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On Chromatic Uniqueness of Complete Complete 6-Partite Graphs

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Abstract

Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. We write $[G] = \{H|H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 6-partite graphs G with $6n+i$ vertices for $i = 0,1,2$ according to the number of 7-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 6-partite graphs G with certain star or matching deleted are obtained.

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1 Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G,\lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted

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by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. Many families of χ-unique graphs are known (see [5,6,7]).

For a graph G, let $V(G)$, $E(G)$, $t(G)$ and $\chi(G)$ be the vertex set, edge set, number of triangles and chromatic number of G, respectively. Let O_n be an edgeless graph with n vertices. Let $Q(G)$ and $K(G)$ be the number of induced subgraph C_4 and complete subgraph K_4 in G. Let S be a set of s edges in G. By $G - S$ (or $G - s$) we denote the graph obtained from G by deleting all edges in S, and $\langle S \rangle$ the graph induced by S. For $t \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_t$, let $K(n_1, n_2, \cdots, n_t)$ be a complete $t-$partite graph with partition sets V_i such that $|V_i| = n_i$ for $i = 1, 2, \cdots, t$. In [2-4,8,9,12-15,17-19], the authors proved that certain families of complete t-partite graphs ($t = 2, 3, 4, 5, 6$) with a matching or a star deleted are χ-unique. In particular, the authors in [2,13-15] determined the chromaticity of complete 6-partite graphs with a matching or a star deleted and leaving the general cases undecided. This paper aims to study the chromaticity of complete 6-partite graphs G with $6n + i$ vertices for $i = 0, 1, 2$ and thus generalize some results in [13-15]. We first characterize certain complete 6-partite graphs G with $6n + i$ vertices for $i = 0, 1, 2$ according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 6-partite graphs with certain star or matching deleted are obtained.

2 Some lemmas and notations

Let $\mathcal{K}^{-s}(n_1, n_2, \cdots, n_t)$ be the family $\{K(n_1, n_2, \cdots, n_t) - S|\ S \subset E(K(n_1, n_2, \cdots, n_t))$ and $|S| = s\}$. For $n_1 \geq s + 1$, we denote by $K_{i,j}^{-K_{1,s}}(n_1, n_2, \cdots, n_t)$ (respectively, $K_{i,j}^{-sK_2}(n_1, n_2, \cdots, n_t)$) the graph in $K^{-s}(n_1, n_2, \cdots, n_t)$ where the s edges in S induced a $K_{1,s}$ with center in V_i and all the end vertices in V_j (respectively, a matching with end vertices in V_i and V_j).

For a graph G and a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of G. Then, we have $P(G, \lambda) = \sum_{r=1}^{p} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - r + 1)$ (see [11]). Therefore, $\alpha(G, r) = \alpha(H, r)$ for each $r = 1, 2, \cdots,$ if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G,x) = \sum_{r=1}^{p} \alpha(G,r)x^r$ is called the σ-polynomial of G (see [1]). Clearly, $P(G,\lambda) = P(H,\lambda)$ implies that $\sigma(G,x) = \sigma(H,x)$ for any graphs G and H.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer to [16].

Lemma 2.1 *(Koh and Teo [6]) Let G and H be two graphs with $H \sim G$, then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G,r) = \alpha(H,r)$ for $r = 1,2,3,4,\cdots$, and $2K(G) - Q(G) = 2K(H) - Q(H)$. Note that $\chi(G) = 3$ then $G \sim H$ implies that $Q(G) = Q(H)$.*

Lemma 2.2 *(Brenti [1]) Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G,x)\sigma(H,x).$$

In particular,

$$\sigma(K(n_1,n_2,\cdots,n_t),x) = \prod_{i=1}^{t} \sigma(O_{n_i},x)$$

Lemma 2.3 *(Brenti [1]) Let $G = K(n_1,n_2,n_3,\cdots,n_t)$ and $\sigma(G,x) = \sum_{r\geq 1} \alpha(G,r)x^r$. Then $\alpha(G,r) = 0$ for $1 \leq r \leq t-1$, $\alpha(G,t) = 1$ and $\alpha(G,t+1) = \sum_{i=1}^{t} 2^{n_i-1} - t$.*

Let $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6$ be positive integers and $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\} = \{x_1,x_2,x_3,x_4,x_5,x_6\}$. If there are two elements x_{i_1} and x_{i_2} in $\{x_1,x_2,x_3,x_4,x_5,x_6\}$ such that $x_{i_2} - x_{i_1} \geq 2$, then $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ is called an *improvement* of $H = K(x_1,x_2,x_3,x_4,x_5,x_6)$.

Lemma 2.4 *(Chen [2]) Suppose $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6$ and $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ is an improvement of $H = K(x_1,x_2,x_3,x_4,x_5,x_6)$. Then*

$$\alpha(H,7) - \alpha(H',7) = 2^{x_{i_2}-2} - 2^{x_{i_1}-1} \geq 2^{x_{i_1}-1}.$$

Let $G = K(n_1,n_2,n_3,n_4,n_5,n_6)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H,7) - \alpha(G,7)$. Clearly, $\alpha'(H) \geq 0$.

Lemma 2.5 *(Chen [2]) Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$. Suppose that min $\{n_i | i = 1, 2, 3, 4, 5, 6\} \geq s + 1 \geq 1$ and $H = G - S$, where S is a set of some s edges of G. Then*

$$s \leq \alpha'(H) = \alpha(H, 7) - \alpha(G, 7) \leq 2^s - 1,$$

$\alpha'(H) = s$ iff the set of end-vertices of any $r \geq 2$ edges in S is not independent in H, and $\alpha'(H) = 2^s - 1$ iff S induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same A_i.

Lemma 2.6 *(Dong et al. [4]) Let n_1, n_2 and s be positive integers with $3 \leq n_1 \leq n_2$, then*

(1) $K_{1,2}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leq s \leq n_2 - 2$,

(2) $K_{2,1}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leq s \leq n_1 - 2$, and

(3) $K^{-sK_2}(n_1, n_2)$ is χ-unique for $1 \leq s \leq n_1 - 1$.

Lemma 2.7 *(Lau and Peng [9]) Let s_i $(1 \leq i \leq t)$ be positive integers. Then*

$$\sum_{i=1}^{t} \binom{s_i}{2} = \binom{\sum_{i=1}^{t} s_i}{2} - \sum_{j=1}^{t} \Big[s_j \sum_{i=j+1}^{t} s_i \Big].$$

For a graph $G \in K^{-s}(n_1, n_2, \cdots, n_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively Type 2 and Type 3) if the vertices of the induced C_4 are in exactly two (respectively three and four) partite sets of $V(G)$. An example of induced C_4 of Types 1, 2 and 3 are shown in Figure 1.

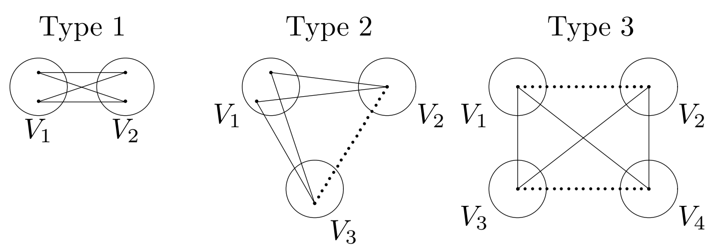

FIGURE 1. Three types of induced C_4

Suppose G is a graph in $K^{-s}(n_1, n_2, \cdots, n_t)$. Let S_{ij} $(1 \leq i \leq t, 1 \leq j \leq t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \geq 0$.

Lemma 2.8 *(Lau and Peng [10]) For integer $t \geq 3$, Let $F = K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) &= Q(F) - \sum_{1 \leq i < j \leq t} (n_i - 1)(n_j - 1)s_{ij} + \binom{s}{2} - \sum_{1 \leq i < j < l \leq t} s_{ij}s_{il} - \\ &\quad \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq k < l \leq t \\ i < k}} s_{ij}s_{kl} + \sum_{1 \leq i < j \leq t} \left[s_{ij} \sum_{k \notin \{i,j\}} \binom{n_k}{2} \right] + \\ &\quad \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq i < k < l \leq t \\ j \notin \{k,l\}}} s_{ij}s_{kl}, \end{aligned}$$

and

$$\begin{aligned} K(G) &= K(F) - \sum_{1 \leq i < j \leq t} \left[s_{ij} \sum_{\substack{1 \leq k < l \leq t \\ \{i,j\} \cap \{k,l\} = \emptyset}} n_k n_l \right] + \\ &\quad \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq i < k < l \leq t \\ j \notin \{k,l\}}} s_{ij}s_{kl}. \end{aligned}$$

By using Lemma 2.8, we obtain the following.

Lemma 2.9 *Let $F = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 6-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) &= Q(F) - \sum_{1 \leq i < j \leq 6} (n_i - 1)(n_j - 1)s_{ij} + \binom{s}{2} - s_{12}(s_{13} + s_{14} + s_{15} + s_{16} + s_{23} + \\ &\quad s_{24} + s_{25} + s_{26}) - s_{13}(s_{14} + s_{15} + s_{16} + s_{23} + s_{34} + s_{35} + s_{36}) - s_{14}(s_{15} + s_{16} + \\ &\quad s_{24} + s_{34} + s_{45} + s_{46}) - s_{15}(s_{16} + s_{25} + s_{35} + s_{45} + s_{56}) - s_{16}(s_{26} + s_{36} + s_{46} + \\ &\quad s_{56}) - s_{23}(s_{24} + s_{25} + s_{26} + s_{34} + s_{35} + s_{36}) - s_{24}(s_{25} + s_{26} + s_{34} + s_{45} + s_{46}) - \\ &\quad s_{25}(s_{26} + s_{35} + s_{45} + s_{56}) - s_{26}(s_{36} + s_{46} + s_{56}) - s_{34}(s_{35} + s_{36} + s_{46}) - \\ &\quad s_{35}(s_{36} + s_{45} + s_{56}) - s_{36}(s_{46} + s_{56}) - s_{45}(s_{46} + s_{56}) - \\ &\quad s_{46}s_{56} + \sum_{1 \leq i < j \leq 6} \left[s_{ij} \sum_{k \notin \{i,j\}} \binom{n_k}{2} \right], \end{aligned}$$

and

$$K(G) = K(F) - \sum_{1 \leq i < j \leq 6} \left[s_{ij} \sum_{\substack{1 \leq k < l \leq 6 \\ \{i,j\} \cap \{k,l\} = \emptyset}} n_k n_l \right] + s_{12}(s_{34} + s_{35} + s_{36} + s_{45} + s_{46} + s_{56}) +$$

$$s_{13}(s_{24}+s_{25}+s_{26}+s_{45}+s_{46}+s_{56})+s_{14}(s_{23}+s_{25}+s_{26}+s_{35}+s_{36}+s_{56})+$$
$$s_{15}(s_{23}+s_{24}+s_{26}+s_{34}+s_{36}+s_{46})+s_{16}(s_{23}+s_{24}+s_{25}+s_{34}+s_{35}+s_{45})+$$
$$s_{23}(s_{45}+s_{46}+s_{56})+s_{24}(s_{35}+s_{36}+s_{56})+s_{25}(s_{34}+s_{36}+s_{46})+$$
$$s_{26}(s_{34}+s_{35}+s_{45})+s_{34}s_{56}+s_{35}s_{46}+s_{36}s_{45}.$$

3 Characterization

In this section, we shall characterize certain complete 6-partite graph $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ according to the number of 7-independent partitions of G where $n_6 - n_1 \leq 4$.

Theorem 3.1 *Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 6-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n$ and $n_6 - n_1 \leq 4$. Define $\theta(G) = [\alpha(G, 7) - 2^{n+1} - 2^n + 6]/2^{n-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(n, n, n, n, n, n)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(n-1, n, n, n, n, n+1)$;

(iii) $\theta(G) = 2$ *if and only if* $G = K(n-1, n-1, n, n, n+1, n+1)$;

(iv) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(n-2, n, n, n, n+1, n+1)$;

(v) $\theta(G) = 3$ *if and only if* $G = K(n-1, n-1, n-1, n+1, n+1, n+1)$;

(vi) $\theta(G) = 3\frac{1}{2}$ *if and only if* $G = K(n-2, n-1, n, n+1, n+1, n+1)$;

(vii) $\theta(G) = 4$ *if and only if* $G = K(n-1, n-1, n, n, n, n+2)$;

(viii) $\theta(G) = 4\frac{1}{4}$ *if and only if* $G = K(n-3, n, n, n+1, n+1, n+1)$;

(ix) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(n-2, n, n, n, n, n+2)$;

(x) $\theta(G) = 5$ *if and only if* $G = K(n-1, n-1, n-1, n, n+1, n+2)$ *or* $G = K(n-2, n-2, n+1, n+1, n+1, n+1)$;

(xi) $\theta(G) = 5\frac{1}{4}$ *if and only if* $G = K(n-3, n, n, n+1, n+1, n+1)$;

(xii) $\theta(G) = 5\frac{1}{2}$ *if and only if* $G = K(n-2, n-1, n, n, n+1, n+2)$;

(xiii) $\theta(G) = 6\frac{1}{2}$ *if and only if* $G = K(n-2, n-1, n-1, n+1, n+1, n+2)$;

(xiv) $\theta(G) = 7$ *if and only if* $G = K(n-2, n-2, n, n+1, n+1, n+2)$;

(xv) $\theta(G) = 8$ *if and only if* $G = K(n-1, n-1, n-1, n-1, n+2, n+2)$*;*

(xvi) $\theta(G) = 8\frac{1}{2}$ *if and only if* $G = K(n-2, n-1, n-1, n, n+2, n+2)$*;*

(xvii) $\theta(G) = 9$ *if and only if* $G = K(n-2, n-2, n, n, n+2, n+2)$*;*

(xviii) $\theta(G) = 10$ *if and only if* $G = K(n-2, n-2, n-1, n+1, n+2, n+2)$*;*

(xix) $\theta(G) = 11$ *if and only if* $G = K(n-1, n-1, n-1, n, n, n+3)$*;*

(xx) $\theta(G) = 12$ *if and only if* $G = K(n-1, n-1, n-1, n-1, n+1, n+3)$*;*

(xxi) $\theta(G) = 13\frac{1}{2}$ *if and only if* $G = K(n-2, n-2, n-2, n+2, n+2, n+2)$*.*

Proof. In order to complete the proof of the theorem, we first give two tables for the θ-value of various complete 6-partite graphs with $6n$ vertices as shown in Tables 1 and 2.

G_i $(1 \le i \le 21)$	$\theta(G_i)$	G_i $(22 \le i \le 42)$	$\theta(G_i)$
$G_1 = K(n, n, n, n, n, n)$	0	$G_{22} = K(n-2, n-2, n, n+1, n+1, n+2)$	7
$G_2 = K(n-1, n, n, n, n, n+1)$	1	$G_{23} = K(n-3, n-1, n+1, n+1, n+1, n+1)$	$5\frac{1}{4}$
$G_3 = K(n-1, n-1, n, n, n+1, n+1)$	2	$G_{24} = K(n-3, n-1, n, n+1, n+1, n+2)$	$7\frac{1}{4}$
$G_4 = K(n-2, n, n, n, n+1, n+1)$	$2\frac{1}{2}$	$G_{25} = K(n-2, n-2, n, n, n+2, n+2)$	9
$G_5 = K(n-1, n-1, n, n, n, n+2)$	4	$G_{26} = K(n-2, n-2, n, n, n+1, n+3)$	13
$G_6 = K(n-2, n, n, n, n, n+2)$	$4\frac{1}{2}$	$G_{27} = K(n-3, n-1, n, n, n+2, n+2)$	$9\frac{1}{4}$
$G_7 = K(n-1, n-1, n-1, n+1, n+1, n+1)$	3	$G_{28} = K(n-3, n-1, n, n, n+1, n+3)$	$13\frac{1}{4}$
$G_8 = K(n-1, n-1, n-1, n, n+1, n+2)$	5	$G_{29} = K(n-4, n, n+1, n+1, n+1, n+1)$	$6\frac{1}{8}$
$G_9 = K(n-2, n-1, n, n+1, n+1, n+1)$	$3\frac{1}{2}$	$G_{30} = K(n-4, n, n, n+1, n+1, n+2)$	$8\frac{1}{8}$
$G_{10} = K(n-2, n-1, n, n, n+1, n+2)$	$5\frac{1}{2}$	$G_{31} = K(n-4, n, n, n, n+2, n+2)$	$10\frac{1}{8}$
$G_{11} = K(n-3, n, n, n+1, n+1, n+1)$	$4\frac{1}{4}$	$G_{32} = K(n-4, n, n, n, n+1, n+3)$	$14\frac{1}{8}$
$G_{12} = K(n-3, n, n, n, n+1, n+2)$	$6\frac{1}{4}$	$G_{33} = K(n-1, n-1, n-1, n-1, n, n+4)$	26
$G_{13} = K(n-1, n-1, n-1, n, n, n+3)$	11	$G_{34} = K(n-2, n-1, n-1, n, n, n+4)$	$26\frac{1}{2}$
$G_{14} = K(n-2, n-1, n, n, n, n+3)$	$11\frac{1}{2}$	$G_{35} = K(n-2, n-2, n, n, n, n+4)$	27
$G_{15} = K(n-3, n, n, n, n, n+3)$	$12\frac{1}{4}$	$G_{36} = K(n-3, n-1, n, n, n, n+4)$	$27\frac{1}{4}$
$G_{16} = K(n-2, n-1, n-1, n+1, n+1, n+2)$	$6\frac{1}{2}$	$G_{37} = K(n-4, n, n, n, n+1, n+3)$	$14\frac{1}{2}$
$G_{17} = K(n-1, n-1, n-1, n-1, n+1, n+3)$	8	$G_{38} = K(n-4, n, n, n, n, n+4)$	$28\frac{1}{8}$
$G_{18} = K(n-1, n-1, n-1, n-1, n+1, n+3)$	12	$G_{39} = K(n-2, n-2, n-1, n+1, n+2, n+2)$	10
$G_{19} = K(n-2, n-1, n-1, n, n+1, n+3)$	$12\frac{1}{2}$	$G_{40} = K(n-2, n-2, n-1, n+1, n+1, n+3)$	14
$G_{20} = K(n-2, n-1, n-1, n, n+2, n+2)$	$8\frac{1}{2}$	$G_{41} = K(n-3, n-1, n-1, n+1, n+2, n+2)$	$10\frac{1}{4}$
$G_{21} = K(n-2, n-2, n+1, n+1, n+1, n+1)$	5	$G_{42} = K(n-3, n-1, n-1, n+1, n+1, n+3)$	$14\frac{1}{4}$

Table 1: Some complete 6-partite graphs with $6n$ vertices and their θ-values.

By the definition of improvement, we have the followings:

(1) G_1 is the improvement of G_2 and G_3 with $\theta(G_2) = 1$;

(2) G_2 is the improvement of G_3, G_4, G_5 and G_6 with $\theta(G_3) = 2$, $\theta(G_4) = 2\frac{1}{2}$, $\theta(G_5) = 4$ and $\theta(G_6) = 4\frac{1}{2}$;

(3) G_3 is the improvement of G_4, G_5, G_7, G_8, G_9 and G_{10} with $\theta(G_4) = 2\frac{1}{2}$, $\theta(G_5) = 4$, $\theta(G_7) = 3$, $\theta(G_8) = 5$, $\theta(G_9) = 3\frac{1}{2}$ and $\theta(G_{10}) = 5\frac{1}{2}$;

G_i $(43 \leq i \leq 69)$	$\theta(G_i)$	G_i $(70 \leq i \leq 95)$	$\theta(G_i)$
$G_{43} = K(n-2, n-1, n-1, n-1, n+2, n+3)$	$15\frac{1}{2}$	$G_{70} = K(n-3, n-3, n+1, n+1, n+2, n+2)$	$12\frac{1}{2}$
$G_{44} = K(n-2, n-1, n-1, n-1, n+1, n+4)$	$27\frac{1}{2}$	$G_{71} = K(n-3, n-3, n+1, n+1, n+1, n+3)$	$16\frac{1}{2}$
$G_{45} = K(n-2, n-2, n-1, n, n+2, n+3)$	16	$G_{72} = K(n-4, n-2, n+1, n+1, n+2, n+2)$	$12\frac{5}{8}$
$G_{46} = K(n-2, n-2, n-1, n, n+1, n+4)$	28	$G_{73} = K(n-4, n-2, n+1, n+1, n+1, n+3)$	$16\frac{5}{8}$
$G_{47} = K(n-3, n-1, n-1, n, n+2, n+3)$	$16\frac{1}{4}$	$G_{74} = K(n-3, n-3, n, n+2, n+2, n+2)$	$14\frac{1}{2}$
$G_{48} = K(n-3, n-1, n-1, n, n+1, n+4)$	$28\frac{1}{4}$	$G_{75} = K(n-3, n-3, n, n+1, n+2, n+3)$	$18\frac{1}{2}$
$G_{49} = K(n-3, n-2, n+1, n+1, n+1, n+2)$	$8\frac{3}{4}$	$G_{76} = K(n-4, n-2, n, n+2, n+2, n+2)$	$14\frac{1}{8}$
$G_{50} = K(n-3, n-2, n, n+1, n+2, n+2)$	$9\frac{3}{4}$	$G_{77} = K(n-4, n-2, n, n+1, n+2, n+3)$	$18\frac{5}{8}$
$G_{51} = K(n-3, n-2, n, n+1, n+1, n+3)$	$14\frac{3}{4}$	$G_{78} = K(n-5, n-1, n+1, n+1, n+2, n+2)$	$13\frac{1}{16}$
$G_{52} = K(n-4, n-1, n+1, n+1, n+1, n+2)$	$9\frac{1}{8}$	$G_{79} = K(n-5, n-1, n+1, n+1, n+1, n+3)$	$17\frac{1}{16}$
$G_{53} = K(n-3, n-2, n, n, n+2, n+3)$	$16\frac{3}{4}$	$G_{80} = K(n-5, n-1, n, n+2, n+2, n+2)$	$15\frac{1}{16}$
$G_{54} = K(n-3, n-2, n, n, n+1, n+4)$	$17\frac{1}{2}$	$G_{81} = K(n-5, n-1, n, n+1, n+2, n+3)$	$19\frac{1}{16}$
$G_{55} = K(n-4, n-1, n, n+1, n+2, n+2)$	$11\frac{1}{8}$	$G_{82} = K(n-6, n+1, n+1, n+1, n+1, n+2)$	$12\frac{1}{32}$
$G_{56} = K(n-4, n-1, n, n, n+2, n+3)$	$17\frac{1}{8}$	$G_{83} = K(n-6, n, n+1, n+1, n+2, n+2)$	$14\frac{1}{32}$
$G_{57} = K(n-4, n-1, n, n+1, n+1, n+3)$	$15\frac{1}{8}$	$G_{84} = K(n-6, n, n+1, n+1, n+1, n+3)$	16
$G_{58} = K(n-4, n-1, n, n, n+1, n+4)$	$29\frac{1}{8}$	$G_{85} = K(n-6, n, n, n+2, n+2, n+2)$	$16\frac{1}{32}$
$G_{59} = K(n-5, n+1, n+1, n+1, n+1, n+1)$	$8\frac{1}{16}$	$G_{86} = K(n-6, n, n, n+1, n+2, n+3)$	$20\frac{1}{32}$
$G_{60} = K(n-5, n, n+1, n+1, n+1, n+2)$	$10\frac{1}{16}$	$G_{87} = K(n-3, n-2, n-2, n+2, n+2, n+3)$	$21\frac{1}{4}$
$G_{61} = K(n-5, n, n, n+1, n+2, n+2)$	$12\frac{1}{16}$	$G_{88} = K(n-4, n-3, n+1, n+2, n+2, n+2)$	$16\frac{3}{8}$
$G_{62} = K(n-5, n, n, n+1, n+1, n+3)$	$16\frac{1}{16}$	$G_{89} = K(n-4, n-3, n+1, n+1, n+2, n+3)$	$20\frac{3}{8}$
$G_{63} = K(n-5, n, n, n, n+2, n+3)$	$18\frac{1}{16}$	$G_{90} = K(n-5, n-2, n+1, n+2, n+2, n+2)$	$16\frac{5}{8}$
$G_{64} = K(n-2, n-2, n-2, n+2, n+2, n+2)$	$13\frac{1}{2}$	$G_{91} = K(n-5, n-2, n+1, n+1, n+2, n+3)$	$20\frac{9}{16}$
$G_{65} = K(n-2, n-2, n-2, n+1, n+2, n+3)$	$17\frac{1}{2}$	$G_{92} = K(n-6, n-1, n+1, n+2, n+2, n+2)$	$17\frac{1}{32}$
$G_{66} = K(n-3, n-2, n-1, n+2, n+2, n+2)$	$13\frac{3}{4}$	$G_{93} = K(n-6, n-1, n+1, n+1, n+2, n+3)$	$21\frac{1}{32}$
$G_{67} = K(n-3, n-2, n-1, n+1, n+2, n+3)$	$17\frac{3}{4}$	$G_{94} = K(n-7, n+1, n+1, n+1, n+2, n+2)$	$16\frac{1}{64}$
$G_{68} = K(n-4, n-1, n-1, n+2, n+2, n+2)$	$14\frac{1}{8}$	$G_{95} = K(n-7, n+1, n+1, n+1, n+1, n+3)$	$20\frac{1}{64}$
$G_{69} = K(n-4, n-1, n-1, n+1, n+2, n+3)$	$18\frac{1}{8}$		

Table 2: Some complete 6-partite graphs with $6n$ vertices and their θ-values.

(4) G_4 is the improvement of G_6, G_9, G_{10}, G_{11} and G_{12} with $\theta(G_6) = 4\frac{1}{2}$, $\theta(G_9) = 3\frac{1}{2}$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{11}) = 4\frac{1}{4}$ and $\theta(G_{12}) = 6\frac{1}{4}$;

(5) G_5 is the improvement of G_6, G_8, G_{10}, G_{13} and G_{14} with $\theta(G_6) = 4\frac{1}{2}$, $\theta(G_8) = 5$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{13}) = 11$ and $\theta(G_{14}) = 11\frac{1}{2}$;

(6) G_6 is the improvement of G_{10}, G_{12}, G_{14} and G_{15} with $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{12}) = 6\frac{1}{4}$, $\theta(G_{14}) = 11\frac{1}{2}$ and $\theta(G_{15}) = 12\frac{1}{4}$;

(7) G_7 is the improvement of G_8, G_9 and G_{16} with $\theta(G_8) = 5$, $\theta(G_9) = 3\frac{1}{2}$ and $\theta(G_{16}) = 6\frac{1}{2}$

(8) G_8 is the improvement of G_{10}, G_{13}, G_{16}, G_{17}, G_{18}, G_{19} and G_{20} with $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{13}) = 11$, $\theta(G_{16}) = 6\frac{1}{2}$, $\theta(G_{17}) = 8$, $\theta(G_{18}) = 12$, $\theta(G_{19}) = 12\frac{1}{2}$ and $\theta(G_{20}) = 8\frac{1}{2}$;

(9) G_9 is the improvement of G_{10}, G_{11}, G_{16}, G_{21}, G_{22}, G_{23} and G_{24} with $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{11}) = 4\frac{1}{4}$, $\theta(G_{16}) = 6\frac{1}{2}$, $\theta(G_{21}) = 5$, $\theta(G_{22}) = 7$, $\theta(G_{23}) = 5\frac{1}{4}$ and $\theta(G_{24}) = 7\frac{1}{4}$;

(10) G_{10} is the improvement of G_{12}, G_{14}, G_{16}, G_{19}, G_{20}, G_{22}, G_{24}, G_{25}, G_{26}, G_{27} and G_{28} with $\theta(G_{12}) = 6\frac{1}{4}$, $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{16}) = 6\frac{1}{2}$, $\theta(G_{19}) = 12\frac{1}{2}$,

$\theta(G_{20}) = 8\frac{1}{2}$, $\theta(G_{22}) = 7$, $\theta(G_{24}) = 13\frac{1}{4}$, $\theta(G_{25}) = 9$, $\theta(G_{26}) = 13$, $\theta(G_{27}) = 9\frac{1}{4}$ and $\theta(G_{28}) = 13\frac{1}{4}$;

(11) G_{11} is the improvement of G_{12}, G_{23}, G_{24}, G_{29} and G_{30} with $\theta(G_{12}) = 6\frac{1}{4}$, $\theta(G_{23}) = 5\frac{1}{4}$, $\theta(G_{24}) = 7\frac{1}{4}$, $\theta(G_{29}) = 6\frac{1}{8}$ and $\theta(G_{30}) = 8\frac{1}{8}$;

(12) G_{12} is the improvement of G_{15}, G_{24}, G_{27}, G_{28}, G_{30}, G_{31} and G_{32} with $\theta(G_{15}) = 12\frac{1}{4}$, $\theta(G_{24}) = 7\frac{1}{4}$, $\theta(G_{27}) = 9\frac{1}{4}$, $\theta(G_{28}) = 13\frac{1}{4}$, $\theta(G_{30}) = 8\frac{1}{8}$, $\theta(G_{31}) = 10\frac{1}{8}$ and $\theta(G_{32}) = 14\frac{1}{8}$;

(13) G_{13} is the improvement of G_{14}, G_{18}, G_{19}, G_{33} and G_{34} with $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{18}) = 12$, $\theta(G_{19}) = 12\frac{1}{2}$, $\theta(G_{33}) = 26$ and $\theta(G_{34}) = 26\frac{1}{2}$;

(14) G_{14} is the improvement of G_{15}, G_{19}, G_{26}, G_{28}, G_{34}, G_{35} and G_{36} with $\theta(G_{15}) = 12\frac{1}{4}$, $\theta(G_{19}) = 12\frac{1}{2}$, $\theta(G_{26}) = 13$, $\theta(G_{28}) = 13\frac{1}{4}$, $\theta(G_{34}) = 26\frac{1}{2}$, $\theta(G_{35}) = 27$ and $\theta(G_{36}) = 27\frac{1}{4}$;

(15) G_{15} is the improvement of G_{28}, G_{36}, G_{37} and G_{38} with $\theta(G_{28}) = 13\frac{1}{4}$, $\theta(G_{36}) = 27\frac{1}{4}$, $\theta(G_{37}) = 14\frac{1}{2}$ and $\theta(G_{38}) = 28\frac{1}{8}$;

(16) G_{16} is the improvement of G_{19}, G_{20}, G_{22}, G_{24}, G_{39}, G_{40}, G_{41} and G_{42} with $\theta(G_{19}) = 12\frac{1}{2}$, $\theta(G_{20}) = 8\frac{1}{2}$, $\theta(G_{22}) = 7$, $\theta(G_{24}) = 7\frac{1}{4}$, $\theta(G_{39}) = 10$, $\theta(G_{40}) = 14$, $\theta(G_{41}) = 10\frac{1}{4}$ and $\theta(G_{42}) = 14\frac{1}{4}$;

(17) G_{17} is the improvement of G_{18}, G_{20} and G_{43} with $\theta(G_{18}) = 12$, $\theta(G_{20}) = 8\frac{1}{2}$ and $\theta(G_{43}) = 15\frac{1}{2}$;

(18) G_{18} is the improvement of G_{19}, G_{33}, G_{43} and G_{44} with $\theta(G_{19}) = 12\frac{1}{2}$, $\theta(G_{33}) = 26$, $\theta(G_{43}) = 15\frac{1}{2}$ and $\theta(G_{44}) = 27\frac{1}{2}$;

(19) G_{19} is the improvement of G_{26}, G_{28}, G_{40}, G_{42}, G_{43}, G_{44}, G_{45}, G_{46}, G_{47} and G_{48} with $\theta(G_{26}) = 13$, $\theta(G_{28}) = 13\frac{1}{4}$, $\theta(G_{40}) = 14$, $\theta(G_{42}) = 14\frac{1}{4}$, $\theta(G_{43}) = 15\frac{1}{2}$, $\theta(G_{44}) = 27\frac{1}{2}$, $\theta(G_{45}) = 16$, $\theta(G_{46}) = 28$, $\theta(G_{47}) = 16\frac{1}{4}$ and $\theta(G_{48}) = 28\frac{1}{4}$;

(20) G_{20} is the improvement of G_{19}, G_{25}, G_{27}, G_{39}, G_{41}, G_{43}, G_{45} and G_{47} with $\theta(G_{19}) = 12\frac{1}{2}$, $\theta(G_{25}) = 9$, $\theta(G_{27}) = 9\frac{1}{4}$, $\theta(G_{39}) = 10$, $\theta(G_{41}) = 10\frac{1}{4}$, $\theta(G_{43}) = 15\frac{1}{2}$, $\theta(G_{45}) = 16$ and $\theta(G_{47}) = 16\frac{1}{4}$;

(21) G_{21} is the improvement of G_{22}, G_{23} and G_{49} with $\theta(G_{22}) = 7$, $\theta(G_{23}) = 5\frac{1}{4}$ and $\theta(G_{49}) = 8\frac{3}{4}$;

(22) G_{22} is the improvement of G_{24}, G_{25}, G_{26}, G_{39}, G_{40}, G_{49}, G_{50} and G_{51} with $\theta(G_{24}) = 7\frac{1}{4}$, $\theta(G_{25}) = 9$, $\theta(G_{26}) = 13$, $\theta(G_{39}) = 10$, $\theta(G_{40}) = 14$, $\theta(G_{49}) = 8\frac{3}{4}$, $\theta(G_{50}) = 9\frac{3}{4}$ and $\theta(G_{51}) = 14\frac{3}{4}$;

(23) G_{23} is the improvement of G_{24}, G_{29}, G_{49} and G_{52} with $\theta(G_{24}) = 7\frac{1}{4}$, $\theta(G_{29}) = 6\frac{1}{8}$, $\theta(G_{49}) = 8\frac{3}{4}$ and $\theta(G_{52}) = 9\frac{1}{8}$;

(24) G_{24} is the improvement of G_{27}, G_{28}, G_{41}, G_{42}, G_{49}, G_{50} and G_{51} with $\theta(G_{27}) = 9\frac{1}{4}$, $\theta(G_{28}) = 13\frac{1}{4}$, $\theta(G_{41}) = 10\frac{1}{4}$, $\theta(G_{42}) = 14\frac{1}{4}$, $\theta(G_{49}) = 8\frac{3}{4}$, $\theta(G_{50}) = 9\frac{3}{4}$ and $\theta(G_{51}) = 14\frac{3}{4}$;

(25) G_{25} is the improvement of G_{26}, G_{39}, G_{45}, G_{50} and G_{53} with $\theta(G_{26}) = 13$, $\theta(G_{39}) = 10$, $\theta(G_{45}) = 16$, $\theta(G_{50}) = 9\frac{3}{4}$ and $\theta(G_{53}) = 16\frac{3}{4}$;

(26) G_{26} is the improvement of G_{35}, G_{40}, G_{45}, G_{46}, G_{51}, G_{53} and G_{54} with $\theta(G_{35}) = 27$, $\theta(G_{40}) = 14$, $\theta(G_{45}) = 16$, $\theta(G_{46}) = 28$, $\theta(G_{51}) = 14\frac{3}{4}$, $\theta(G_{53}) = 16\frac{3}{4}$ and $\theta(G_{54}) = 17\frac{1}{2}$;

(27) G_{27} is the improvement of G_{28}, G_{31}, G_{41}, G_{47}, G_{50}, G_{53}, G_{55} and G_{56} with $\theta(G_{28}) = 13\frac{1}{4}$, $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{41}) = 10\frac{1}{4}$, $\theta(G_{47}) = 16\frac{1}{4}$, $\theta(G_{50}) = 9\frac{3}{4}$, $\theta(G_{53}) = 16\frac{3}{4}$, $\theta(G_{55}) = 11\frac{1}{8}$ and $\theta(G_{56}) = 17\frac{1}{8}$;

(28) G_{28} is the improvement of G_{36}, G_{37}, G_{42}, G_{47}, G_{48}, G_{51}, G_{53}, G_{54}, G_{56}, G_{57} and G_{58} with $\theta(G_{36}) = 27\frac{1}{4}$, $\theta(G_{37}) = 14\frac{1}{2}$, $\theta(G_{42}) = 14\frac{1}{4}$, $\theta(G_{47}) = 16\frac{1}{4}$, $\theta(G_{48}) = 28\frac{1}{4}$, $\theta(G_{51}) = 14\frac{3}{4}$, $\theta(G_{53}) = 16\frac{3}{4}$, $\theta(G_{54}) = 17\frac{1}{2}$, $\theta(G_{56}) = 17\frac{1}{8}$, $\theta(G_{57}) = 15\frac{1}{8}$ and $\theta(G_{58}) = 29\frac{1}{8}$;

(29) G_{29} is the improvement of G_{30}, G_{52}, G_{59} and G_{60} with $\theta(G_{30}) = 8\frac{1}{8}$, $\theta(G_{52}) = 9\frac{1}{8}$, $\theta(G_{59}) = 8\frac{1}{16}$ and $\theta(G_{60}) = 10\frac{1}{16}$;

(30) G_{30} is the improvement of G_{31}, G_{32}, G_{52}, G_{55}, G_{57}, G_{60}, G_{61} and G_{62} with $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{32}) = 14\frac{1}{8}$, $\theta(G_{52}) = 9\frac{1}{8}$, $\theta(G_{55}) = 11\frac{1}{8}$, $\theta(G_{57}) = 15\frac{1}{8}$, $\theta(G_{60}) = 10\frac{1}{16}$, $\theta(G_{61}) = 12\frac{1}{16}$ and $\theta(G_{62}) = 16\frac{1}{16}$;

(31) G_{31} is the improvement of G_{32}, G_{55}, G_{56}, G_{61} and G_{63} with $\theta(G_{32}) = 14\frac{1}{8}$, $\theta(G_{55}) = 11\frac{1}{8}$, $\theta(G_{56}) = 17\frac{1}{8}$, $\theta(G_{61}) = 12\frac{1}{16}$ and $\theta(G_{63}) = 18\frac{1}{16}$;

(32) G_{39} is the improvement of G_{40}, G_{45}, G_{50}, G_{64}, G_{65}, G_{66} and G_{67} with $\theta(G_{40}) = 14$, $\theta(G_{45}) = 16$, $\theta(G_{50}) = 9\frac{3}{4}$, $\theta(G_{64}) = 13\frac{1}{2}$, $\theta(G_{65}) = 17\frac{1}{2}$, $\theta(G_{66}) = 13\frac{3}{4}$ and $\theta(G_{67}) = 17\frac{3}{4}$;

(33) G_{41} is the improvement of G_{42}, G_{47}, G_{50}, G_{55}, G_{66}, G_{67}, G_{68} and G_{69} with $\theta(G_{42}) = 14\frac{1}{4}$, $\theta(G_{47}) = 16\frac{1}{4}$, $\theta(G_{50}) = 9\frac{3}{4}$, $\theta(G_{55}) = 11\frac{1}{8}$, $\theta(G_{66}) = 13\frac{3}{4}$, $\theta(G_{67}) = 17\frac{3}{4}$, $\theta(G_{68}) = 14\frac{1}{8}$ and $\theta(G_{69}) = 18\frac{1}{8}$;

(34) G_{49} is the improvement of G_{50}, G_{51}, G_{52}, G_{70}, G_{71}, G_{72} and G_{73} with $\theta(G_{50}) = 9\frac{3}{4}$, $\theta(G_{51}) = 14\frac{3}{4}$, $\theta(G_{52}) = 9\frac{1}{8}$, $\theta(G_{70}) = 12\frac{1}{2}$, $\theta(G_{71}) = 16\frac{1}{2}$, $\theta(G_{72}) = 12\frac{5}{8}$ and $\theta(G_{73}) = 16\frac{5}{8}$;

(35) G_{50} is the improvement of G_{51}, G_{53}, G_{55}, G_{66}, G_{67}, G_{70}, G_{72}, G_{74}, G_{75}, G_{76} and G_{77} with $\theta(G_{51}) = 14\frac{3}{4}$, $\theta(G_{53}) = 16\frac{3}{4}$, $\theta(G_{55}) = 11\frac{1}{8}$, $\theta(G_{66}) = 13\frac{3}{4}$, $\theta(G_{67}) = 17\frac{3}{4}$, $\theta(G_{70}) = 12\frac{1}{2}$, $\theta(G_{72}) = 12\frac{5}{8}$, $\theta(G_{74}) = 14\frac{1}{2}$, $\theta(G_{75}) = 18\frac{1}{2}$, $\theta(G_{76}) = 14\frac{1}{8}$ and $\theta(G_{77}) = 18\frac{5}{8}$;

(36) G_{52} is the improvement of G_{55}, G_{57}, G_{60}, G_{72}, G_{73}, G_{78} and G_{79} with $\theta(G_{55}) = 11\frac{1}{8}$, $\theta(G_{57}) = 15\frac{1}{8}$, $\theta(G_{60}) = 10\frac{1}{16}$, $\theta(G_{72}) = 12\frac{5}{8}$, $\theta(G_{73}) = 16\frac{5}{8}$, $\theta(G_{78}) = 13\frac{1}{16}$ and $\theta(G_{79}) = 17\frac{1}{16}$;

(37) G_{55} is the improvement of G_{56}, G_{57}, G_{61}, G_{68}, G_{69}, G_{72}, G_{76}, G_{77}, G_{78}, G_{80} and G_{81} with $\theta(G_{56}) = 17\frac{1}{8}$, $\theta(G_{57}) = 15\frac{1}{8}$, $\theta(G_{61}) = 12\frac{1}{16}$, $\theta(G_{68}) = 14\frac{1}{8}$, $\theta(G_{69}) = 18\frac{1}{8}$, $\theta(G_{72}) = 12\frac{5}{8}$, $\theta(G_{76}) = 14\frac{1}{8}$, $\theta(G_{77}) = 18\frac{5}{8}$, $\theta(G_{78}) = 13\frac{1}{16}$, $\theta(G_{80}) = 15\frac{1}{16}$ and $\theta(G_{81}) = 19\frac{1}{16}$;

(38) G_{59} is the improvement of G_{60} and G_{82} with $\theta(G_{60}) = 10\frac{1}{16}$ and $\theta(G_{82}) = 12\frac{1}{32}$;

(39) G_{60} is the improvement of G_{61}, G_{62}, G_{78}, G_{80}, G_{82}, G_{83} and G_{84} with $\theta(G_{61}) = 12\frac{1}{16}$, $\theta(G_{62}) = 16\frac{1}{16}$, $\theta(G_{78}) = 13\frac{1}{16}$, $\theta(G_{80}) = 15\frac{1}{16}$, $\theta(G_{82}) = 12\frac{1}{32}$, $\theta(G_{83}) = 14\frac{1}{32}$ and $\theta(G_{84}) = 16$;

(40) G_{61} is the improvement of G_{62}, G_{63}, G_{78}, G_{80}, G_{81}, G_{83}, G_{85} and G_{86} with $\theta(G_{62}) = 16\frac{1}{16}$, $\theta(G_{63}) = 18\frac{1}{16}$, $\theta(G_{78}) = 13\frac{1}{16}$, $\theta(G_{80}) = 15\frac{1}{16}$, $\theta(G_{81}) = 19\frac{1}{16}$, $\theta(G_{83}) = 14\frac{1}{32}$, $\theta(G_{85}) = 16\frac{1}{32}$ and $\theta(G_{86}) = 20\frac{1}{32}$;

(41) G_{64} is the improvement of G_{65}, G_{66} and G_{87} with $\theta(G_{65}) = 17\frac{1}{2}$, $\theta(G_{66}) = 13\frac{3}{4}$ and $\theta(G_{87}) = 21\frac{1}{4}$;

(42) G_{70} is the improvement of G_{71}, G_{72}, G_{74}, G_{75}, G_{88} and G_{89} with $\theta(G_{71}) = 16\frac{1}{2}$, $\theta(G_{72}) = 12\frac{5}{8}$, $\theta(G_{74}) = 14\frac{1}{2}$, $\theta(G_{75}) = 18\frac{1}{2}$, $\theta(G_{88}) = 16\frac{3}{8}$ and $\theta(G_{89}) = 20\frac{3}{8}$;

(43) G_{72} is the improvement of G_{73}, G_{76}, G_{77}, G_{78}, G_{88}, G_{89}, G_{90} and G_{91} with $\theta(G_{73}) = 16\frac{5}{8}$, $\theta(G_{76}) = 14\frac{1}{8}$, $\theta(G_{77}) = 18\frac{5}{8}$, $\theta(G_{78}) = 13\frac{1}{16}$, $\theta(G_{88}) = 16\frac{3}{8}$, $\theta(G_{89}) = 20\frac{3}{8}$, $\theta(G_{90}) = 16\frac{5}{8}$ and $\theta(G_{91}) = 20\frac{9}{16}$;

(44) G_{78} is the improvement of G_{79}, G_{80}, G_{81}, G_{83}, G_{91}, G_{92} and G_{93} with $\theta(G_{79}) = 17\frac{1}{16}$, $\theta(G_{80}) = 15\frac{1}{16}$, $\theta(G_{81}) = 19\frac{1}{16}$, $\theta(G_{83}) = 14\frac{1}{32}$, $\theta(G_{91}) = 20\frac{9}{16}$, $\theta(G_{92}) = 17\frac{1}{32}$ and $\theta(G_{93}) = 21\frac{1}{32}$;

(45) G_{82} is the improvement of G_{83}, G_{84}, G_{94} and G_{95} with $\theta(G_{83}) = 14\frac{1}{32}$, $\theta(G_{84}) = 16$, $\theta(G_{94}) = 16\frac{1}{64}$ and $\theta(G_{95}) = 20\frac{1}{64}$;

Hence, by Lemma 2.4 and the above arguments, we know (i) to (xxi) holds. Thus the proof is completed.

Similarly to the proof of Theorem 3.1, we can obtain Theorems 3.2 and 3.3.

Theorem 3.2 *Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 6-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n + 1$ and $n_6 - n_1 \leq 4$. Define $\theta(G) = [\alpha(G, 7) - 5 \cdot 2^{n-1} - 2^n + 6]/2^{n-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(n, n, n, n, n, n + 1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(n - 1, n, n, n, n + 1, n + 1)$;

(iii) $\theta(G) = 2$ *if and only if* $G = K(n - 1, n - 1, n, n + 1, n + 1, n + 1)$;

(iv) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(n - 2, n, n, n + 1, n + 1, n + 1)$;

(v) $\theta(G) = 3$ *if and only if* $G = K(n - 1, n, n, n, n, n + 2)$;

(vi) $\theta(G) = 3\frac{1}{2}$ *if and only if* $G = K(n - 2, n - 1, n + 1, n + 1, n + 1, n + 1)$;

(vii) $\theta(G) = 4$ *if and only if* $G = K(n - 1, n - 1, n, n, n + 1, n + 2)$;

(viii) $\theta(G) = 4\frac{1}{4}$ *if and only if* $G = K(n - 3, n, n + 1, n + 1, n + 1, n + 1)$;

(ix) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(n - 2, n, n, n, n + 1, n + 2)$;

(x) $\theta(G) = 5$ *if and only if* $G = K(n - 1, n - 1, n - 1, n + 1, n + 1, n + 2)$;

(xi) $\theta(G) = 5\frac{1}{2}$ *if and only if* $G = K(n - 2, n - 1, n, n + 1, n + 1, n + 2)$;

(xii) $\theta(G) = 7$ *if and only if* $G = K(n - 1, n - 1, n - 1, n, n + 2, n + 2)$ *or* $G = K(n - 2, n - 2, n + 1, n + 1, n + 1, n + 2)$;

(xiii) $\theta(G) = 7\frac{1}{2}$ *if and only if* $G = K(n - 2, n - 1, n, n, n + 2, n + 2)$;

(xiv) $\theta(G) = 8\frac{1}{2}$ *if and only if* $G = K(n - 2, n - 1, n, n + 1, n + 1, n + 2)$;

(xv) $\theta(G) = 9$ *if and only if* $G = K(n - 2, n - 2, n, n + 1, n + 2, n + 2)$;

(xvi) $\theta(G) = 10$ *if and only if* $G = K(n - 1, n - 1, n, n, n, n + 3)$;

(xvii) $\theta(G) = 11$ *if and only if* $G = K(n - 1, n - 1, n - 1, n, n + 1, n + 3)$;

(xviii) $\theta(G) = 12$ *if and only if* $G = K(n - 2, n - 2, n - 1, n + 2, n + 2, n + 2)$;

(xix) $\theta(G) = 14$ *if and only if* $G = K(n - 1, n - 1, n - 1, n - 1, n + 2, n + 3)$.

Theorem 3.3 *Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 6-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n + 2$ and $n_6 - n_1 \leq 4$. Define $\theta(G) = [\alpha(G, 6) - 2^{n+2} + 6]/2^{n-1}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(n, n, n, n, n+1, n+1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(n-1, n, n, n+1, n+1, n+1)$;

(iii) $\theta(G) = 2$ *if and only if* $G = K(n, n, n, n, n, n+2)$ *or* $G = K(n-1, n-1, n+1, n+1, n+1, n+1)$;

(iv) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(n-2, n, n+1, n+1, n+1, n+1)$;

(v) $\theta(G) = 3$ *if and only if* $G = K(n-1, n, n, n, n+1, n+2)$;

(vi) $\theta(G) = 4$ *if and only if* $G = K(n-1, n-1, n, n+1, n+1, n+2)$;

(vii) $\theta(G) = 4\frac{1}{4}$ *if and only if* $G = K(n-3, n+1, n+1, n+1, n+1, n+1)$;

(viii) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(n-2, n, n, n+1, n+1, n+2)$;

(ix) $\theta(G) = 5$ *if and only if* $G = K(n-1, n-1, n-1, n-1, n+3, n+3)$;

(x) $\theta(G) = 5\frac{1}{2}$ *if and only if* $G = K(n-2, n-1, n+1, n+1, n+1, n+2)$;

(xi) $\theta(G) = 6$ *if and only if* $G = K(n-1, n-1, n, n, n+2, n+2)$;

(xii) $\theta(G) = 6\frac{1}{2}$ *if and only if* $G = K(n-2, n, n, n, n+2, n+2)$;

(xiii) $\theta(G) = 7$ *if and only if* $G = K(n-1, n-1, n-1, n+1, n+2, n+2)$;

(xiv) $\theta(G) = 7\frac{1}{2}$ *if and only if* $G = K(n-2, n-1, n, n+1, n+2, n+2)$;

(xv) $\theta(G) = 9$ *if and only if* $G = K(n-1, n, n, n, n, n+3)$ *or* $G = K(n-2, n-2, n+1, n+1, n+2, n+2)$;

(xvi) $\theta(G) = 10$ *if and only if* $G = K(n-1, n-1, n, n, n+1, n+3)$;

(xvii) $\theta(G) = 10\frac{1}{2}$ *if and only if* $G = K(n-2, n-1, n-1, n+2, n+2, n+2)$;

(xviii) $\theta(G) = 11$ *if and only if* $G = K(n-1, n-1, n-1, n+1, n+1, n+3)$ *or* $G = K(n-2, n-2, n, n+2, n+2, n+2)$;

(xix) $\theta(G) = 13$ *if and only if* $G = K(n-1, n-1, n-1, n, n+2, n+3)$.

4 Chromatically closed 6-partite graphs

In this section, we obtained several χ-closed families of graphs from the graphs in Theorem 3.1 to 3.3 with a set S of s edges deleted.

Theorem 4.1 *The family of graphs* $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5, n_6)$ *where* $n_1 + n_2 + n_3 + n_4 + n_5 = 6n$, $n_6 - n_1 \le 4$ *and* $n_1 \ge s + 10$ *is* χ*-closed except the graphs* $\{\mathcal{K}^{-s}(n-1, n-1, n-1, n, n+1, n+2), \mathcal{K}^{-s}(n-2, n-2, n+1, n+1, n+1, n+1)\}$.

Proof. By Theorem 3.1, there are 21 cases to consider. Denote each graph in Theorem 3.1 $(i), (ii), \cdots, (xxi)$ by $G_1, G_2, \cdots, G_{21}$, respectively. Suppose $H \sim G_i - S$. It suffices to show that $H \in \{G_i - S\}$. Let $\{B_1, B_2, B_3, B_4, B_5, B_6\}$ be 6-independent partition of H, $|B_i| = P_i$, $i = 1, 2, 3, 4, 5, 6$, $F_i = (p_1, p_2, p_3, p_4, p_5, p_6)$. Then there exists $S' \subseteq e(F)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s \ge 0$.

Case (i). Let $G = G_1$ with $n \ge s + 2$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n, n, n, n, n, n)$. By Lemma 2.5, we have

$$\alpha(G - S, 7) = \alpha(G, 7) + \alpha'(G - S) \text{ with } s \le \alpha'(G - S) \le 2^s - 1,$$
$$\alpha(F - S', 7) = \alpha(F, 7) + \alpha'(F - S') \text{ with } 0 \le s' \le \alpha'(F - S').$$

Hence,

$$\alpha(F - S', 7) - \alpha(G - S, 7) = \alpha(F, 7) - \alpha(G, 7) + \alpha'(F - S') - \alpha'(G - S).$$

By the definition, $\alpha(F, 7) - \alpha(G, 7) = 2^{n-2}(\theta(F) - \theta(G))$. By Theorem 3.1, $\theta(F) \ge 0$. Suppose $\theta(F) > 0$, then

$$\begin{aligned} \alpha(F - S', 7) - \alpha(G - S, 7) &\ge 2^{n-2} + \alpha'(F - S') - \alpha'(G - S) \\ &\ge 2^s + \alpha'(F - S') - 2^s + 1, \\ &\ge 1, \end{aligned}$$

contradicting $\alpha(F - S', 7) = \alpha(G - S, 7)$. Hence, $\theta(F) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathcal{K}^{-s}(n, n, n, n, n, n)$.

Case (ii). Let $G = G_2$ with $n \ge s + 3$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n - 1, n, n, n, n, n + 1)$. By Lemma 2.5, we have

$$\alpha(G - S, 7) = \alpha(G, 7) + \alpha'(G - S) \text{ with } s \le \alpha'(G - S) \le 2^s - 1,$$
$$\alpha(F - S', 7) = \alpha(F, 7) + \alpha'(F - S') \text{ with } 0 \le s' \le \alpha'(F - S').$$

Hence,

$$\alpha(F-S',7)-\alpha(G-S,7)=\alpha(F,7)-\alpha(G,7)+\alpha'(F-S')-\alpha'(G-S).$$

By the definition, $\alpha(F,7)-\alpha(G,7)=2^{n-2}(\theta(F)-\theta(G))$. Suppose $\theta(F)\neq\theta(G)$. Then, we consider two subcases.

Subcase (a). $\theta(F)<\theta(G)$. By Theorem 3.1, $F=G_1$ and $H=G_1-S'\in\{G_1-S'\}$. However, $G-S\notin\{G_1-S'\}$ since by Case (i) above, $\{G_1-S'\}$ is χ-closed, a contradiction.

Subcase (b). $\theta(F)>\theta(G)$. By Theorem 3.1, $\alpha(F,7)-\alpha(G,7)\geq 2^{n-2}$. So,

$$\begin{aligned}\alpha(F-S',7)-\alpha(G-S,7) &\geq 2^{n-2}+\alpha'(F-S')-\alpha'(G-S)\\ &\geq 2^{s}+\alpha'(F-S')-2^{s}+1,\\ &\geq 1,\end{aligned}$$

contradicting $\alpha(F-S',7)=\alpha(G-S,7)$. Hence, $\theta(F)-\theta(G)=0$ and so $F=G$ and $s=s'$. Therefore, $H\in\mathcal{K}^{-s}(n-1,n,n,n,n,n+1)$.

Using Table 1, we can prove (iii) to (xxi) except (x) in a similar way. This completes the proof.

Similarly, we can prove Theorems 4.2 and 4.3.

Theorem 4.2 *The family of graphs* $\mathcal{K}^{-s}(n_1,n_2,n_3,n_4,n_5,n_6)$ *where* $n_1+n_2+n_3+n_4+n_5+n_6=6n+1$, $n_6-n_1\leq 4$ *and* $n_1\geq s+7$ *is* χ*-closed except the graphs* $\{\mathcal{K}^{-s}(n-1,n-1,n-1,n,n+2,n+2),\mathcal{K}^{-s}(n-2,n-2,n+1,n+1,n+1,n+2)\}$.

Theorem 4.3 *The family of graphs* $\mathcal{K}^{-s}(n_1,n_2,n_3,n_4,n_5,n_6)$ *where* $n_1+n_2+n_3+n_4+n_5+n_6=6n+2$, $n_6-n_1\leq 4$ *and* $n_1\geq s+7$ *is* χ*-closed except the graphs* $\{\mathcal{K}^{-s}(n,n,n,n,n,n+2),\mathcal{K}^{-s}(n-1,n-1,n+1,n+1,n+1,n+1)\}$, $\{\mathcal{K}^{-s}(n-1,n,n,n,n,n+3),\mathcal{K}^{-s}(n-2,n-2,n+1,n+1,n+2,n+2)\}$ *and* $\{\mathcal{K}^{-s}(n-1,n-1,n-1,n+1,n+1,n+3),\mathcal{K}^{-s}(n-2,n-2,n,n+2,n+2,n+2)\}$.

5 Chromatically unique 6-partite graphs

The following results give several families of chromatically unique complete 6-partite graphs having $6n$ vertices with a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$ and a matching sK_2, respectively.

Theorem 5.1 *The graphs $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5,n_6)$ where $n_1+n_2+n_3+n_4+n_5+n_6=6n$, $n_6-n_1\le 4$ and $n_1\ge s+10$ are χ-unique for $1\le i\neq j\le 6$ except the graphs $\{\mathcal{K}^{-s}(n-1,n-1,n-1,n,n+1,n+2),\mathcal{K}^{-s}(n-2,n-2,n+1,n+1,n+1,n+1)\}$.*

Proof. By Theorem 3.1, there are 21 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\cdots,(xiv)$ by $G_1,G_2,\cdots,G_{21}$, respectively. The proof for each graph obtained from G_i $(i=1,2,\cdots,21)$ is similar, so we only give the detail proof for the graphs obtained from G_2 below.

By Lemma 2.5 and Case 2 of Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)=\{K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)|(i,j)\in$ {(1,2),(2,1),(1,6),(6,1),(2,3),(2,6),(6,2)} is χ-closed for $n\ge s+3$. Note that
$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1))=t(G_2)-s(4n+1)$ for $(i,j)\in\{(1,2),(2,1)\}$,
$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1))=t(G_2)-4sn$ for $(i,j)\in\{(1,6),(6,1)\}$,
$t(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n,n+1))=t(G_2)-4sn$,
$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1))=t(G_2)-s(4n-1)$ for $(i,j)\in\{(2,6),(6,2)\}$.
By Lemmas 2.2 and 2.6, we conclude that $\sigma(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1))\neq\sigma(K_{j,i}^{-K_{1,s}}(n-1,n,n,n,n,n+1))$ for each $(i,j)\in\{(1,2),(1,6),(2,6)\}$. We now show that $K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n,n+1)$ and $K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)$ for $(i,j)\in\{(1,6),(6,1)\}$ are not χ-equivalent. We have

$$\begin{aligned}Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n,n+1)) &= Q(G_2)-s(n-1)^2+\binom{s}{2}+s\Bigg[\binom{n-1}{2}+\\ &\quad \binom{n}{2}+\binom{n}{2}+\binom{n+1}{2}\Bigg];\\ Q(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)) &= Q(G_2)-sn(n-2)+\binom{s}{2}+4s\binom{n}{2} \text{ for } (i,j)\in\{(1,6),(6,1)\};\end{aligned}$$

with

$$Q\Big(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n,n+1)\Big)-Q\Big(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)\Big)=0$$

and that

$$\begin{aligned}K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n,n+1)) &= K(G_2)-s(6n^2-1);\\ K(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)) &= K(G_2)-6sn^2\\ &\quad \text{for } (i,j)\in\{(1,6),(6,1)\};\end{aligned}$$

with

$$K\Big(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n,n+1)\Big)-K\Big(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)\Big)=s$$

This means that $2K(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)) - Q(K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)) \neq 2K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n,n+1)) - Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n,n,n+1))$ for $(i,j) \in \{(1,6),(6,1)\}$, contradicting Lemma 2.1. Hence, $K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)$ is χ-unique where $n \geq s+3$ for $1 \leq i \neq j \leq 6$.

The proof is thus complete.

Theorem 5.2 *The graphs* $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5,n_6)$ *where* $n_1+n_2+n_3+n_4+n_5+n_6 = 6n$, $n_6 - n_1 \leq 4$ *and* $n_1 \geq s+10$ *are* χ*-unique except the graphs* $\{\mathcal{K}^{-s}(n-1,n-1,n-1,n,n+1,n+2), \mathcal{K}^{-s}(n-2,n-2,n+1,n+1,n+1,n+1)\}$.

Proof. By Theorem 3.1, there are 21 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\cdots,(xxi)$ by $G_1,G_2,\cdots,G_{21}$, respectively. For a graph $K(p_1,p_2,p_3,p_4,p_5,p_6)$, let $S = \{e_1,e_2,\cdots,e_s\}$ be the set of s edges in $E(K(p_1,p_2,p_3,p_4,p_5,p_6))$ and let $t(e_i)$ denote the number of triangles containing e_i in $K(p_1,p_2,p_3,p_4,p_5,p_6)$. The proofs for each graph obtained from G_i $(i = 1,2,\cdots,21)$ are similar, so we only give the proof of the graph obtained from G_2 as follows.

Suppose $H \sim G = K_{1,2}^{-sK_2}(n-1,n,n,n,n,n+1)$ for $n \geq s+3$. By Theorem 4.1 and Lemma 2.1, $H \in \mathcal{K}^{-s}(n-1,n,n,n,n,n+1)$ and $\alpha'(H) = \alpha'(G) = s$. Let $H = F - S$ where $F = K(n-1,n,n,n,n,n+1)$. Clearly, $t(e_i) \leq 4n+1$ for each $e_i \in S$. So,

$$t(H) \geq t(F) - s(4n+1),$$

with equality holds only if $t(e_i) = 4n+1$ for all $e_i \in S$. Since $t(H) = t(G) = t(F) - s(4n+1)$, the equality above holds with $t(e_i) = 4n+1$ for all $e_i \in S$. Therefore each edge in S has an end-vertex in V_1 and another end-vertex in V_j $(2 \leq j \leq 5)$. Moreover, S must induce a matching in F. Otherwise, equality does not hold or $\alpha'(H) > s$. By Lemma 2.8, we obtain

$$\begin{aligned} Q(H) - 2K(G) &= Q(F) - s(n-2)(n-1) + \binom{s}{2} + \\ &\quad + s\left[\binom{n}{2} + \binom{n}{2} + \binom{n}{2} + \binom{n+1}{2}\right] - 2\left[K(F) - s(6n^2+3n)\right] \\ &\geq Q(H) - 2K(H); \end{aligned}$$

the equality holds if and only if $s = s_{1j}$ for $2 \leq j \leq 5$. Therefore, we have $\langle S \rangle \cong sK_2$ with $H \cong G$.

Thus the proof is complete.

Similarly to the proofs of Theorems 5.1 and 5.2, we can prove Theorems 5.3 to 5.6.

Theorem 5.3 *The graphs* $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$ *where* $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n+1$, $n_6 - n_1 \leq 4$ *and* $n_1 \geq s+7$ *are* χ*-unique for* $1 \leq i \neq j \leq 6$ *except the graphs* $\{K_{i,j}^{-K_{1,s}}(n-1, n-1, n-1, n, n+2, n+2), K_{i,j}^{-K_{1,s}}(n-2, n-2, n+1, n+1, n+1, n+2)\}$.

Theorem 5.4 *The graphs* $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$ *where* $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n+2$, $n_6 - n_1 \leq 4$ *and* $n_1 \geq s+7$ *are* χ*-unique for* $1 \leq i \neq j \leq 6$ *except the graphs* $\{K_{i,j}^{-K_{1,s}}(n, n, n, n, n, n+2), K_{i,j}^{-K_{1,s}}(n-1, n-1, n+1, n+1, n+1, n+1)\}$, $\{K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+3), K_{i,j}^{-K_{1,s}}(n-2, n-2, n+1, n+1, n+2, n+2)\}$ *and* $\{K_{i,j}^{-K_{1,s}}(n-1, n-1, n-1, n+1, n+1, n+3), K_{i,j}^{-K_{1,s}}(n-2, n-2, n, n+2, n+2, n+2)\}$.

Theorem 5.5 *The graphs* $K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4, n_5, n_6)$ *where* $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n + 1$, $n_6 - n_1 \leq 4$ *and* $n_1 \geq s + 7$ *are* χ*-unique except the graphs* $\{K_{1,2}^{-sK_2}(n-1, n-1, n-1, n, n+2, n+2), K_{1,2}^{-sK_2}(n-2, n-2, n+1, n+1, n+1, n+2)\}$.

Theorem 5.6 *The graphs* $K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4, n_5, n_6)$ *where* $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n + 2$, $n_6 - n_1 \leq 4$ *and* $n_1 \geq s + 7$ *are* χ*-unique except the graphs* $\{K_{1,2}^{-sK_2}(n, n, n, n, n, n+2), K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1, n+1)\}$, $\{K_{1,2}^{-sK_2}(n-1, n, n, n, n, n+3), K_{1,2}^{-sK_2}(n-2, n-2, n+1, n+1, n+2, n+2)\}$ *and* $\{K_{1,2}^{-sK_2}(n-1, n-1, n-1, n+1, n+1, n+3), K_{1,2}^{-sK_2}(n-2, n-2, n, n+2, n+2, n+2)\}$.

Remark: This paper generalized some results in papers [13,14,15].

Problems: (1) Study the chromaticity of the graphs $K^{-s}(n-1, n-1, n-1, n, n+1, n+2)$ and $K^{-s}(n-2, n-2, n+1, n+1, n+1, n+1)$.
(2) Study the chromaticity of the graphs $K^{-s}(n-1, n-1, n-1, n, n+2, n+2)$ and $K^{-s}(n-2, n-2, n+1, n+1, n+1, n+2)$.
(3) Study the chromaticity of the graphs $K^{-s}(n, n, n, n, n, n+2)$, $K^{-s}(n-1, n-1, n+1, n+1, n+1, n+1)$, $K^{-s}(n-1, n, n, n, n, n+3)$, $K^{-s}(n-2, n-2, n+1, n+1, n+2, n+2)$, $K^{-s}(n-1, n-1, n-1, n+1, n+1, n+3)$ and $K^{-s}(n-2, n-2, n, n+2, n+2, n+2)$.

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