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On Differential Subordination and Superordination for Univalent Function Involving New Operator

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ABSTRACT: The goal of this paper is to explore some of the features of differential subordination of analytic univalent functions in an open unit disc. It also seeks to shed light on geometric features such as coefficient inequality, Hadamard product qualities, and the Komatu integral operator. Some intriguing results for third-order differential subordination and superordination of analytic univalent functions were obtained. Then, using the convolution of two linear operators, certain results of third order differential subordination involving linear operators were reported. We use features of the Komatu integral operator to analyze and study third-order subordinations and superordinations in relation to the convolution. Finally, several results for third order differential subordination in the open unit disk using generalized hypergeometric function were obtained by using the convolution operator.

Keywords: Univalent Function, Convex function, Differential Subordination, Differential Superordination, Derivative Operator, Komatu Integral Operator, Hadamard Product.

1. INTRODUCTION

Rogosinski [1] developed established the basic results on subordination, which can be traced back to Littlewood [2]. Recently, Srivastava and Owa [3] used subordination to study the intriguing characteristics of the generalized hypergeometric function. Miller and Mocanu [4] published an article on differential subordinations, which is a generalization of differential inequalities. The major focus of this paper is on the differential subordination and superordination of univalent functions in an open unit disk.

Let $L = \{r \in \mathbb{C} : |r| < 1\}$ a disc with an open unit in $\mathbb{C}$. Let $H(L)$ be a class of analytical functions in L with let $L[e,i]$ as a subclass of $H(L)$ of the form

$$k(r) = e + e_l r^l + e_{l+1} r^{l+1} + \cdots,$$

Such that $e \in \mathbb{C}, l \in \mathbb{N} = \{1,2,\ldots\}, H_0 \equiv H[0,1]$ and $H \equiv H[1,1]$. Let $\mathscr{A}$ denote the class of form's analytic functions:

$$k(r) = r + \sum_{\imath=2}^{\infty} e_\imath r^\imath, \quad (e_\imath \geq 0, \imath \in \mathbb{N} = \{1,2,3,\ldots\}, r \in \mathscr{L}) \tag{1}$$

Let

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$$g(r)=r+\sum_{l=2}^{\infty} c_l r^l,(r\in\mathscr{L}) \tag{2}$$

The Hadamard product for $k(r)$ and $g(r)$ in $\mathscr{A}$ is given by

$$k(r)*g(r)=r+\sum_{l=2}^{\infty} e_\iota c_l r^\iota\cdot(r\in\mathscr{L}) \tag{3}$$

Then, we let $k(r)$ and $g(r)$ to be analytic functions in L.The function $k(r)$ is subordinate to a function $g(r)$ or $g(r)$ is superordinate to $k(r)$, if and only if there exists a Schwarz function $z(r)$ analytic in L, with $z(0)=0$ and $|z(r)|<1,(r\in\mathscr{L})$, such that

$$k(r)=g(z(r))$$

written as

$$k\prec g \text{ or } k(r)\prec g(r),(r\in\mathscr{L}).$$

Furthermore, if the function k is univalent in L, the following equivalence $k(r)\prec g(r)$ is obtained if and only if $k(0)=g(0)$ and $k(\mathscr{L})\subset g(\mathscr{L})$ [5–8]. A function $k(r)$ is said to be starlike (convex) in $\mathscr{L}$ if it meets the following criteria:

$$\left\{\Re e\left\{\frac{rk'(r)}{k(r)}\right\}>0,k(r)\neq 0\right\},\left\{\Re e\left\{1+\frac{rk''(r)}{k'(r)}\right\}>0\right\}, \text{ respectively (See [5]).}$$

2. BASIC CONCEPTS

Definition 2.1. *[9] Let $\psi:\mathbb{C}^4\times\mathscr{L}\to\mathbb{C}$ and let h be univalent in L. If b is analytic in L and meets the third-order differential subordination requirement:*

$$\psi\left(b(r),rb'(r),r^2b''(r),r^3b'''(r);r\right)\prec h(r),(r\in\mathscr{L}) \tag{4}$$

then b is known as a differential subordination solution.

Definition 2.2. *[10] Let $\psi:\mathbb{C}^4\times\mathscr{L}\to\mathbb{C}$ with h is analytic in L. If b and $\psi\left(b(r),rb'(r),r^2b''(r),r^3b'''(r);r\right)$ are univalent in L and meets the third-order differential subordination requirement:*

$$h(r)\prec\psi\left(b(r),rb'(r),r^2b''(r),r^3b'''(r);r\right),(r\in\mathscr{L}) \tag{5}$$

then b is known as a differential superordination solution.

Definition 2.3. *[9] Let Ω and Δ be any sets in $\mathbb{C}$ and let $b(r)$ be an analytic function in the open unit disk L with $b(0)=e$ and let $\psi(a,s,d,u;r):\mathbb{C}^4\times\mathscr{L}\to\mathbb{C}$. The core concerns of this theory is the generalizations of the implication:*

(i) *$\left\{\psi\left(b(r),rb'(r),r^2b''(r),r^3b'''(r);r\right),(r\in\mathscr{L})\right\}\subset\Omega\Rightarrow b(\mathscr{L})\subset\Delta$. In [11], the authors coined the term differential superordination to describe the dual dilemma of differential subordination.*

(ii) *$\Omega\subset\left\{\psi\left(b(r),rb'(r),r^2b''(r),r^3b'''(r);r\right),(r\in\mathscr{L})\right\}\Rightarrow\Delta\subset b(\mathscr{L})$.*

We present a derivation operator $M^{\lambda}_{\beta_1,\beta_2}:\mathscr{A}\to\mathscr{A}$ for k given by Eljamal and Darus in [12], which we use for the function $k\in\mathscr{A}$, if a natural number is positive λ is fixed and $0\le\beta_1\le\beta_2$.

$$M^{\lambda}_{\beta_1,\beta_2}k(r)=r+\sum_{l=2}^{\infty}\left(\frac{1+(\beta_1+\beta_2)(\iota-1)}{1+\beta_2(\iota-1)}\right)^{\lambda} e_\iota r^\iota. \tag{6}$$

From (6), we get

$$w\left(M^{\lambda}_{\beta_1,\beta_2}k(r)\right)'=(\beta_1+\beta_2)M^{\lambda}_{\beta_1,\beta_2}k(r)-((\beta_1+\beta_2)+1)M^{\lambda+1}_{\beta_1,\beta_2}k(r). \tag{7}$$

For function $k \in \mathscr{A}$, the Komatu integral operator $I_m^\lambda : \mathscr{A} \to \mathscr{A}$ and $\lambda \geq 0, m \in \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1,2,3,\ldots\}$ in the following manner [13]:

$$I_m^\lambda k(r) = \frac{m^\lambda}{\Gamma(\lambda)} \int_0^1 t^{m-2} \left(\log \frac{1}{t}\right)^{\lambda-1} k(r)(tr)dt, \tag{8}$$

The gamma function is symbolized by Γ. Thus, we get

$$I_m^\lambda k(r) = r + \sum_{l=2}^{\infty} \left(\frac{m\iota}{m+\iota-1}\right)^\lambda e_\iota r^\iota. \tag{9}$$

For $\lambda, \mathrm{e} \geq 0$, we obtain

$$I_m^\lambda \left(I_m^\alpha k(r)\right) = I_m^{\lambda+\alpha} k(r)$$

From (9) we have

$$r\left(I_m^\lambda k(r)\right)' = mI_m^\lambda k(r) - (m+1)I_m^{\lambda+1} k(r). \tag{10}$$

For function $k \in \mathscr{A}$, the operator $E_{\beta_1,\beta_2}^{\lambda,m} : \mathscr{A} \to \mathscr{A}$ is defined by the Hadamard product of the operator $M_{\beta_1,\beta_2}^\lambda$ and the operator I_m^λ, such that

$$E_{\beta_1,\beta_2}^{\lambda,m} k(r) = \left(D_m^\lambda * M_{\beta_1,\beta_2}^\lambda\right) k(r), \quad (r \in \mathscr{L})$$

and

$$E_{\beta_1,\beta_2}^{\lambda,m} k(r) = r + \sum_{\iota=2}^{\infty} \left(\frac{m\left(1+(\beta_1+\beta_2)(\iota-1)\right)}{(m+\beta_2(\iota-1))}\right)^\lambda e_\iota r^\iota. \tag{11}$$

From (11) we obtain

$$w\left(E_{\beta_1,\beta_2}^{\lambda,m} k(r)\right)' = \mu E_{\beta_1,\beta_2}^{\lambda,m} k(r) - (\mu+1) E_{\beta_1,\beta_2}^{\lambda+1,m} k(r). \tag{12}$$

The following are the operator's special cases. $E_{\beta_1,\beta_2}^{\lambda,m}$.

1. When $m = 1$, incorporate the derivative operator $M_{\beta_1,\beta_2}^\lambda$ given by Eljamal and Darus in [12]:-
   (a) When $\beta_1 = 1, \beta_2 = 0, M_{\beta_1,\beta_2}^\lambda$ reduces to M^λ Salagean is the one who introduces it [14].
   (b) When $\beta_2 = 0, M_{\beta_1,\beta_2}^\lambda$ reduces to $M_{\beta_1}^\lambda$ Al-Oboudi is the one who introduces it [15].
2. When $\beta_1 = 0, \beta_2 = 1$, inclued the Komatu integral operator I_m^λ given by [13] is included. The operator $I_m^\lambda k(w)$ is connected to the study of Flett on multiplier transformation [16] Jung et al. [17] and Liu [18] analysed several interesting concepts that were examined using the operator I_m^λ.

The definitions and lemmas listed below are required to prove our primary conclusions.

Definition 2.4. *[9] The set of analytic and injective functions on* $\overline{\mathscr{L}}/E(q)$, *is denoted by the letter* Q, *where* $E(q) = \{x \in \partial\mathscr{L}; \lim_{r\to x} q(r) = \infty\}$, *and such that* $q'(x) \neq 0$ *for* $x \in \partial\mathscr{L}/E(q)$. *The subclass of* Q *for which* $q(0) = e$ *is referred to as* $Q(e)$.

Definition 2.5. *[9] Assume Ω is a set in $\mathbb{C}, q(r) \in Q$ with l to have a positive value. The category of admissible functions $\Psi_l[\Omega, q]$ includes all of the functions $\psi : \mathbb{C}^4 \times \mathscr{L} \to \mathbb{C}$ that meet the admission criteria*

$$\psi(a,s,d,u;r) \notin \Omega$$

whenever

$$a = q(x), \quad s = yxq(x), \quad \Re e\left\{1 + \frac{d}{s}\right\} \geq y\Re e\left\{1 + \frac{xq''(x)}{q'(x)}\right\},$$

and

$$\Re e\left\{\frac{u}{s}\right\} \geq y^2 \Re e\left\{\frac{x^2 q'''(x)}{q'(x)}\right\},$$

and $r \in \mathscr{L}, x \in \partial\mathscr{L}/E(q)$ and $y \geq \iota$.

Definition 2.6. *[10] Assume Ω is a set in $\mathbb{C}, q \in H[e,n]$ and $q'(r) \neq 0$. The category of admissible functions $\Psi'[\Omega, q]$ includes all of the functions $\psi : \mathbb{C}^4 \times \mathscr{L} \to \mathbb{C}$ that meet the admission criteria*

$$\psi(a,s,d,u;x) \in \Omega$$

whenever

$$a = q(r), \quad s = \frac{rq'(r)}{y}, \quad \Re e\left\{1 + \frac{d}{s}\right\} \leq \frac{1}{y}\Re e\left\{1 + \frac{rq''(r)}{q'(r)}\right\}, \text{ for } r \in \mathscr{L}$$

and

$$\Re e\left\{\frac{u}{s}\right\} \geq \frac{1}{y^2}\Re e\left\{\frac{r^2 q'''(r)}{q'(r)}\right\},$$

for $r \in \mathscr{L}, x \in \partial\mathscr{L}$ and $y \geq \iota \geq 1$.

Lemma 2.7. *[9] Let $\psi \in \Psi_l[\Omega, q]$ with $q(0) = e$. If the analytic function*

$$b(r) = e + e_l r^l + e_{t+1} r^{l+1} + \cdots, (r \in \mathscr{L})$$

meets the following criteria for inclusion $\psi\left(b(r), rb'(r), r^2 b''(r), r^3 b'''(r); r\right) \in \Omega$, then

$$b(r) \prec q(r)$$

Lemma 2.8. *[10] Let $\psi \in \Psi_t[\Omega, q]$ wtih $q(0) = e$. If $p \in Q(e)$ and*

$$\psi\left(b(r), rb'(r), r^2 b''(r), r^3 b'''(r); r\right)$$

is univalent in L, then

$$\Omega \subset \psi\left(b(r), rb'(r), r^2 b''(r), r^3 b'''(r); r\right),$$

implies

$$q(r) \prec b(r).$$

We will analyse the set of functions that can be used if operator $E^{\lambda,m}_{\beta_1,\beta_2} k(r)$ is used as defined by (11), we will gain a few outcomes of differential subordination and superordination of Oros [19, 20]. Ali et al. [21–23] and Cho et al. [24] have made some intriguing advances in differential subordination and superordination for several operators are connected.

3. THE MAIN RESULTS

Definition 3.1. *Assume Ω is a set in $\mathbb{C}, q \in Q_0 \cap H[0,m]$. The category of admissible functions $\Phi_{\mathrm{L}}[\Omega, q]$ includes all of the functions $\phi : \mathbb{C}^4 \times \mathscr{L} \to \mathbb{C}$ that meet the admission criteria*

$$\phi(m,v,c,n;r) \notin \Omega,$$

whenever

$$m = q(x), v = \frac{yxq'(x) + (\mu+1)q(x)}{\mu}, \Re\left\{\frac{\mu^2 c - (\mu+1)^2 m}{\mu v - (\mu+1)m} - 2\mu - 3\right\} \geq y\Re\left\{1 + \frac{xq''(x)}{q'(x)}\right\},$$

and

$$\Re\left\{\frac{\mu^3 n - \left(3\mu^3 + 9\mu^2\right)c + (3\mu+8)(\mu+1)^2 m}{\mu - (v+m) - m} - \left(3\mu^2 + 3\mu + 12\right)\right\} \geq y^2 \Re\left\{\frac{x^2 q'''(x)}{q'(x)}\right\},$$

for $r \in \mathscr{L}, x \in \partial\mathscr{L}/E(q)$ and $y \geq \iota$.

Theorem 3.2. *Assume $\phi \in \Phi_l[\Omega, q]$. If $k \in \mathscr{A}$ and $q \in Q_0$ fulfill the following requirements*

$$\Re\left\{\frac{xq''(x)}{q'(x)}\right\} \geq o, \left|\frac{E^{\lambda,m}_{\beta_1,\beta_2} k(r)}{q'(x)}\right| \leq y,$$

and

$$\left\{\phi\left(E^{\lambda,m}_{\beta_1,\beta_2} k(r), E^{\lambda+1,m}_{\beta_1,\beta_2} k(r), E^{\lambda+2,m}_{\beta_1,\beta_2} k(r), E^{\lambda+3,m}_{\beta_1,\beta_2} k(r); r\right); r \in \mathscr{L}\right\} \subset \Omega, \tag{13}$$

then

$$E^{\lambda,m}_{\beta_1,\beta_2} k(r) \prec q(r).$$

Proof. Let $g(r) \in \mathscr{L}$ define by

$$g(r) = E^{\lambda_1 m}_{\beta_1,\beta_2} k(r). \tag{14}$$

Based on the relationship (12) with the source (14), we have

$$E^{\lambda+1,m}_{\beta_1,\beta_2} k(r) = \frac{rg'(r) + (\mu+1)g(r)}{\mu}. \tag{15}$$

Then,

$$E^{\lambda+2,m}_{\beta_1,\beta_2} k(r) = \frac{r^2g''(r) + (2\mu+3)rg'(r) + (\mu+1)^2 g(r)}{\mu^2}, \tag{16}$$

and

$$E^{\lambda+3,m}_{\beta_1,\beta_2} k(r) = \frac{r^3g'''(r) + (3\mu+9)r^2g''(r) + \left(3\mu^2+3\mu+12\right) rg'(r) + (\mu+1)^3 g(r)}{\mu^3}, \tag{17}$$

Define the change from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$m(a,s,d,u) = a, v(a,s,d,u) = \frac{m+(\mu+1)a}{\mu}, c(a,s,d,u) = \frac{d+(2\mu+3)s+(\mu+1)^2 a}{\mu^2},$$

and

$$n(a,s,d,u) = \frac{u+(3\mu+9)d+\left(3\mu^2+3\mu+12\right)s+(\mu+1)^3 a}{\mu^3}.$$

Let

$$\begin{aligned} &\psi(a,s,d,u;r) = \phi(m,v,c,n;r) = \\ &\phi\left(a, \tfrac{m+(\mu+1)a}{\mu}, \tfrac{d+(2\mu+3)s+(\mu+1)^2 a}{\mu^2}, \tfrac{u+(3\mu+9)d+\left(3\mu^2+3\mu+12\right)s+(\mu+1)^3 a}{\mu^3}; r\right). \end{aligned} \tag{18}$$

The proof will use Lemma 1.7.Using equations (14) and (16) from (18), we have

$$\psi\left(g(r), rg'(r), r^2g''(r), r^3g'''(r); r\right) = \phi\left(E^{\lambda,m}_{\beta_1,\beta_2} k(r), E^{\lambda+1,m}_{\beta_1,\beta_2} k(r), E^{\lambda+2,m}_{\beta_1,\beta_2} k(r), E^{\lambda+3,m}_{\beta_1,\beta_2} k(r); r\right). \tag{19}$$

Therefore, (13) we have

$$\psi\left(g(r), rg'(r), r^2g''(r), r^3g'''(r); r\right) \in \Omega. \tag{20}$$

$$1 + \frac{d}{s} = \frac{\mu^2 c - (\mu+1)^2 m}{\mu v - (\mu+1)m} - 2\mu - 3$$

and
$$\frac{u}{s} = \frac{\mu^3 n - \left(3\mu^3 + 9\mu^2\right) c + (3\mu + 8)(\mu + 1)^2 m}{\mu - (v + m) - m} - \left(3\mu^2 + 3\mu + 12\right).$$
Because admissibility criterion has been met for $\psi \in \Psi_l[\Omega, q]$, by Lemma 1.7.
$$g(r) \prec q(r), \quad \text{or} \quad E^{\lambda,m}_{\beta_{1},\beta 2} k(r) \prec q(r).$$

Theorem 3.3. *Assume $\phi \in \Phi_l[h, q]$ with $q(0) = 1$. If the function $k \in \mathscr{A}$ and $q \in Q_0$ fulfill the following requirements*
$$\Re\left\{\frac{xq''(x)}{q'(x)}\right\} \geq o, \left|\frac{E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r)}{q'(x)}\right| \leq y.$$

and
$$\phi\left(E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r), E^{\lambda+1,m}_{\beta_{1},\beta_{2}} k(r), E^{\lambda+2,m}_{\beta_{1},\beta_{2}} k(r), E^{\lambda+3,m}_{\beta_{1},\beta_{2}} k(r); r\right) < h(r), \tag{21}$$

then
$$E^{\lambda_\beta,m}_{\beta_{1},\beta_{2}} k(r) \prec q(r).$$

The following is a continuation of Theorem 2.2to the case, such that $q(r)$ on $\partial\mathscr{L}$ has unclear behaviour.

Corollary 3.4. *If the function $k \in \mathscr{A}$ such that $\Omega \in \mathbb{C}, q(r)$ be univalent in $\mathscr{L}$ and $q(0) = 1$. Let $\phi \in \Phi_\iota\left[\Omega, q_\rho\right]$ for some $\rho \in (0, 1)$, where $q_\rho(r) = q(\rho r)$ and*
$$\phi\left(E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r), E^{\lambda+1,m}_{\beta_{1},\beta 2} k(r), E^{\lambda+2,m}_{\beta_{1},\beta 2} k(r), E^{\lambda+3,m}_{\beta_{1},\beta 2} k(r); r\right) \in \Omega,$$

then
$$E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r) \prec q(r).$$

Proof. By Theorem 2.2, we have $E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r) \prec q(\rho r)$. The outcome of the subordination connection is now known
$$q_\rho(r) \prec q(r).$$

Theorem 3.5. *Assume $h(r)$ with $q(r)$ be univalent in L, such that $q(0) = 1$ with $q_\rho(r) = q(\rho r)$ and $h_\rho(r) = h(\rho r)$. Assume $\phi : \mathbb{C}^4 \times \mathscr{L} \to \mathbb{C}$ fulfils one of requirements:*
(1) $\phi \in \Phi_l\left[h, q_\rho\right]$, for some $\rho \in (0, 1)$; or
(2) there is a $\rho_0 \in$ as a result $\phi \in \Phi_{\llcorner}\left[h_\rho, q_\rho\right]$, for all $\rho \in (\rho_0, 1)$. If the function $k \in \mathscr{A}$ satisfy (21), then
$$E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r) \prec q(r).$$

Proof. In this instance (1). We get Theorem 2.2 by applying it $E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r) \prec q_\rho(r)$, since $q_\rho(r) \prec q(r)$, we deduce
$$E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r) \prec q(r).$$

Case (2). If we let $g_\rho(r) = E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r) = E^{\lambda,m}_{\beta_{1},\beta_{2}} k(\rho r) = g(\rho r)$, then
$$\phi\left(g_\rho(r), rg'_\rho(r), r^2 g''_\rho(r), r^3 g'''_\rho(r); \rho r\right) = \phi\left(g(\rho r), rg'(\rho r), r^2 g''(\rho r), r^3 g'''(\rho r); \rho r\right) \in h_\rho(\mathscr{L}).$$

Using Theorem 2.2 and comment (20), $z(r) = \rho r$ is any mapping L in to L, we get $g_\rho(r) \prec q_\rho(r)$ for some $\rho \in (\rho_0, 1)$. By allowing $\rho \to 1^-$, we obtain
$$g(r) \prec q(r)$$

Hence,
$$E^{\lambda,m}_{\beta_{1},\beta_{2}} k(r) \prec q(r).$$

For the following outcome, strongest dominant of the differential subordination is necessary (21).

Theorem 3.6. *Let $h(r)$ be univalent in L and let $\phi : \mathbb{C}^4 \times \mathscr{L} \to \mathbb{C}$. Assume the differential equation is*

$$\phi\left(\begin{array}{c} q(r), \frac{rq'(r)+(\mu+1)q(r)}{\mu}, \frac{r^2q''(r)+(2\mu+3)rq'(r)+(\mu+1)^2q(r)}{\mu^2} \\ \frac{r^3q'''(r)+(3\mu+9)r^2q''(r)+\left(3\mu^2+3\mu+12\right)rq'(r)+(\mu+1)^3q(r)}{\mu^3}; r \end{array}\right) = h(r), \tag{22}$$

has a solution $q(r)$ such that $q(0) = 0$, fulfil one of the requirements:

(1) $q(r) \in Q_0$ and $\phi \in \Phi_\iota[h,q]$.

(2) $q(r)$ is univalent in L and $\phi \in \Phi_\iota\left[h,q_\rho\right]$ for some $\rho \in (0,1)$; or

(3) $q(r)$ is univalent in L, there is a $\rho_0 \in (0,1)$ and $\phi \in \Phi_\iota\left[h_\rho,q_\rho\right]$ for all $\rho \in (\rho_0,1)$. If the function $k \in \mathscr{A}$ satisfies (21), then $E^{\lambda,m}_{\beta_1,\beta_2}k(r) \prec q(r)$ and $q(r)$ is the most powerful.

Proof. We may derive that $q(r)$ is a dominant of $q(r)$ by applying Theorems 2.3 and Theorems 2.5. Because $q(r)$ satisfies (22), it is a solution of (21) and will thus be dominated by all dominants of (21). As a result, $q(r)$ is the most dominant of (21).

In this particular instance, $q(r) = Mr, M > 0$, and following Definition 1.5, the category of admissible functions $\Phi_\iota[\Omega,q]$ referred to as $\Phi_\iota[\Omega,M]$ is described further down.

Definition 3.7. *Assume Ω is a set in $\mathbb{C}, \Re\{m\} > 0, \lambda \geq 1$ with $M > 0$. The category of functions that are admissible $\Phi_l[\Omega,M]$ includes all of the functions $\phi : \mathbb{C}^4 \times \mathscr{L} \to \mathbb{C}$ that meet the admission criteria:*

$$\phi\left(\begin{array}{c} Me^{i\theta}, \frac{y+(\mu+1)Me^{i\theta}}{\mu}, \frac{L+\left[(2\mu+3)y+(\mu+1)^2Me^{i\theta}\right]}{\mu^2} \\ \frac{N+(3\mu+9)L+\left[\left(3\mu^2+3\mu+12\right)y+(\mu+1)^3Me^{i\theta}\right]}{\mu^3} \end{array}\right) \notin \Omega, \tag{23}$$

whenever $\theta \in \mathbb{R}, \Re\left(Le^{i\theta}\right) \geq y(y-1)M, y \geq 2$.

Corollary 3.8. *Assume $\phi \in \Phi_l[\Omega,M]$. If $k \in \mathscr{A}$ meet the following criteria for inclusion*

$$\left|E^{\lambda,m}_{\beta_1,\beta_2}k(r)\right| \leq yM, (M > 0, y \geq 2)$$

and

$$\phi\left(E^{\lambda,m}_{\beta_1,\beta_2}k(r), E^{\lambda+1,m}_{\beta_1,\beta_2}k(r), E^{\lambda+2,m}_{\beta_1,\beta_2}k(r), E^{\lambda+3,m}_{\beta_1,\beta_2}k(r); r\right) \in \Omega,$$

then

$$E^{\lambda,m}_{\beta_1,\beta_2}k(r) < Mr.$$

Let us look at a unique situation, $\Omega = q(\mathscr{L}) = \{r : |r| < M\}$, the class $\Phi_\iota[\Omega,M]$ is referred to as $\Phi_l[M]$.

Corollary 3.9. *Let $\phi \in \Phi_\iota[M]$. If the function $k \in \mathscr{A}$ fulfils the following criteria:*

$$\left|E^{\lambda,m}_{\beta_1,\beta_2}k(r)\right| \leq yM, (M > 0, y \geq 2)$$

and

$$\left|\phi\left(E^{\lambda,m}_{\beta_1,\beta_2}k(r), E^{\lambda+1,m}_{\beta_1,\beta_2}k(r), E^{\lambda+2,m}_{\beta_1,\beta_2}k(r), E^{\lambda+3,m}_{\beta_1,\beta_2}k(r); r\right)\right| < M,$$

then

$$\left|E^{\lambda,m}_{\beta_1,\beta_2}k(r)\right| < M.$$

Definition 3.10. *Assume Ω is a set in $\mathbb{C}, q \in Q_0 \cap H[0,p], q'(r) \neq 0$ with $\Re\{m\} > 0, \lambda \geq 1$. The category of admissible functions $\Phi_\iota[\Omega,q]$ includes all of the functions $\phi : \mathbb{C}^4 \times \mathscr{L} \to \mathbb{C}$ that meet the admission criterion:*

$$\phi(m,v,c,n;r) \in \Omega,$$

therefore,

$$m = q(r), v = \frac{rq'(r)/y + (\mu+1)q(r)}{\mu}, \Re\left\{\frac{\mu^2 c - (\mu+1)^2 m}{\mu v - (\mu+1)m} - 2\mu + 3\right\} \leq \frac{1}{y}\Re\left\{1 + \frac{rq''(r)}{q'(r)}\right\},$$

and

$$\Re\left\{\frac{\mu^3 n - \left(3\mu^3 + 9\mu^2\right)c + (3\mu+8)(\mu+1)^2 m}{\mu - (v+m) - m} - \left(3\mu^2 + 3\mu + 12\right)\right\} \geq \frac{1}{y^2}\Re\left\{\frac{r^2 q'''(r)}{q'(r)}\right\},$$

for $\lambda \geq 1, y \geq p$ with $r \in \mathscr{L}$. .

Theorem 3.11. *Assume $\phi \in \Phi_l[\Omega,q]$. If $k \in \mathscr{A}$ and $E^{\lambda,m(r)}_{\beta^{\lambda,m}_{1,\beta_2}} \in Q_0$ with $q'(r) \neq 0$ meet the criteria listed below*

$$\Re\left\{\frac{rq''(r)}{q'(r)}\right\} \geq 0, \quad \left|\frac{E^{\lambda,m}_{\beta_1,\beta_2} k(r)}{q'(r)}\right| \leq y,$$

and the function $\phi\left(E^{\lambda_2}_{\beta^{\lambda,m}_{1,\beta_2}} k(r), E^{\lambda+1,m}_{\beta_1,\beta_2} k(r), E^{\lambda+2,m}_{\beta_1,\beta_2} k(r), E^{\lambda+3,m}_{\beta_1,\beta_2} k(r); r\right)$ is univalent in L, then

$$\Omega \subset \left\{\phi\left(E^{\lambda_2}_{\beta^{\lambda,m}_{1,\beta_2}} k(r), E^{\lambda+1,m}_{\beta_1,\beta_2} k(r), E^{\lambda+2,m}_{\beta_1,\beta_2} k(r), E^{\lambda+3,m}_{\beta_1,\beta_2} k(r); r\right); r \in \mathscr{L}\right\}, \tag{24}$$

implies

$$q(r) < E^{\lambda,m}_{\beta_1,\beta_2} k(r).$$

Proof. By using (19) and (24), we get

$$\Omega \subset \psi\left(g(r), rg'(r), r^2 g''(r), r^3 g'''(r); r\right), (r \in \mathscr{L}).$$

We can see from (18) that the admissibility criterion for $\phi \in \Phi'_\iota[\Omega,q]$ is the same as that for in Definition 1.6. As a result, and following Lemma 1.8, we obtain

$$q(r) < g(r) \quad \text{or} \quad q(r) < E^{\lambda,m}_{\beta_1,\beta_2} k(r), (r \in \mathscr{L}).$$

Theorem 3.12. *Let $h(r)$ be analytic on L and $\phi \in \Phi'_\iota[h,q]$. If the function $k \in \mathscr{A}$ and $E^{\lambda,m}_{\beta_1,\beta_2} k(r) \in Q_0$ and $\phi : \mathbb{C}^4 \times \mathscr{L} \to \mathbb{C}$ with $\phi\left(E^{\lambda,m}_{\beta_1,\beta_2} k(r), E^{\lambda+1,m}_{\beta_1,\beta_2} k(r), E^{\lambda+2,m}_{\beta_1,\beta_2} k(r), E^{\lambda+3,m}_{\beta_1,\beta_2} k(r); r\right)$ is univalent in L, then*

$$h(r) \prec \phi\left(E^{\lambda,m}_{\beta_1,\beta_2} k(r), E^{\lambda+1,m}_{\beta_1,\beta_2} k(r), E^{\lambda+2,m}_{\beta_1,\beta_2} k(r), E^{\lambda+3,m}_{\beta_1,\beta_2} k(r); r\right) \tag{25}$$

which implies that

$$q(r) \prec E^{\lambda,m}_{\beta_1,\beta_2} k(r).$$

Proof. We obtain relationship (25) by employing

$$h(r) = \Omega \subset \phi\left(E^{\lambda,m}_{\beta_1,\beta_2}k(r), E^{\lambda+1,m}_{\beta_1,\beta_2}k(r), E^{\lambda+2,m}_{\beta_1,\beta_2}k(r), E^{\lambda+3,m}_{\beta_1,\beta_2}k(r); r\right).$$

and from Theorem 2.11., we have

$$q(r) \prec E^{\lambda,m}_{\beta_1,\beta_2}k(r).$$

We combine Theorem 2.3 with Theorem 2.12, and obtain the sandwich-type Theorem shown below.

Theorem 3.13. *Assume $h_1(r), q_1(r)$ are analytic functions in $\mathscr{L}$, $h_2(r)$ is a univalent function in $\mathscr{L}$, $q_2(r) \in Q_0, q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_l[h_2, q_2] \cap \Phi'_l[h_1, q_1]$. If the function $k \in \mathscr{A}$ and $E^{\lambda,m}_{\beta_1,\beta_2}k(r) \in Q_0 \cap H[0,P]$ and $\phi\left(E^{\lambda,m}_{\beta_1,\beta_2}k(r), E^{\lambda+1,m}_{\beta_1,\beta_2}k(r), E^{\lambda+2,m}_{\beta_1,\beta_2}k(r), E^{\lambda+3,m}_{\beta_1,\beta_2}k(r); r\right)$ is a univalent in L, then*

$$h_1(r) \prec \phi\left(E^{\lambda,m}_{\beta_1,\beta_2}k(r), E^{\lambda+1,m}_{\beta_1,\beta_2}k(r), E^{\lambda+2,m}_{\beta_1,\beta_2}k(r), E^{\lambda+3,m}_{\beta_1,\beta_2}k(r); r\right) \prec h_2(r)$$

implies

$$q_1(r) \prec E^{\lambda,m}_{\beta_1,\beta_2}k(r) \prec q_2(r).$$

4. CONCLUSIONS

The subclass of analytical univalent function related to the idea differential subordination is investigated. We examined the differential subordination and superordination results involving a specific class of univalent functions defined on the open unit disc space of univalent functions. We obtained several properties of subordinations and superordinations connected with the Hadamard product notion using attributes of the Komatu integral operator. We studied the various features of subordinations and superordinations utilizing properties of the generalized derivative operator and proved certain theorems. We also obtained conclusions on third-order differential subordination using a linear operator.

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