Using Quasi-Subordination to Solve the Fekete-Szego Problem for a Subclass of Meromorphic Functions

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ABSTRCT- *For subclasses of meromorphic functions formed on the open unit disk in the complex plane, constraints for the Fekete-Szegö coefficient functional associated with quasi-subordination have been found.*

Keywords: *Analytic Function, Meromorphic Function, Convex function, Quasi-Subordination, Fekete-Szego problem.*

2010MathematicsSubjectClassification: 05A30, 30C45.

I. INTRODUCTIONAND DEFINITIONS

Let $\mathfrak{I} = \{w \in \mathbb{C} : |w| < 1\}$ be an open unit disc in $\mathbb{C}$. Let $H(\mathfrak{I})$ be the class of analytic functions in $\mathfrak{I}$ and consider $\mathfrak{I}[a, i]$ to be a subclass of $H(\mathfrak{I})$of the form

$$h(w) = a + a_i w^i + a_{i+1} w^{i+1} + \cdots ,$$

where $a \in \mathbb{C}$ and $i \in \mathbb{N} = \{1,2, \dots\}$. Let the class of all meromorphic functions be Σ of the form

$$h(w) = w^{-1} + \sum_{i=0}^{\infty} a_i w^i, \quad (w \in \mathfrak{I}^*) \qquad (1)$$

such that

$$\mathfrak{I}^* = \{w : w \in \mathbb{C} \ and \ 0 < |w| < 1\} = \mathfrak{I} \backslash \{0\}.$$

The Hadamrd product for two functions in Σ, such that

$$k(w) = w^{-1} + \sum_{i=0}^{\infty} c_i w^i, \quad (w \in \mathfrak{I}^*) \qquad (2)$$

is given by

$$h(w) * k(w) = w^{-1} + \sum_{i=0}^{\infty} a_i c_i w^i. \quad (w \in \mathfrak{I}^*)(3)$$

The subclass $\Sigma^*(\gamma)$ of the class $\Sigma$are meromorphcallystarlike functions of the γ order. A function $h \in \Sigma^*(\gamma)$ of the kind (1) if

$$\Re e\left\{-\frac{wh'(w)}{h(w)}\right\} > \gamma. \qquad (w \in \mathfrak{I}^*)$$

Pommerenke [29] introduced and researched the class $\Sigma^*(\gamma)$(see also Miller [25]).

Now, we let $h(w)$ and $k(w)$ be analytic function in $\mathfrak{I}$.The function $h(w)$is said to be subordinate to a function $k(w)$ or $k(w)$ is said to be superordinate to $h(w)$, if and only if there exists a Schwarz function $z(w)$ analytic in $\mathfrak{I}$, with $z(0) = 0$ and $|z(w)| < 1, (w \in \mathfrak{I})$, such that

$$h(w) = k(z(w)),$$

written as

$$h \prec k \ or \ h(w) \prec k(w), \quad (w \in \mathfrak{I}).$$

Furthermore, if the function h is univalent in $\mathfrak{I}$, then we get the following equivalence $h(w) \prec k(w)$if and only if$h(0) = k(0)$and $h(\mathfrak{I}) \subset k(\mathfrak{I})$ [26].

Let $\mathcal{g}(w)$ be an analytic function on $\mathfrak{I}$ that satisfies $\mathcal{g}(0) = 1$ and $\mathcal{g}'(0) > 0$, mapping $\mathfrak{I}$ onto a region that is starlike with respect to 1 and symmetric with respect to the real axis. Let $\Sigma^*(\gamma)$ be the set of functions $h \in \Sigma$ for which

$$-\frac{wh'(w)}{h(w)} \prec g(w).$$

Silverman et al. [34] proposed and investigated the $\Sigma^*(g)$ class (see also [6, 14]). When $g(w) = \frac{1+(1-2\gamma)w}{1-w}(0 \le \gamma < 1)$, the class $\Sigma^*(\gamma)$ is a subclass of $\Sigma^*(g)$.

Robertson [33] developed the notion of quasi subordination in 1970. The function $h(w)$ is quasi-subordinate to $k(w)$ for two analytic functions h and k, as written:

$$h(w) \prec_q k(w),$$

if analytic functions φ and z exist with $|\varphi(w)| \le 1$, $z(0) = 0$, and $|z(w)| < 1$, then

$$h(w) = \varphi(w)k(z(w)).$$

When $\varphi(w) = 1, h(w) = k(z(w))$, indicating that $h(w) \prec k(w)$ in D. It's also worth noting that if $z(w) = w$, then $h(w) = \varphi(w)k(w)$, and h is majorized by k, as written $h(w) \ll k(w)$ in D. As a result, it is self-evident that quasi-subordination is a generalization of both subordination and majorization. For works on quasi-subordination, see [5, 15, 16, 17, 23, 32]. In this study, g is assumed to be analytic in D, with $g(0) = 1$.

Definition 1. Let $\Sigma_q^*(g)$be the class of function $h(\mathrm{w}) \in \Sigma$ satisfying the quasi-subordination

$$-\frac{wh'(w)}{h(w)} - 1 \prec_q g(w) - 1. \qquad (4)$$

Mohd and Darus [27] created and researched the class $S_q^*(g)$, which is the meromorphic analogue of the class $\Sigma_q^*(g)$, which is made up of function $h(w)$ of the type $w + \sum_{i=2}^{\infty} a_i w^i$, for which

$$\frac{wh'(w)}{h(w)} - 1 \prec_q g(w) - 1. \qquad (5)$$

Now, we define the following class, which is inspired by [24, 33].

Definition 2. For $d \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$and $\mu \in \mathbb{C}\backslash(0,1]$. Let the class $N_q(\mu, g)$, consists of function

$h \in \Sigma$ satisfying the quasi-subordination

$$\frac{1}{d}\left[\frac{wh' + (1+2\mu)w^2h'' + \mu w^3 h'''}{wh' + \mu w^2 h''} - 1\right] \prec_q g(w) - 1. \qquad (6)$$

The i-th coefficient of a meromorphic function $h \in \Sigma$ is known to be restricted by i (see [13]). The coefficient bounds provide details about the function's geometric features. Many writers have looked at the Fekete-Szego coefficient bounds for different classes [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 20, 18, 19, 22, 30, 31]. The coefficient estimates for the functions in the above-mentioned class are obtained in this study.

Let$\mho$ be the class of functions of the form

$$z(w) = z_1 w + z_2 w^2 + z_3 w^3 + \cdots,$$

satisfying $|z(w)| < 1$ for $w \in \Im$.

We'll need the lemma below to back up our findings.

Lemma 3. [21]: If $\varphi \in P$ then $|r_i| \le 2$ for each i, where P is the family of all functions φ analytic in $\Im$ for which $\Re\{\varphi(w)\} > 0$,

$$\varphi(w) = r_0 + r_1 w + r_2 w^2 + r_3 w^3 + \cdots. \quad (7)$$

Lemma 4. [21]: If $\varphi(w) = r_0 + r_1 w + r_2 w^2 + r_3 w^3 + \cdots$ for $w \in \Im$. Is function with positive real part in $\Im$ and τ is complex number, then

$$|z_2 - \tau z_1^2| \le 2max\{1; |2\tau - 1|\}. \qquad (8)$$

Lemma 5. [21]: If $z \in \mho$, then for any complex number τ

$$|z_2 - \tau z_1^2| \le max\{1; |\tau|\}. \qquad (9)$$

The result is sharp for the functions $z(w) = w$ or $z(w) = w^2$.

II. MAIN RESULTS

Throughout, let $g(w) = 1 + E_1 w + E_2 w^2 + E_3 w^3 + \cdots, E_1 > 0$ and $\varphi(w) = r_0 + r_1 w + r_2 w^2 + r_3 w^3 + \cdots$.

Theorem 1: If $h(w)$ given by (1) belongs to $\Sigma_q^*(g)$, then

$$|a_0| \le E_1 \ ,$$

$$|a_1| \le \frac{E_1}{2}\left[1 + \left|z_2 + \left(\frac{E_2}{E_1} - r_0 E_1\right) z_1^2\right|\right], \quad (10)$$

and τ is any complex number,

$$|a_{1-}\tau a_0^2| \le \frac{E_1}{2}\left[1 + max\left\{1, + \left|\frac{E_2}{E_1}\right| + E_1|1 - 2\tau|\right\}\right]. \qquad (11)$$

Proof. If $h(w)$ belongs to $\Sigma_q^*(\mathscr{g})$, then there are analytic functions $z(w)$ and $\varphi(w)$, with $z(0) = 0, |z(w)| < 1$ and $|\varphi(w)| < 1$ such that

$$-\frac{wh'(w)}{h(w)} - 1 = \varphi(w)[\mathscr{g}(z(w)) - 1].$$

Since

$$-\frac{wh'(w)}{h(w)} = 1 - a_0 w + (a_0^2 - 2a_1)w^2 + \cdots,$$

$$\mathscr{g}(z(w)) = 1 + E_1 z_1 w + (E_1 z_2 + E_2 z_1^2)w^2 + (E_1 z_3 + 2E_2 z_1 z_2 + E_3 z_1^3)w^3 + \cdots,$$

and

$$\varphi(w)[\mathscr{g}(z(w)) - 1] = r_0 E_1 z_1 w + (r_0 E_1 z_2 + r_0 E_2 z_1^2 + r_1 E_1 z_1)w^2 \ldots, \quad (12)$$

then

$$a_0 = -r_0 E_1 z_1 \quad ,$$

$$a_1 = -\frac{r_0 E_1}{2}\left[z_2 + \frac{r_1}{r_0} z_1 + \left(\frac{E_2}{E_1} - r_0 E_1\right) z_1^2\right],$$

and since $\varphi(w)$ is analytic and bounded in D, we get [28]

$$|r_i| \le 1 - |r_i|^2 \le 1, \quad (i > 0).$$

Using this fact, as well as the well-known inequality, $|z_1| < 1$, we obtain

$$|a_0| \le E_1 \ ,$$

$$|a_1| \le \frac{E_1}{2}\left[1 + \left|z_2 + \left(\frac{E_2}{E_1} - r_0 E_1\right) z_1^2\right|\right].$$

Thus,

$$a_1 - \tau a_0^2 = \frac{r_0 E_1}{2}\left[z_2 + \frac{r_1}{r_0} z_1 + \left(\frac{E_2}{E_1} - r_0 E_1 + 2\tau r_0 E_1\right) z_1^2\right],$$

and

$$|a_1 - \tau a_0^2| \le \frac{|r_0| E_1}{2}\left[\left|\frac{r_1}{r_0} z_1\right| + \left|z_2 + \left(\frac{E_2}{E_1} - r_0 E_1 + 2\tau r_0 E_1\right) z_1^2\right|\right].$$

Since

$$|r_i| \le 1 - |r_i|^2 \le 1, \quad (i > 0) \text{and} |z_1| < 1$$

Then, we have

$$|a_1 - \tau a_0^2| \le \frac{E_1}{2}\left[1 + \left|z_2 + \left(\frac{E_2}{E_1} - r_0 E_1 + 2\tau r_0 E_1\right) z_1^2\right|\right].$$

After applying Lemma 1.5 to the result (11) for the functions, the result is sharp

$$-\frac{wh'(w)}{h(w)} - 1 = \varphi(w)[\mathscr{g}(2w^2) - 1],$$

and

$$-\frac{wh'(w)}{h(w)} - 1 = \varphi(w)[\mathscr{g}(w) - 1].$$

The proof of Theorem 2.1 is now complete.

Remark2: We get the result given by Silverman et al.[34] by putting $\varphi(w) = 1$ in Theorem 2.1.

Theorem 3: If $h(w)$ belongs to Σ satisfies

$$-\frac{wh'(w)}{h(w)} - 1 \ll \mathscr{g}(w) - 1,$$

if τ is a complex number, then

$$|a_1 - \tau a_0^2| \le \frac{E_1}{2}\left[1 + \left|\frac{E_2}{E_1}\right| + E_1|1 - 2\tau|\right]. \quad (13)$$

Proof. In the proof of Theorem 2.1, we get the result by choosing $z(w) = w$.

Theorem 4: If $h(w)$ given by (1) belongs to $N_q(\mu, \mathscr{g})$, $\mu \in \mathbb{C}\backslash(0,1]$, then

$$|a_0| \le \frac{dE_1}{2(1+\mu)} \ ,$$

$$|a_1| \le \frac{d}{6(1+2\mu)}(E_1 + max\{E_1, E_1^2 + |E_2|\}), \quad (14)$$

and τ is any complex number,

$$|a_1 - \tau a_0^2| \le \frac{d}{6(1+2\mu)}\left(E_1 + max\left\{E_1, \left|1 - \frac{3(1+2\mu)}{2(1+\mu)^2}\tau\right| E_1^2 + |E_2|\right\}\right). \quad (15)$$

Proof. If $h(w) \in N_q(\mu, \mathscr{g}), \mu \in \mathbb{C}\backslash(0,1]$, then there are analytic functions $z(w)$ and $\varphi(w)$, with $z(0) = 0, |z(w)| < 1$ and $|\varphi(w)| < 1$ such that

$$\frac{1}{d}\left[\frac{wh' + (1+2\mu)w^2 h'' + \mu w^3 h'''}{wh' + \mu w^2 h''} - 1\right] = \varphi(w)[\mathscr{g}(z(w)) - 1]. \quad (16)$$

Since

$$\frac{wh' + (1+2\mu)w^2 h'' + \mu w^3 h'''}{wh' + \mu w^2 h''} - 1 = 2(1+\mu)a_0 w + (-4(1+\mu)^2 a_0^2 + 6(1+2\mu)a_1)w^2 + \cdots,$$

$$\mathcal{g}(z(w)) - 1 = E_1 z_1 w + (E_1 z_2 + E_2 z_1^2) w^2 + (E_1 z_3 + 2E_2 z_1 z_2 + E_3 z_1^3) w^3 + \cdots,$$

and

$$\varphi(w)[\mathcal{g}(w) - 1] = r_0 E_1 z_1 w + (r_0 E_1 z_2 + r_0 E_2 z_1^2 + r_1 E_1 z_1) w^2 \ldots,$$

by from (16), we gat

$$a_0 = \frac{d r_0 E_1 z_1}{2(1+\mu)} \ ,$$

$$a_1 = \frac{d}{6(1+2\mu)} (r_0 E_1 z_2 + r_0 (E_2 + E_1^2 r_0) z_1^2 + r_1 E_1 z_1),$$

and since $\varphi(w)$ is analytic and bounded in D, we get [28]

$$|r_i| \le 1 - |r_i|^2 \le 1, \ \ (i > 0).$$

Using this fact, as well as the well-known inequality, $|z_1| < 1$, we obtain

$$|a_0| \le \frac{dE_1}{2(1+\mu)} \ ,$$

$$|a_1| \le \frac{d}{6(1+2\mu)} (E_1 + max\{E_1, E_1^2 + |E_2|\}).$$

Thus,

$$a_{1-}\tau a_0^2 = \frac{d}{6(1+2\mu)} \left(r_1 E_1 z_1 + r_0 \left(E_1 z_2 + \left(E_2 + r_0 E_1^2 - \frac{3(1+2\mu)}{2(1+\mu)^2} \mu r_0 E_1^2 \right) z_1^2 \right) \right),$$

and

$$|a_{1-}\tau a_0^2| \le \frac{d}{6(1+2\mu)} \left(|r_1 E_1 z_1| + \left| r_0 E_1 \left(z_2 - \left(\frac{3(1+2\mu)}{2(1+\mu)^2} \mu r_0 E_1 - r_0 E_1 - \frac{E_2}{E_1} \right) z_1^2 \right) \right| \right).$$

Again applying

$$|r_i| \le 1 - |r_i|^2 \le 1, \ \ (i > 0) \text{and} |z_1| < 1$$

Then, we have

$$|a_{1-}\tau a_0^2| \le \frac{dE_1}{6(1+2\mu)} \left(1 + \left| z_2 - \left(-\left(1 - \frac{3(1+2\mu)}{2(1+\mu)^2} \mu \right) r_0 E_1 - \frac{E_2}{E_1} \right) z_1^2 \right| \right).$$

After applying Lemma 1.5 to

$$\left| z_2 - \left(-\left(1 - \frac{3(1+2\mu)}{2(1+\mu)^2} \mu \right) r_0 E_1 - \frac{E_2}{E_1} \right) z_1^2 \right|$$

yields

$$|a_{1-}\tau a_0^2| \le \frac{dE_1}{6(1+2\mu)} \left(1 + max \left\{ 1, \left| -\left(1 - \frac{3(1+2\mu)}{2(1+\mu)^2} \tau \right) r_0 E_1 - \frac{E_2}{E_1} \right| \right\} \right).$$

Take note of this

$$\left| -\left(1 - \frac{3(1+2\mu)}{2(1+\mu)^2} \tau \right) r_0 E_1 - \frac{E_2}{E_1} \right| \le |r_0| E_1 \left| 1 - \frac{3(1+2\mu)}{2(1+\mu)^2} \tau \right| + \left| \frac{E_2}{E_1} \right|,$$

as a result, we can deduce that

$$|a_{1-}\tau a_0^2| \le \frac{d}{6(1+2\mu)} \left(E_1 + max \left\{ E_1, \left| 1 - \frac{3(1+2\mu)}{2(1+\mu)^2} \tau \right| E_1^2 + |E_2| \right\} \right).$$

Remark 5: By putting $\varphi(w) = 1$ and $d = 1$, The above will be reduced to a $|a_1|$ estimate.

Theorem 6: If $h(w)$ belongs to Σ satisfies

$$\frac{1}{d} \left[\frac{wh' + (1+2\mu) w^2 h'' + \mu w^3 h'''}{wh' + \mu w^2 h''} - 1 \right] \ll \mathcal{g}(w) - 1, \qquad (17)$$

the following inequalities arise as a result of this:

$$|a_0| \le \frac{dE_1}{2(1+\mu)} \ ,$$

$$|a_1| \leq \frac{d}{6(1+2\mu)}(E_1, E_1^2 + |E_2|),$$

and τ is any complex number,

$$|a_{1-}\tau a_0^2| \leq \frac{d}{6(1+2\mu)}\left(E_1 + \left|1 - \frac{3(1+2\mu)}{2(1+\mu)^2}\tau\right| E_1^2 + |E_2|\right).$$

Proof. In the proof of Theorem 1, we get the result by choosing $z(w) = w$.

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