly in the next part. While main results are stated in part 3. Finally, some conclusions and future work are stated in the last part 4.

2. Preliminaries

2.1. Definitions of some fractional derivatives

In order to use the concept of fractional derivative in the central and non-central moments, we state here some useful definitions of fractional derivative and integral which are varies from each to other by their ease of computation and concluding results.

2.1.1. Caputo definition

For real valued function $h(x)$, Caputo derivative of order ν is given as:

$$D_*^{\nu} h(t) = \frac{1}{\Gamma(\nu - L)} \int_0^t \frac{h^{(L)}(\tau)}{(t-\tau)^{1-(m-\nu)}} d\tau \tag{2.1}$$

where $((L-1) < \nu < L), L$ is integer and ν is real number.

2.1.2. Marchaud definition

The Marchaud definition for a function $h(x)$ of order $\nu \in R$ is given as:

$$D_t^\nu h(t) = \frac{1}{\Gamma(1-\nu)} \int_{-\infty}^{x} (\frac{h(x)-h(\tau)}{(x-\tau)^{1+\tau}} d\tau \tag{2.2}$$

This definition coincides with the Riemann-Liouville version.

2.1.3. Conformable derivative

Khalil et al.[4] present the conformable fractional derivative of the function $h(x)$ of real order α depends on origin definition of derivative which depends on limit as following:

$$D_t^\nu f(t) = \lim_{\zeta \to 0} \frac{h(t+\zeta t^{1-\nu}) - h(t)}{\zeta} \tag{2.3}$$

for all $t > 0, \nu \in (0,1)$

2.2. Shifted Power Law Distribution Function

The shifted power law distribution is a general form associated with the factor $k \geq 0$ with the following standard form:

$$Pr(x) = (\frac{\alpha-1}{k+x_0})(\frac{k+x}{k+x_0})^{-\alpha}, \alpha > 1 \tag{2.4}$$

where x_0 is the minimum value for x .

The above function $Pr(x)$ in 2.4 may written as following:

$$\begin{aligned} Pr(x) &= (\frac{\alpha-1}{k+x_0})(\frac{k+x}{k+x_0})^{-\alpha} \\ &= (\frac{\alpha-1}{k+x_0})(\frac{1}{k+x_0})^{-\alpha}(k+x)^{-\alpha} \\ &= (\frac{\alpha-1}{k+x_0})(\frac{1}{k+x_0})^{-\alpha}(k+x)^{-\alpha}\frac{x^{-\alpha}}{x^{-\alpha}} \\ &= (\frac{\alpha-1}{k+x_0})(\frac{1}{k+x_0})^{-\alpha}(\frac{k}{x}+1)^{-\alpha}x^{-\alpha} \end{aligned}$$

$$= C(\frac{k}{x}+1)^{-\alpha}x^{-\alpha} \tag{2.5}$$

where $C = (\frac{\alpha-1}{k+x_0})(\frac{1}{k+x_0})^{-\alpha}$

3. Methodology

3.1. Fractional Moments

The moment of random variable X of higher order k is usually determined by using the characteristic function CDF of X which represent the accumulated probability of X for all $t \leq x$. In this paper the Fourier transform (FT) have been handled as following:

$$Fq = F(\nu) \tag{3.1}$$

which is the characteristic distribution function (CDF) of $q(\nu)$ gives as:

$$F(x) = E(e^{ixt}) = \int_{-\infty}^{\infty} e^{ixt} q(x) dx \tag{3.2}$$

where $x \in R$ and $G(X) = X^k$ with $k = 1, 2, 3, \ldots.$ is the n^{th} moment.

It is convenient to simplify the work with the fractional order and generalization to the complex order by the Taylor expansion which used in association with this property:

$$E(iX^n) = \frac{d^n \phi(x)}{dx^n}|_{x=0} \tag{3.3}$$

and the FT of the function $q(x)$ is given by:

$$F(\nu) = \sum_{n=0}^{\infty} E[(iX^n)] \frac{\nu^n}{n!} \tag{3.4}$$

In [6] a formula has been presented in order to find the derivative of arbitrary order using the Caputo definition of fractional derivative with the cooperation of the closed form proposed in [6] . The following relation is proved in [7] which refers to the moment of order α based on some characteristics of expectation function and Fourier transform.

$$M_\alpha = \frac{d^\alpha F(t)}{du^\alpha}|_{u=0} \tag{3.5}$$

3.2. Results and Discussions

Here, the form prepared above in the case of shifted power law type of functions. Recall the convolution operation of two functions which reads:

$$D_*^\alpha h(t) = \frac{t^{1-\alpha})}{\Gamma(1-\alpha)} * \frac{dh(t)}{dt} \tag{3.6}$$

after applying FT for 3.6 we obtain:

$$F[D_*^\alpha h(\tau)] = F[\frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)} * \frac{dg(\tau)}{d\tau}]$$

$$F[D_*^\alpha h(t)] = F[\frac{\tau^{1-\alpha})}{\Gamma(1-\alpha)}] * F[\frac{dh(\tau)}{d\tau}] \tag{3.7}$$

Then the FT of fractional derivative 3.5 reads:

$$F[D_*^\alpha h(\tau)](\omega) = (i\omega)^\alpha F[\frac{dh(\tau)}{d\tau}] \tag{3.8}$$

So, we get in 3.8 a single closed form for using Caputo derivative for FT. Using 3.8 for general shifted power law distribution function to get the following derivative: For positive values of C, α and k, we get:

$$\frac{dPr(t)}{dt} = -\alpha C(k+t)^{-\alpha-1} \tag{3.9}$$

$$F[D_*^\alpha Pr(t)](\omega) = (i\omega)^\alpha F[\frac{dPr(t)}{dt}] \tag{3.10}$$

$$
\begin{aligned}
&= (i\omega)^\alpha F[-\alpha C(k+t)^{-\alpha-1}] \\
&= (i\omega)^\alpha - \alpha C F[(k+t)^{-\alpha-1}] \\
&= (i\omega)^\alpha - \alpha C\sqrt{2\pi}(-i)^\alpha \delta^{-\alpha-1}(k+\omega), \alpha > 0
\end{aligned}
$$

where $\delta(\omega)$ is the Dirac delta function. The desired result of finding the closed form using the FT in finding the moment of fractional order gives in 3.10.

4. Conclusions and Future Work

The proposed form in this paper may be useful in computation manner for different applications. Some improvement may be required for this form to get more accurate results. Also for seeking of comparison one may study another forms or schemes and compare results with the form prepared in this paper.

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