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Injective Modules, Projective Modules and Their Relationships to Some Rings

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Abstract. The main objective of this study is to introduce several facts about the relationship of injective and projective modules with two rings namely H-ring and CO-H-ring. We extend some important results given H-ring and CO-H-ring. As applications we obtain some new results when M is a multiplication and faithful R-module, then R is H-ring. Also we investigate that any divisible module over Euclidean domain R gives R is H-ring. It is important to point out the fact that previous ring can be obtained from some modules over another ring namely QF-ring. Finally, we show that if M is a free module over the ring R, then R is CO-H-ring. To prove these results we have had to develop new methods.

INTRODUCTION

Here we introduce the concept of injective module and projective module over H-ring and CO-H-ring. In [1], “the author defined injective module by an R − module M is said to be injective if and only if for any monomorphism f:$A \to B$ such that A and B are two modules and any monomorphism $g: A \to M$, there exists a homomorphism h:B→M such that f∘h=g”,. The reader is referred to Robert’s Book [1] for these modules. “The first important contribution in this direction is due to Osofsky who considered rings over which all cyclic modules are injective” [2]. “As is well known, an R-module M is called regular if given any element m∈ M there exists f ∈ Hom (M, R) with (mf)m = m” [3]. More information about regularity in [4,5]. “Any one can find addition characterization of many concepts such as perfect ring, flat module and multiplication module in” [6]. “An R-module M is R-torsion free if rm = 0 for r ∈ R and m ∈ M, then either r = 0 or m = 0 and an R-module M is R-divisible if rM = M for all 0 ≠r ∈ R” [7]. Many properties about Artinian module was studied in [8]. Recently, several articles have appeared that support the importance of modules and their relationship to other algebraic concepts [9 -15]**.** The purpose of this paper is to give similar characterizations of H-rings and COH-rings R in terms of injective and projective modules.

INJECTIVE AND PROJECTIVE MODULES AND H-RING

In this section, we studied several relationships between injective, projective modules and H-ring.

(*) “Every injective R-module is a lifting. Also, if R is Artinian ring and every f.g. injective R-module is lifting, then R is H-ring” [16].

Theorem 1. Let M be an R-module. If M is a regular, then R is H-ring.

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Proof. Firstly, we know that M is regular module iff for all m ∈M, f(m) ∈Hom_R(M, R), then mf(m)=m. Or, iff for all m ∈M, ∃ N ≤M ∋M=m⨁N. If f(m)=e, then mf(m)=me=m. So, f(rm)=rf(m)=re and hence $r_1e=r_2e$ ($r_1m=r_2m$). Therefore, $r_1mf(m)=r_2mf(m)$ and then $r_1em=r_2em$ (one to one). Thus M is injective and by (*) R is H-ring.

Theorem 2. Let R be a perfect regular ring. If M is a f. generated module, then R is H-ring.

Proof. Since R is regular ring, then M is flat. But R is perfect ring. So M is projective with f.g. property we obtain M=$\bigoplus\sum e_i$. Thus M is injective and so lifting. Thus R is H-ring.
Now we study the relationship between multiplication faithful module and H-ring.

Theorem 3. Let M be an R-module. If M is a multiplication and faithful R-module, then R is H-ring.

Proof. Suppose that M is a multiplication R-module. We should prove that M is a torsion free. Take M is not torsion free. Then M is a torsion (∃m∈M is a torsion element, ∃c∈R and 0≠m∈M∋cm=0) such that c one of the regular element in R (non-zero divisors). There exists an ideal I ∈ R ∋Rm=Im, because M is a multiplication module. Then cI≠0, I≠0. So cIM=crm and hence cIM=0 (cI≠0). So M is unfaithful module and this contradiction. Therefore M is a torsion free (M≈I). Hence there exists a homomorphism is one to one (M is injective module). Thus R is H-ring.

The next results explain how to using divisible module and his relationship with injective module in order to obtain H-ring.

Lemma 1. Let M be a module over P.I.D. So M is injective iff it is divisible.

Proof. ⟹
Let A be injective. Not that Rr is an ideal of R. We define f:Rr→A by f(br)=ba and f(cr)=ba, a∈A. Now every two elements b, c ∈R, we get

$$\begin{aligned} f(br+cr)&=f((b+c)r)\\ &=(b+c)a\\ &=ba+ca\\ &=f(br)+f(cr) \end{aligned}$$

and

$$\begin{aligned} f(b(cr))&=f((bc)r)\\ &=(bc)a\\ &=b(ca)\\ &=bf(cr). \end{aligned}$$

Hence f is R-homomorphism. M is injective, so ∃ g: R→A ∋ g| Rr = f. Then

$$\begin{aligned} a&=1.a\\ &=f(1.r)\\ &=g(1.r)\\ &=rg(1) \ni g(1) \in A. \end{aligned}$$

Thus A is a divisible module.
⟸ Suppose M is a divisible and f:I→M is an R-homo. and I is an ideal. We have R is P.I.D. , then I=Rb. Since M is a divisible, then ∃a∈M∋f(b)=ba. Define g:R→M by g(r)=ra ∋g is an R-homo. Every bc ∈R, then g(bc)=(st)a=c(ba)=cf(b)=f(cb). Thus M is injective module.

Theorem 4. If R be a Euclidean domain and M is a divisible over R, then R is H-ring.

Theorem 5. Suppose M is divisible over Dedekind domain R with f. g. prime ideals. So R is H-ring.

Corollary 1. Let R be a P.I.D. ring. If M is a divisible R-module, then R is H-ring

Example 1. Q is an injective Z-module, so Z is H-ring.

Example 2. Z is not injective Z-module, so Z is not H-ring.

Corollary 2. The following are equivalent:
1- R is regular ring.
2- Every cyclic R-module is a flat.
3- Every simple R-module M is injective and R is H-ring.

CO-H-RINGS

In this section, we introduce the main conditions and some properties of CO-H-rings. Therefore we need to start with the following property:

(##) Every projective R-module is C_1-module$\Longrightarrow$ Every R-module is a direct sum of a projective and singular.
Also, if R is Noetherian ring whose indecomposable injective module are f.g R-module and every f.g. projective R-module is C_1-module, so R is co-H-ring.
Recall that a ring R is called Quasi-Frobenius (QF-ring) if every projective module is injective$\Leftrightarrow$every injective module is projective.
Also, R is called QF-ring $\Leftrightarrow$ every injective module is discrete $\Leftrightarrow$ every injective module is quasi-discrete $\Leftrightarrow$every projective module is continuous $\Leftrightarrow$ every projective module is quasi-continuous.

Lemma 1. [16]. "A ring R is QF-ring iff it is H-ring with Z(R)=J(R) such that J(R) is a Jacobson radical and Z(R) is a singular ideal of R iff co-H-ring".

From [7], we have the definitions of C_1, C_2, C_3, D_1, D_2 and D_3 and by the following.
(**C_1**) "A module *M* is an extending module if and only if every closed sub-module in *M* is direct summand of *M*".
(**C_2**) "If a sub-module X of M is isomorphic to a direct summand of M, then X is a direct summand".
(**C_3**) "If M_1 and M_2 are direct summands of M with $M_1 \cap M_2=0$, then $M_1 \oplus M_2$ is a direct summand of M".
(**D_1**) "M is a lifting module".
(**D_2**) "If X is a sub-module of M such that M=X is isomorphic to a direct summand of M, then X is a direct summand of M".
(**D_3**): If M_1 and M_2 are direct summands of M with.
$M = M_1+M_2$, then $M_1 \cap M_2$ is a direct summand of M".

Lemma 2. If M is a free R-module, then R is CO-H-ring.

Proof.
Take F free on S. We consider f: A→B is a homomorphism. For all $x \in S$, if $a_x \in A$;

$$j(x)= a_x\ldots\ldots (1).$$

If $x \in F$, $g(x) \in B$ and f: A→B is onto, so $a_x \in A$;

$$f(a_x)=g(x)\ldots\ldots (2).$$

Since F is a free on S; ∃ a unique homo. h:F→A;

$$h\circ i=j\ldots\ldots\ldots (3).$$

To prove f∘h=g.?
Let $x \in F$. So, $x=\sum r_k x_k$, x_k in S, r_k in R; k=1, 2, …, n. F=⟨S⟩). Now

$$(f\circ h)\ (x)= (f\circ h)\ (\sum r_k x_k)$$

$$=f(h(\sum r_k x_k))$$

$$=f(\sum r_k h(x_k)).$$

$$=f(\sum r_k (h(i(x_k))).$$

Now

$$(f\circ h)= f(\sum r_k((h\circ i)(x_k)))$$

$$=f(\sum r_k(j(x))) \text{ (by 3)}$$

$$=f(\sum r_k a_{xk}) \text{ (by 1)}$$

$$= \sum r_k f(a_{xk}).$$

$$= \sum r_k g(x_k) \text{ (by 2)}$$

$$=g\sum r_k (xk).$$

So

$$f\circ h=g(x).$$

Thus, M is projective and so is injective. So M is lifting module. Thus, R is H-ring. Thus, R is a CO-H-ring.

Theorem 6. Suppose that M is a free R-module over R with C_1 and C_2 properties, then R is QF-ring and so is CO-H-ring.

Proof.
We have M is a free. So M is a projective. But M satisfies C_1 and $C_{2,}$ then it is continuous module. So R is a QF-ring and then CO-H-ring.

Theorem 7. Let R be a ring. If M is a injective R-module with C_1 and C_3 properties, then R is QF-ring and so is CO-H-ring.

Proof. Same proof Theorem. If M satisfies C_1 and C_3, then it is semi-continuous.

Corollary 3. Let R be a P.I.D. ring. If M is a divisible R-module with D_1 and D_2 properties, then R is QF-ring and so is CO-H-ring.

Proof. Since R is a P.I.D. ring and M is a divisible module, then M is injective. So M is projective. We have M satisfies D_1 and D_2 properties imply M is a semi-perfect module. So R is a QF-ring. Thus R is a CO-H-ring.

Corollary 4. If M is a regular module over P.I.D with D_1 and D_3 properties, then R is QF-ring and so is CO-H-ring.

Proof. Since M is a regular module, then it is injective and hence it is a projective module. Also, from D_1 and D_3 properties, we obtain quasi-semi-perfect. So R is a QF-ring. Thus R is a CO-H-ring.
Let R be a Dedekind domain and a nonzero ideal I_1 be a (FI) of R. If M is a divisible and I_2 is an integral ideal, then M is injective (R CO-H-ring).
Theorem 8. Let R be a Dedekind domain with finitely generated prime ideals. If M is a divisible, then it is injective. So R is a QF-ring (R is CO-H-ring).

Corollary 5. A divisible module M over Euclidean domain R is injective and hence R is CO-H-ring.

Proof. Let $I_1 \lhd R$. So $I_1=\{0\}=()$ or let $0\neq \alpha$ in I_1 $\exists d(\alpha)$ is a least, so any β in I_1 we get $\beta=q\,\alpha+r$ and $r=0$ or $d(r)<d(\alpha)$. But $r=q-\beta\,\alpha$ in I_1. Since $d(\alpha)$ is a minimal and $r=0$, then $\alpha|\beta$ and $I_1=(\alpha)$. Hence R is a principal ideal domain with M divisible module, we obtain M is an injective module and so M is quasi-semi perfect. Thus R is a QF-ring and hence is CO-H-ring.

Corollary 6. Let R be a ring. If M is a multiplication and torsion free R-module, then there exists an ideal I of R s.t $M\approx I$ ($\exists$ f homomorphism is one to one). So M is injective module and then R is H-ring (R is a CO-H-ring).

CONCLUSION

Injective and projective modules are two of important concepts in module theory. In this paper we have defined injective module as an algebraic structure. Some basic properties have been introduced. It has been shown that if R is a ring and M is a free R-module with C_1 and C_2 properties, then R is QF-ring and so is CO-H-ring. The main result is if R is a Dedekind domain with finitely generated prime ideals and M is a divisible R-module, then it is an injective with R is a QF-ring, so R is CO-H-ring. Also, we proved that if R is a ring and M is a multiplication with faithful R-module, then R is H-ring. Finally, we investigated many results about H-ring and CO-H-ring by using others modules such as divisible and regular modules.

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