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Certain Geometric Properties of Meromorphic Functions Defined by a New Linear Differential Operator

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Abstract. In the present paper, we investigate a new linear operator through the Hadamard product between the fundamental hypergeometric function and the Mittag Leffler function. Furthermore, the geometric properties of a new meromorphic function subclass will be investigated.

Keywords: Mittag-Leffler, Meromorphic functions, Starlike functions, Convex functions, Differential operator, q-hypergeometric functions, Hadamard product.

1. Introduction and Definitions

Let $\mathfrak{I} = \{w \in \mathbb{C} : |w| < 1\}$ be an open unit disc in $\mathbb{C}$. Let $H(\mathfrak{I})$ be the class of analytic functions in $\mathfrak{I}$ and consider $\mathfrak{I}[a, i]$ to be a subclass of $H(\mathfrak{I})$ of the form

$$h(w) = a + a_i w^i + a_{i+1} w^{i+1} + \cdots,$$

where $a \in \mathbb{C}$ and $i \in \mathbb{N} = \{1,2,\ldots\}$. Let the class of all analytic functions be Σ of the form

$$h(w) = w^{-1} + \sum_{i=0}^{\infty} a_i w^i, \quad (w \in \mathfrak{I}^*) \tag{1}$$

such that

$$\mathfrak{I}^* = \{w : w \in \mathbb{C} \; and \; 0 < |w| < 1\} = \mathfrak{I} \backslash \{0\}.$$

The subclasses $\Sigma^*(\gamma)$ and $\Sigma_j(\gamma)$ of the class Σ are meromorphcally starlike functions of the γ order and meromorphcally convex functions of the γ order respectvely. A function $h \in \Sigma^*(\gamma)$ of the kind (1) if

$$\Re e\left\{-\frac{wh'(w)}{h(w)}\right\} > \gamma, \qquad (w \in \mathfrak{I}^*)$$

and $h \in \Sigma_j(\gamma)$ if

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$$\Re e\left\{-\left(\frac{wh''(w)}{h'(w)}+1\right)\right\} > \gamma. \qquad (w \in \Im^*)$$

The Hadamrd product for two functions in Σ, such that

$$k(w) = w^{-1} + \sum_{i=0}^{\infty} c_i\, w^i, \ \ (w \in \Im^*) \tag{2}$$

is given by

$$h(w) * k(w) = w^{-1} + \sum_{i=0}^{\infty} a_i\, c_i\, w^i. \ \ (w \in \Im^*) \tag{3}$$

For parameters $a_j, c_n, p(j = 1, \dots, l, n = 1, \dots, q, c_n \in \mathbb{C} \setminus \{0, -1, -2, \dots\})$, the p-hypergeometric function ${}_l\Upsilon_q$ is defined by following (see [6,17]):

$${}_l\Upsilon_q\big(a_{1,} \dots, a_l; c_1, \dots, c_q\,; p; w\big) = \sum_{i=0}^{\infty} \frac{(a_1,p)_i \dots (a_l,p)_i}{(p,p)_i (c_1,p)_i \dots (c_q,p)_i} \times \left[(-1)^i p^{\binom{i}{2}}\right]^{1+q-1} w^i, \tag{4}$$

with $l \le q + 1; l, q \in \mathbb{N} \cup \{0\}$; $\mathbb{N} = \{1,2, \dots\}$ and $\binom{i}{2} = \frac{i(i-1)}{2}$, where$(a, p)_i$ is the p-anlogue of the symbol of Pochhammer $(a)_i$ is defined in terms of

$$(a, p)_i = \begin{cases} (1-a)(1-ap)(1-ap^2)\dots\big(1-ap^{i-1}\big) & for\ i \in \mathbb{N} \\ 1 & if\ i = 0 \end{cases}. \tag{5}$$

Heine introduced the hypergeometric series described by (4) in 1846, and it was known as the Heines series at the time. Further information on q-theory would be available for readers in [10,11,20]. It is obvious that

$$\lim_{p \to 1-}\left[\,{}_l\Upsilon_q(p^{a_{1,}} \dots, p^{a_l}; p^{c_1}, \dots, p^{c_q}; p; (p-1)^{1+q-1} w)\right] = {}_l\mathrm{F}_q\big(a_{1,} \dots, a_l; c_1, \dots, c_q\,; w\big), \tag{6}$$

where ${}_l\mathrm{F}_q\big(a_{1,} \dots, a_l; c_1, \dots, c_q\,; w\big)$ the generalized hypergeometric function represents (as shown in[12,19]). Riemann, Gauss, Euler, and others conducted detailed studies of hypergeomtric functions a few hundred years ago. The emphasis in this field is primarily on the structural beauty of this theory and its specific applications, which include dynamic systems, mathematical mechanics, numerical analysis, and combinatorics. Based on this, hypergeomtric functions are used in a variety of fields. consisting of geometric function theory one example that can be associated with hypergeomtric functions is the well-known Dziok-Srivestava operator [6,7] defined by the Hadamard.

The p-hypergeometric function defined in (1.4), for $w \in \Im, |p| < 1$ and $l = q + 1$, take the form below:

$${}_l\Upsilon_q\big(a_{1,} \dots, a_l; c_1, \dots, c_q\,; p; w\big) = \sum_{i=0}^{\infty} \frac{(a_1,p)_i \dots (a_l,p)_i}{(p,p)_i (c_1,p)_i \dots (c_q,p)_i} w^i, \tag{7}$$

which converges absolutely in the open unit disk $\Im$.

The p-analog of the Liu-Srivastava operator was recently introduced with reference to the function ${}_l\Upsilon_q\big(a_{1,} \dots, a_l; c_1, \dots, c_q\,; p; w\big)$ for meromorphic functions $h \in \Sigma$ consisting in the type of functions (1) (see Darus and Aldwby[1],Janani and Murugusundaramorthy[17]), as outlined beneath.

$${}_l\Phi_q\big(a_{1,} \dots, a_l; c_1, \dots, c_q\,; p; w\big)h(w) = w^{-1}\,{}_l\Upsilon_q\big(a_{1,} \dots, a_l; c_1, \dots, c_q\,; p; w\big) * h(w)$$

$$= w^{-1} + \sum_{i=0}^{\infty} \frac{\prod_{j=1}^{l}(a_j, p)_{i+1}}{(p,p)_{i+1} \prod_{n=1}^{q}(c_n, p)_{i+1}} a_i\, w^i. \quad (8)$$

For convenience, we write

$${}_l\Phi_q\left(a_{1,} \ldots, a_l; c_1, \ldots, c_q\,; p; w\right)h(w) = {}_l\Phi_q\left(a_{i,}; c_n; p; w\right)h(w).$$

Before going on, we state the famous Mittag–Leffler function $\mathrm{T}_\nu(w)$,introduced by Mittag–Leffler [8,16], as well as the generalisation of Wiman [23] $\mathrm{T}_{\nu,\tau}(w)$ given respctively as follows:

$$\mathrm{T}_\nu(w) = \sum_{i=0}^{\infty} \frac{w^i}{\Gamma(\nu i + 1)}, \quad (9)$$

and

$$\mathrm{T}_{\nu,\tau}(w) = \sum_{i=0}^{\infty} \frac{w^i}{\Gamma(\nu i + \tau)}, \quad (10)$$

where $\nu, \tau \in \mathbb{C}, Re(\nu) > 0 \; and \; Re(\tau) > 0$.

In recent years, there has been a growing emphasis on Mittag-Leffler-type roles that focus on the development of possibilities for their probability application, applied problems, statistical and distribution theory, and so on. More information on how the Mittag-Leffler functions are used can be found in [3,21]. In most of our Mittag-Leffler function work, we investigate geometric properties such as convexity, close-to-convexity, and starlikeness, for more knowledge on the Mittag–Leffler function refer to see [4], and [18] presents results relevant to partial.

The function defined by (10) does not fall within class Σ. Based on this case, the function is then normlized as follows:

$$\mathrm{E}_{\nu,\tau}(w) = w^{-1}\Gamma(\tau)\mathrm{T}_{\nu,\tau}(w) = w^{-1} + \sum_{i=0}^{\infty} \frac{\Gamma(\tau)}{\Gamma(\nu(i+1)+\tau)} w^i\,, \quad (11)$$

Using the function $\mathrm{E}_{\nu,\tau}(w)$ given by (11), the new operator $\mathcal{F}_\tau^{\nu,m}\left[\, a_{i,}, c_n, \alpha\right]: \Sigma \to \Sigma$ is defined as follows in terms of the Hadamard product:

$$\mathcal{F}_\tau^{\nu,0}\left[\, a_{i,}, c_n, \alpha\right]h(w) = {}_l\Phi_q\left(a_{i,}; c_n; p; w\right)h(w) * \mathrm{E}_{\nu,\tau}(w),$$

$$\mathcal{F}_\tau^{\nu,1}\left[\, a_{i,}, c_n, \alpha\right]h(w) =$$
$$(1-\alpha)\left({}_l\Phi_q\left(a_{i,}; c_n; p; w\right)h(w) * \mathrm{E}_{\nu,\tau}(w)\right) + \alpha w \left({}_l\Phi_q\left(a_{i,}; c_n; p; w\right)h(w) * \mathrm{E}_{\nu,\tau}(w)\right)',$$

$$\vdots$$

$$\mathcal{F}_\tau^{\nu,m}\left[\, a_{i,}, c_n, \alpha\right]h(w) = \mathcal{F}_\tau^{\nu,1}\left(\mathcal{F}_\tau^{\nu,m-1}\left[\, a_{i,}, c_n, \alpha\right]h(w)\right). \quad (12)$$

If $h \in \Sigma$, then from (12) we deduce that

$$\mathcal{F}_\tau^{\nu,m}\left[\, a_{i,}, c_n, \alpha\right]h(w) = w^{-1} + \sum_{i=0}^{\infty} [1 + (i-1)\alpha]^m \Omega_{(i+1,\nu,\tau)}\left(a_l, c_q\right) a_i w^i\,, \quad (13)$$

where

$$\Omega_{(i+1,\nu,\tau)}(a_l, c_q) = \frac{\prod_{j=1}^{l}(a_j, p)_{i+1}}{(p,p)_{i+1}\prod_{n=1}^{q}(c_n, p)_{i+1}}\left(\frac{\Gamma(\tau)}{\Gamma(\nu(i+1)+\tau)}\right). \quad (14)$$

Remark 1.1 It can be shown that, when the parameters are defined, $l, q, \alpha, m, \nu, \tau, p, a_1, \dots, a_l$ and $c_1, \dots, c_q$, it is observed that the defined operator $\mathcal{F}_{\tau}^{\nu,m}[a_{l,}, c_q, \alpha]h(w)$ leads to different operators. For further illustration, examples are given.

1- For $l = 1, \alpha = 1, q = 0, \nu = 0, \tau = 1, a_1 = p\ and\ p \to 1$ we have the operator $\mathrm{N}^m h(w)$ studied by El–Ashwh and Aouf [2].
2- For $\tau = 1, \nu = 0, m = 0, a_j = p^{a_j}, c_n = p^{c_n}, a_i > 0, c_n > 0, (j = 1, \dots, l; n = 1, \dots, q, l = q+1) and\ p \to 1$ we get the operator $\mathrm{I}_{l,q}[a_{i,}, c_n]h(w)$ researched by Liu and Srivastava[15].
3- For $l = 2, m = 0, q = 1, \nu = 0, \tau = 1, a_2 = p\ and\ p \to 1$ we get the operator $\mathrm{H}[a_{1,}, c_1]h(w)$ studied by Liu and Srivstava [13].
4- For $l = 1, m = 0, q = 0, \nu = 0, \tau = 1, a_1 = \alpha + 1\ and\ p \to 1$ we get the operator $\mathrm{M}^{\alpha}h(w)$ it was introduced by Ganigi and Uralegadi[22], and then Yang[24] generalized it.

For instance,Challab et al.[5],Elrifai et al.[9],Lashin[14],Liu and Srivstava[15] and others have explored a range of meromorphic function subclasses. Our implementation of the new subclass $\mathcal{H}_{\nu,\tau}^{m}[a_{l,}, c_q, \alpha; C, D, d]$ of Σ, which includes operator $\mathcal{F}_{\tau}^{\nu,m}[a_{l,}, c_q, \alpha]h(w)$, has inspired these works, and is seen as follows:

Definition 1.2 For functions $h \in \Sigma$ in the class $\mathcal{H}_{\nu,\tau}^{m}[a_{l,}, c_q, \alpha; C, D, d]$, and $-1 \le D < C \le 1$; $a_l, c_q, \alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$; $l \le q+1; l, q \in \mathbb{N} \cup \{0\}$; $\mathbb{N} = \{1,2,\dots\}$ if it satisfies the inequality

$$\left| \frac{\dfrac{w\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,}, c_q, \alpha]h(w)\right)^{u}}{\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,}, c_q, \alpha]h(w)\right)^{u-1}} + 1}{D\dfrac{w\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,}, c_q, \alpha]h(w)\right)^{u}}{\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,}, c_q, \alpha]h(w)\right)^{u-1}} + [d(C-D)+D]} \right| < 1. \quad (15)$$

Let Σ^* denote the subclass of Σ consjsting of functions of the form:

$$h(w) = w^{-1} + \sum_{i=0}^{\infty} |a_i| w^i, \quad (w \in \mathfrak{I}^*) \quad (16)$$

Now, we define the class $\mathcal{H}_{\nu,\tau}^{m,*}[a_{l,}, c_q, \alpha; C, D, d]$ by

$$\mathcal{H}_{\nu,\tau}^{m,*}[a_{l,}, c_q, \alpha; C, D, d] = \mathcal{H}_{\nu,\tau}^{m}[a_{l,}, c_q, \alpha; C, D, d] \cap \Sigma^*. \quad (17)$$

2. After removing the denominator from (19) and going to allow

This section describes the task of obtaining sufficient conditions for (16) the function to be given within class $\mathcal{H}_{\nu,\tau}^{m,*}[a_{l,}, c_q, \alpha; C, D, d]$, and demonstrates that this requirement is needed for functions in that class. Furthermore, for class $\mathcal{H}_{\nu,\tau}^{m,*}[a_{l,}, c_q, \alpha; C, D, d]$,growth and distortion limits, linear combinations,closure theorems and extreme points are presented.

In our very first Theorem, for functions h in $\mathcal{H}_{\nu,\tau}^{m,*}[a_{l,}, c_q, \alpha; C, D, d]$, we'll start with the conditions that are both necessary and adequate.

Theorem 2.1 A function $h(w)$ of the form (16) is in the class $\mathcal{H}_{\nu,\tau}^{m,*}[a_{l,}, c_q, \alpha; C, D, d]$ if and only if

$$\sum_{i=0}^{\infty}\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)|a_i| \\ \leq D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big). \tag{18}$$

Proof. Suppose the inequality (18) is valid, we get

$$\left|\frac{\dfrac{w\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u}}{\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u-1}}+1}{D\dfrac{w\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u}}{\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u-1}}+[d(C-D)+D]}\right|$$

$$=\left|\frac{w\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u}+\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u-1}}{Dw\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u}+[d(C-D)+D]\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u-1}}\right|$$

$$=\left|\frac{(u-1)(-1)^u(u-1)!+\sum_{i=0}^{\infty}\frac{i!}{(i-u)}\left[1+\frac{1}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)a_iw^{i+1}}{\big(D(u-1)-d(C-D)\big)(-1)^u(u-1)!+\sum_{i=0}^{\infty}\frac{i!}{(i-u)}\left[D+\frac{d(C-D)+D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)a_iw^{i+1}}\right|$$

$$<1, \qquad (w\in\mathfrak{I}^*)$$

Then, we have $h(w)\in\mathcal{H}_{\nu,\tau}^{m,*}[a_{l,}, c_q, \alpha; C, D, d]$, by the maximum modulus theorem.
Conversely, assume that $h(w)$ is with $h(w)$ of the form (16) in class $\mathcal{H}_{\nu,\tau}^{m,*}[a_{l,}, c_q, \alpha; C, D, d]$, then we discover from (15) that

$$\left|\frac{w\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u}+\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u-1}}{Dw\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u}+[d(C-D)+D]\left(\mathcal{F}_{\tau}^{\nu,m}[a_{l,},c_q,\alpha]h(w)\right)^{u-1}}\right|$$

$$=\left|\frac{(u-1)(-1)^u(u-1)!+\sum_{i=0}^{\infty}\frac{i!}{(i-u)}\left[1+\frac{1}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)a_iw^{i+1}}{\big(D(u-1)-d(C-D)\big)(-1)^u(u-1)!+\sum_{i=0}^{\infty}\frac{i!}{(i-u)}\left[D+\frac{d(C-D)+D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)a_iw^{i+1}}\right|$$

$$<1, \qquad (w\in\mathfrak{I}^*) \tag{19}$$

since the above inequality is valid for all $w\in\mathfrak{I}^*$, on the real axis, pick w values. After removing the denominator from (19) and going to allow $w\to 1^-$ by real values, we obtian

$$\sum_{i=0}^{\infty}\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)|a_i| \\ \leq D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big).$$

As a result, the desired inequality (18) of *Theorem 2.1.* is obtained.

Corollary 2.2 If the function$h(w)$ given by (16) is in the class $\mathcal{H}_{\nu,\tau}^{m,*}\left[a_{l,}, c_q, \alpha; C, D, d\right]$ then

$$|a_i| \leq \frac{D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right)}{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\left(i-(u-1)\right)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}\left(a_l, c_q\right)}, (i \geq 1)$$

the result is sharp for the function

$$h(w) = w^{-1} + \frac{D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right)}{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\left(i-(u-1)\right)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}\left(a_l, c_q\right)} w^i. \quad (20)$$

For functions belonging to class $\mathcal{H}_{\nu,\tau}^{m,*}\left[a_{l,}, c_q, \alpha; C, D, d\right]$, Growth and distortion limits will be in the following outcome:

Theorem 2.3 If a function $h(w)$ of the kind (16) is in the class$\mathcal{H}_{\nu,\tau}^{m,*}\left[a_{l,}, c_q, \alpha; C, D, d\right]$ then for $|w| = r$, we get

$$\frac{1}{r} - \frac{D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\left(1-(u-1)\right)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\left(a_l, c_q\right)} r$$

$$\leq |h(w)| \leq$$

$$\frac{1}{r} + \frac{D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\left(1-(u-1)\right)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\left(a_l, c_q\right)} r, \quad (21)$$

and

$$\frac{1}{r^2} - \frac{D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\left(1-(u-1)\right)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\left(a_l, c_q\right)}$$

$$\leq |h'(w)| \leq$$

$$\frac{1}{r^2} + \frac{D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\left(1-(u-1)\right)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\left(a_l, c_q\right)}. \quad (22)$$

Proof. By *Theorem 2.1*

$$\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\left(1-(u-1)\right)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\left(a_l, c_q\right)\sum_{i=0}^{\infty}|a_i|$$

$$\leq \sum_{i=0}^{\infty}\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\left(i-(u-1)\right)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}\left(a_l, c_q\right)|a_i|$$

$$\leq D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right),$$

which yields:

$$\sum_{i=0}^{\infty}|a_i| \le \frac{D(u-1)+(u-1)!\big(|d|(C-D)-(u-1)\big)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\big(1-(u-1)\big)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\big(a_l,c_q\big)}.$$

Therefore,

$$|h(w)| \le \frac{1}{|w|}+|w|\sum_{i=0}^{\infty}|a_i|$$

$$\le \frac{1}{|w|}+\frac{D(u-1)+(u-1)!\big(|d|(C-D)-(u-1)\big)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\big(1-(u-1)\big)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\big(a_l,c_q\big)}|w|,$$

and

$$|h(w)| \ge \frac{1}{|w|}-|w|\sum_{i=0}^{\infty}|a_i|$$

$$\ge \frac{1}{|w|}-\frac{D(u-1)+(u-1)!\big(|d|(C-D)-(u-1)\big)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\big(1-(u-1)\big)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\big(a_l,c_q\big)}|w|.$$

Now, by distinguishing both sides of (16) in relation to w, we get:

$$|h'(w)| \le \frac{1}{|w|^2}+\sum_{i=0}^{\infty}|a_i|$$

$$\le \frac{1}{|w|^2}+\frac{D(u-1)+(u-1)!\big(|d|(C-D)-(u-1)\big)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\big(1-(u-1)\big)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\big(a_l,c_q\big)},$$

and

$$|h'(w)| \ge \frac{1}{|w|^2}-\sum_{i=0}^{\infty}|a_i|$$

$$\ge \frac{1}{|w|^2}-\frac{D(u-1)+(u-1)!\big(|d|(C-D)-(u-1)\big)}{\frac{1}{(1-u)!}\left[\frac{2-(u-1)-D\big(1-(u-1)\big)-d(C-D)-D}{1-(u-1)}\right]\Omega_{(2,\nu,\tau)}\big(a_l,c_q\big)}.$$

Next, for functions in class $\mathcal{H}_{\nu,\tau}^{m,*}\big[a_{l,},c_q,\alpha;C,D,d\big]$, we evaluate the radius of meromorphic star-likeness and convexity of order γ.

Theorem 2.4 Let the function $h(w)$ given by (16) be in the class $\mathcal{H}_{\nu,\tau}^{m,*}\big[a_{l,},c_q,\alpha;C,D,d\big]$. Thus, we have:

(i) h is meromorphically starlike of the γ order in the disc $|w|=r_1$, so

$$\Re e\left\{-\frac{wh'(w)}{h(w)}\right\}>\gamma, \qquad (|w|=r_1, 0\le\gamma<1)$$

where

$$r_1 = \inf_{i\ge 1}\left\{\frac{(1-\gamma)\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}\big(a_l,c_q\big)}{(i+\gamma)\big[D(u-1)+(u-1)!\big(|d|(C-D)-(u-1)\big)\big]}\right\}^{\frac{1}{i+1}}. \quad (23)$$

(ii) h is meromorphically convex of the γ order in the disc $|w| = r_2$, so

$$\Re e\left\{-\left(\frac{wh''(w)}{h'(w)}+1\right)\right\} > \gamma, \qquad (|w| = r_2, 0 \le \gamma < 1)$$

where
$r_2 =$

$$\inf_{i\ge 1}\left\{\frac{(1-\gamma)\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D(i-(u-1))-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)}{i(i+\gamma)[D(u-1)+(u-1)!(|d|(C-D)-(u-1))]}\right\}^{\frac{1}{i+1}}. \quad (24)$$

Proof. (i) We can obtain from the description (16):

$$\left|\frac{\frac{wh'(w)}{h(w)}+1}{\frac{wh'(w)}{h(w)}-1+2\gamma}\right| < \frac{\sum_{i=0}^{\infty}(i+1)|a_i||w|^{i+1}}{2(1-\gamma)-\sum_{i=0}^{\infty}(i-1+2\gamma)|a_i||w|^{i+1}}.$$

Then,

$$\left|\frac{\frac{wh'(w)}{h(w)}+1}{\frac{wh'(w)}{h(w)}-1+2\gamma}\right| < 1, \qquad (0 \le \gamma < 1)$$

if

$$\sum_{i=0}^{\infty}\left(\frac{i+\gamma}{i-\gamma}\right)|a_i||w|^{i+1} < 1. \qquad (25)$$

As a result of *Theorem 2.1*, the inequality (25) is valid if

$$\left(\frac{i+\gamma}{i-\gamma}\right)|w|^{i+1}$$

$$< \frac{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D(i-(u-1))-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)}{D(u-1)+(u-1)!(|d|(C-D)-(u-1))},$$

then

$$|w| < \left\{\frac{(1-\gamma)\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D(i-(u-1))-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)}{(i+\gamma)[D(u-1)+(u-1)!(|d|(C-D)-(u-1))]}\right\}^{\frac{1}{i+1}}.$$

The last inequality leads us to the disk $|w| = r_1$ immediately, where r_1 is supplied by (23).

(ii) We discover from (16) that, to demonstrate the second statement of *Theorem 2.4*,

$$\left|\frac{\frac{wh''(w)}{h'(w)}+2}{\frac{wh''(w)}{h'(w)}+2\gamma}\right| < \frac{\sum_{i=0}^{\infty} i(i+1)|a_i||w|^{i+1}}{2(1-\gamma)-\sum_{i=0}^{\infty} i(i-1+2\gamma)|a_i||w|^{i+1}}.$$

Consequently, we obtain the desired inequality:

$$\left|\frac{\frac{wh''(w)}{h'(w)}+2}{\frac{wh''(w)}{h'(w)}+2\gamma}\right|<1,\quad (0\le\gamma<1)$$

if

$$\sum_{i=0}^{\infty} i\left(\frac{i+\gamma}{i-\gamma}\right)|a_i||w|^{i+1}<1. \tag{26}$$

As a result of *Theorem 2.1*, the inequality (26) is valid if

$$i\left(\frac{i+\gamma}{i-\gamma}\right)|w|^{i+1}$$

$$<\frac{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}\big(a_l,c_q\big)}{D(u-1)+(u-1)!\big(|d|(C-D)-(u-1)\big)},$$

then

$$|w|<\left\{\frac{(1-\gamma)\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}\big(a_l,c_q\big)}{i(i+\gamma)\big[D(u-1)+(u-1)!\big(|d|(C-D)-(u-1)\big)\big]}\right\}^{\frac{1}{i+1}}.$$

The last inequality leads us to the disk$|w|=r_2$immediately,where r_2 is supplied by (24).

It can now evaluate the extreme points and the closure theorems of class $\mathcal{H}_{\nu,\tau}^{m,*}\left[a_{l,},c_q,\alpha;C,D,d\right]$.

Theorem 2.5 Under convex, linear combinations, class $\mathcal{H}_{\nu,\tau}^{m,*}\left[a_{l,},c_q,\alpha;C,D,d\right]$ is closed.

Proof. Let

$$h_j(w)=w^{-1}+\sum_{i=0}^{\infty}\left|a_{i,j}\right|w^i,\quad (j=1,2)$$

are in $\mathcal{H}_{\nu,\tau}^{m,*}\left[a_{l,},c_q,\alpha;C,D,d\right]$. It is sufficient to demonstrate that f is a function defined by

$$f(w)=(1-c)h_1(w)+ch_2(w),\quad (0\le c\le 1)$$

is in the class $\mathcal{H}_{\nu,\tau}^{m,*}\left[a_{l,},c_q,\alpha;C,D,d\right]$, since

$$f(w)=w^{-1}+\sum_{i=0}^{\infty}\left[(1-c)\left|a_{i,1}\right|+c\left|a_{i,2}\right|\right]w^i.\ \ (0\le c\le 1) \tag{27}$$

In view of *Theorem 2.1*, we have:

$$\sum_{i=0}^{\infty}\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}\left[(1-c)\left|a_{i,1}\right|+c\left|a_{i,2}\right|\right]$$

$$= (1-c)\sum_{i=0}^{\infty}\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}|a_{i,1}|$$

$$+ c\sum_{i=0}^{\infty}\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}|a_{i,2}|$$

$$\leq (1-c)\big[D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big)\big]+c\big[D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big)\big]$$

$$= \big[D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big)\big]$$

which shows that $f(w) \in \mathcal{H}_{\nu,\tau}^{m,*}[\,a_{l,}, c_q, \alpha; C, D, d]$.

Theorem 2.6 Let $h_0(w) = \frac{1}{w}$ *and*

$$h_i(w) = \frac{1}{w} + \frac{D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big)}{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}}w^i\,.\,(i\geq 1)$$

Then $h(w) \in \mathcal{H}_{\nu,\tau}^{m,*}[\,a_{l,}, c_q, \alpha; C, D, d]$ if and only if it can be expressed in the form

$$h(w) = \sum_{i=0}^{\infty} v_i h_i(w), \quad \left(v_i \geq 0 \; and \sum_{i=0}^{\infty} v_i = 1\right) \tag{28}$$

Proof. If we express the $h(w)$ function in the form given by (28), then

$$h_i(w) = w^{-1} + \sum_{i=0}^{\infty} v_i \frac{D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big)}{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}}w^i$$

and for this function, we have:

$$\sum_{i=0}^{\infty}\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}(a_l,c_q)$$
$$\times v_i \frac{D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big)}{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\big(i-(u-1)\big)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m\Omega_{(i+1,\nu,\tau)}}$$

$$= \sum_{i=0}^{\infty} v_i\big[D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big)\big]$$

$$= (1-v_0)\big[D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big)\big] \leq D(u-1)+(u-1)!\,\big(|d|(C-D)-(u-1)\big).$$

The condition (18) is satisfied. Thus, $h(w) \in \mathcal{H}_{\nu,\tau}^{m,*}[\,a_{l,}, c_q, \alpha; C, D, d]$.

Conversely, we suppose that $h(w) \in \mathcal{H}_{\nu,\tau}^{m,*}\left[\, a_{l,}, c_q, \alpha; C, D, d\right]$, since

$$|a_i| \leq \frac{D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right)}{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\left(i-(u-1)\right)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m \Omega_{(i+1,\nu,\tau)}}, (i \geq 1)$$

we set

$$v_i = \frac{\frac{i!}{(i-u)!}\left[\frac{i-(u-1)+1-D\left(i-(u-1)\right)-d(C-D)-D}{i-(u-1)}\right][1+(i-1)\alpha]^m \Omega_{(i+1,\nu,\tau)}}{D(u-1)+(u-1)!\left(|d|(C-D)-(u-1)\right)} |a_i|, (i \geq 1)$$

and

$$v_0 = 1 - \sum_{i=0}^{\infty} v_i \ ,$$

so it follows that

$$h(w) = \sum_{i=0}^{\infty} v_i h_i(w)\, .$$

The proof is complete.

3. Conclusions

The association between the analytical nature of complex functions with various properties and geometric behaviour has been strengthened through introduced a new subclass of meromorphic function that is related to both the Mittag-Leffler function and the q-hypergeomtric function, which is inspired by this approach. In relation to this subclass, we have obtained adequate and required conditions. It also goes over linear combinations, the distortion principle, and their properties.

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