(الحلول الموجبة للمعادلات التفاضلية العادية الغير خطية)

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(Positive Solutions for Nonlinear Ordinary Differential Equations)

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(الملخص)

سندرس في هذا البحث الحلول الموجبة للمعادلات التفاضلية العادية الغير الخطية من الرتبة الثانية مع شروط حدودية والمعرفة على مجموعة الأعداد الحقيقية الموجبة مع الصفر و تحديد القيمة الذاتية (λ) التي تجعل المسألة تمتلك حلا . وقد توصلنا إلى تحديد هذه القيمة الذاتية (λ) والتي عندها تمتلك المسألة على الأقل حلا واحدا موجبا .

Keywords : Positive Solutions , Boundary Value Problems , Eigenvalue .

ABSTRACT . In this research we shall study the positive solutions for nonlinear ordinary differential equations from second order with boundary conditions defined at positive real numbers with zero and determining the values of the eigenvalue (λ) which the problems has obtained the solution . We get to determine the eigenvalue which this problem has at least one positive solution .

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1. INTERODUCTION

Many of researches studies the positive solution for a boundary value problems which has the nonlinear ordinary differential equations , see [1] , [3] , [4] , and in this research we will study the boundary value problems which has the differential equations from nonlinear ordinary differential equations from second order with boundary conditions and defined at positive real numbers with zero$\{ R^+ \cup \{0\} \}$.

The present paper will search for a solution for the following boundary problem :-

$$\left.\begin{aligned} & y'' + \lambda\, h(t)\, g(y) = 0 \qquad , \quad a < t < b \\ & \alpha\, y(a) - \beta\, y'(a) = 0 \\ & \gamma\, y(b) + \delta\, y'(b) = 0 \end{aligned}\right\} \quad ....(1.1)$$

Let the following conditions are satisfied :-

1- $h : [a , b] \to [0 , \infty)$ is continuous and does not vanish identically on any subinterval ,

2 - $g : [0 , \infty) \to [0 , \infty)$ is continuous ,

3 - $g_0 = \lim_{x\to 0^+} \frac{g(x)}{x}$ and $g_\infty = \lim_{x\to\infty} \frac{g(x)}{x}$ exist ,

4 - α , β , γ , δ such that α and β are not both zero , γ and δ are not both zero and
$R = \alpha\gamma + \alpha\delta + \beta\gamma > 0$, and

$\infty <$ b < 5 - $0 \le a$

Many researcher have focused on this problem a many which [2] , [7] , [8] , with in the period in which $0 < t < 1$. In this research , we shall study this problem in the period $0 \le a < t < b < \infty$.
The first we shall found the Green's function of the boundary problem then we define the cone in Banach space then we application the fixed point theorem to determine the eigenvalue (λ) which the boundary value problem will have at least one positive solution. .
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2- BACKGROUND SECTION

The Green's function , whose kernel, G (t , s) , for

$$- y'' = 0$$
$$\alpha\, y(a) - \beta\, y'(a) = 0$$
$$\gamma\, y(b) + \delta\, y'(b) = 0$$

Is

$$G(t,s) = \begin{cases} \frac{1}{R}(\alpha t + \beta)\ (\gamma + \delta - \gamma s) & a \le t \le s \le b \\ \frac{1}{R}(\alpha s + \beta)\ (\gamma + \delta - \gamma t) & a \le s \le t \le b \end{cases} \qquad \text{..........(2.1)}$$

from which $G(t , s) > 0$ on $(a , b) \times (a , b)$, ...(2.2)

$$G(t , s) \le G(s , s) = \frac{1}{R}(\alpha s + \beta)\ (\gamma + \delta - \gamma s) \quad , \quad a \le t \le b , a \le s \le b , \quad \text{......(2.3)}$$

Let $D = \min\left\{ \frac{\gamma + 4\delta}{4(\gamma+\delta)}, \frac{\alpha + 4\beta}{4(\alpha+\beta)} \right\}$

We have from [6] :

$$G(t , s) \ge DG(s , s) = D\, \frac{1}{R}(\ s\ )\ (\gamma + \delta - \gamma s) \qquad \frac{2a+1}{4} \le t \le \frac{2b+1}{4}, a \le s \le b , \text{...(2.4)}$$

Now we state the Krasnosel'skii fixed point theorem to obtain solutions of (1.1) , for eigen value λ .

THEOREM 1 [5]. Let B a Banach space , and let P be a cone in B . Assume that ,
N , K are open subsets of B with and let $0 \in N \subset \overline{N} \subset K$ $T : P \cap (\overline{K} \setminus N) \to P$
be a completely continuous operator such that , either

1- $\| Tu \| \le \| u \|$, $u \in P \cap \partial N$, and $\| Tu \| \ge \| u \|$, $u \in P \cap \partial K$, or

2- $\| Tu \| \ge \| u \|$, $u \in P \cap \partial N$, and $\| Tu \| \le \| u \|$, $u \in P \cap \partial K$

Then T has a fixed point in $P \cap (\overline{K} \setminus N)$.

3. DETERMINING THE SOLUTIONS

From [8] we note that y(t) is a solution of (1.1) if and only if ,

$$y(t) = \lambda \int_a^b G(t,s)\,h(s)\,g(y(s))\,ds \quad , \quad a \le t \le b$$

let B = C[a , b] , with norm , $\|x\| = \sup_{a \le t \le b} |x(t)|$

We define a cone J by :

$$J = \left\{ x \in B : x(t) \ge 0 \text{ on } [a,b] \ , \min_{\frac{2a+1}{4} \le t \le \frac{2b+1}{4}} x(t) \ge D\|x\| \right\}$$

Where $D = \min\left\{ \dfrac{\gamma + 4\delta}{4(\gamma+\delta)}, \dfrac{\alpha+4\beta}{4(\alpha+\beta)} \right\}$

And , let the number $k \in [a,b]$ be defined by □

$$\int_{\frac{2a+1}{4}}^{\frac{2b+1}{4}} G(k,s)\,h(s)\,ds = \max \int_{\frac{2a+1}{4}}^{\frac{2b+1}{4}} G(t,s)\,h(s)\,ds \qquad \text{............(3.1)}$$

THEOREM 2 . Assume that conditions (1 -5) are satisfied .Then , for each λ satisfying

$$\frac{1}{\left(D \int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\,h(s)\,ds\right) g_\infty} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\,h(s)\,ds\right) g_0} \qquad \text{.......... (3.2)}$$

there exists at least one solution of (1.1) in J .

Proof. Let λ be given as in (3.2) . Now , let $\varepsilon > 0$ be chosen such that

$$\frac{1}{\left(D \int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\,h(s)\,ds\right)(g_\infty - \varepsilon)} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\,h(s)\,ds\right)(g_0 + \varepsilon)} \qquad \text{...........(3.3)}$$

Define an integral operator $T : P \to B$ by

$$Ty(t) = \lambda \int_a^b G(t,s)\, h(s)\, g(y(s))\, ds \quad , \quad y \in J \qquad \ldots\ldots\ldots(3.4)$$

We seek a fixed point of T in the cone J.

, Notice from (2.2) that , for $y \in J$, $Ty(t) \geq 0$ on $[a,b]$. Also , for $y \in J$, we have from (2.3) that

$$Ty(t) = \lambda \int_a^b G(t,s)\, h(s)\, g(y(s))\, ds$$

$$\leq \lambda \int_a^b G(s,s)\, h(s)\, g(y(s))\, ds$$

Hence : $\quad ||Ty|| \leq \lambda \int_a^b G(s,s)\, h(s)\, g(y(s))\, ds \qquad \ldots\ldots\ldots(3.5)$

And next , if $y \in J$, we have by (2.4) and (3.5) ,

$$\min_{\frac{2a+1}{4} \leq t \leq \frac{2b+1}{4}} Ty(t) = \min_{\frac{2a+1}{4} \leq t \leq \frac{2b+1}{4}} \lambda \int_a^b G(t,s)\, h(s)\, g(y(s))\, ds$$

$$\geq D \lambda \int_a^b G(s,s)\, h(s)\, g(y(s))\, ds$$

$$\geq D \|Ty\|$$

Hence , $T : p \to p$. is completely continuous.

Now, turning to g_0 , there exist an $R_1 > 0$ such that $g(x) \leq (g_0 + \varepsilon)\, x$, for $0 < x \leq R_1$.

So , choosing $y \in J$ with $|| y || = R_1$, we have from (2.3) and (3.3)

$$Ty(t) \leq \lambda \int_a^b G(s,s)\, h(s)\, g(y(s))\, ds$$

$$\leq \lambda \int_a^b G(s,s)\, h(s)\, (g_0 + \varepsilon)\, y(s) ds$$

$$\leq \lambda \int_a^b G(s,s)\, h(s)\, ds\, (g_0 + \varepsilon)\, y(s) \|y\|$$

$$\leq \|y\|$$

So that , $\|Ty\| \leq \|y\|$. So , if we set $\Omega_1 = \{x \in B|\ ||x|| < R_1\}$

then

$$||Ty|| \leq ||y|| \text{ , for } y \in J \cap \partial\Omega_1 . \qquad \ldots\ldots\ldots(3.6)$$

Next , considering g_∞ , there exist an $R_2 > 0$ such that $g(x) \geq (g_\infty - \varepsilon)\, x$,for all $x > R_2$.

Let $R_3 = \max \{2R_1 , R_2/D\}$ and let $\Omega_2 = \{ x \in B \mid ||x|| < R_3\}$.

If $y \in J$ with $||y|| = R_3$, then $\min_{\frac{2a+1}{4} \leq t \leq \frac{2b+1}{4}} y(t) \geq D\|y\| = DR_3 \geq R_2$, and we have from (3.1) and (3.3) that :

$$\begin{aligned} Ty(h) &= \lambda \int_a^b G(k,s)\, h(s)\, g(y(s))\, ds \\ &\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\, h(s)\, g(y(s))\, ds \\ &\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\, h(s)\, (g_\infty - \varepsilon)\, y(s) ds \\ &\geq \frac{\lambda}{D} \int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\, h(s)\, ds\, (g_\infty - \varepsilon)\, \|y\| \\ &\geq \|y\| \end{aligned}$$

Thus , $\|Ty\| \geq \|y\|$. Hence ,

$$||Ty|| \geq ||y|| \text{ , } \quad \text{for } y \in J \cap \partial\Omega_2 \qquad \ldots\ldots\ldots(3.7)$$

Applying (1) of theorem 1 to (3.6) and (3.7) yields that T has a fixed point y(t) $\in J \cap (\overline{\Omega_2} \setminus \Omega_1)$. As such , y(t) is a desired solution of 1.1 for the given λ . Further , since G (t , s) > 0 , it follows that y(t) > 0 for a < t < b . This completes the proof of the theorem .

<u>THEOREM 3 .</u> Assume that condition (1 – 5) are satisfied . Then , for each λ satisfying

$$\frac{1}{(D\int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\,h(s)\,ds)g_0} < \lambda < \frac{1}{(\int_a^b G(s,s)\,h(s)\,ds)g_\infty} \qquad \ldots\ldots\ldots.(3.8)$$

there exists at least one solution of (1.1) in J .

Proof. Let λ be given as in (3.8) . Now , let $\varepsilon > 0$ be chosen such that

$$\frac{1}{(D\int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\,h(s)\,ds)(g_0-\varepsilon)} < \lambda < \frac{1}{(\int_a^b G(s,s)h(s)\,ds)(g_\infty+\varepsilon)} \qquad \ldots\ldots\ldots(3.9)$$

Let T be the cone preserving , completely continuous operator that was define by (3.4) . Beginning with g_0 , there exist an $R_4 > 0$ such that $g(x) \geq (g_0 - \varepsilon)\, x$, for $0 < x \leq R_4$.

So , for $y \in J$ with $\| y \| = R_4$, we have from (3.1) and (3.9) so , for

$$\begin{aligned} Ty(k) &= \lambda\int_a^b G(k,s)\,h(s)\,g(y(s))\,ds \\ &\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\,h(s)\,g(y(s))\,ds \\ &\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\,h(s)\,g_0-\varepsilon)\,y(s)ds \\ &\geq D\lambda \int_{(2a+1)/4}^{(2b+1)/4} G(k,s)\,h(s)\,ds\,(g_0-\varepsilon)\,\|y\| \\ &\geq \|y\| \end{aligned}$$

Thus , $\|Ty\| \geq \|y\|$. So , if we let

$$\Omega_3 = \{x \in B \mid \|x\| < R_4\}$$

then

$$||Ty|| \geq ||y|| \text{ for } y \in J \cap \partial\Omega_3 \quad \ldots\ldots. (3.10)$$

It remains to consider g_∞ ,there exist an $R_5 > 0$ such that $g(x) \leq (g_\infty + \varepsilon)\, x$, for all $x > R_5$

There are the two cases :-

1- g is bounded

2- g is unbounded .

For case (**1**) , suppose $R_6 > 0$ is such that $g(x) \leq R_6$, for all $0 < x < \infty$.

Let $R_7 = \max\{2R_4 , R_6\ \lambda\int_a^b G(s,s)\, h(s)\, g(y(s))\, ds\}$. Then , for $y \in J$ with $||y|| = R_7$ we have from (2.3)

and (3.2)

$$\begin{aligned} Ty(t) &= \lambda\int_a^b G(t,s)\, h(s)\, g(y(s))\, ds\} \\ &\leq \lambda R_6 \int_a^b G(s,s)\, h(s)\, ds \\ &\leq \|y\| \end{aligned}$$

So that $\|Ty\| \leq \|y\|$. So if $\Omega_4 = \{x \in B|\ ||x|| < R_7\}$

then

$$||Ty|| \leq ||y|| \text{ , for } y \in J \cap \partial\Omega_4 \quad \ldots\ldots\ldots.(3.11)$$

For case (**2**) , let $R_8 > \max\{2R_4 , R_5\}$ be such that $g(x) \leq g(R_8)$, for $0 < x \leq R_8$.

Choosing $y \in J$ with $||y|| = R_8$ and we have from (2.3),(3.2) and (3.9)

$$\begin{aligned} Ty(t) &= \lambda\int_a^b G(t,s)\, h(s)\, g(y(s))\, ds\} \\ &\leq \lambda \int_a^b G(s,s)\, h(s)\, g(y(s))\, ds \\ &\leq \lambda \int_a^b G(s,s)\, h(s)\, g(R_8)\, ds \\ &\leq \lambda \int_a^b G(s,s)\, h(s)\, ds\, (g_\infty + \varepsilon) R_8 \end{aligned}$$

But

$$\lambda \int_a^b G(s,s)\,h(s)\,ds\,(g_\infty + \varepsilon)R_8 = \lambda \int_a^b G(s,s)\,h(s)\,ds\,(g_\infty + \varepsilon)\|y\|$$

Therefore

$$Ty(t) \le \lambda \int_a^b G(s,s)\,h(s)\,ds\,(g_\infty + \varepsilon)\|y\|$$

and so $\|Ty\| \le \|y\|$. For this case , if we let

$$\Omega_4 = \{x \in B \mid \|x\| < R_8\}$$

then

$$\|Ty\| \le \|y\| \quad , \qquad \text{for } y \in J \cap \partial\Omega_4 \qquad \ldots\ldots\ldots(3.12)$$

Thus , in either of the case , an applying of part (2) of theorem 1 to (3.10),(3.11) and (3.12) yields that T has a fixed point $y(t) \in J \cap (\overline{\Omega_4} \setminus \Omega_3)$. As such , y(t) is a desired solution of (1.1) for the given λ . Further , since $G(t, s) > 0$, it follows that $y(t) > 0$ for $a < t < b$. This completes the proof of the theorem .

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