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On Chromatic Uniqueness of Certain 6-Partite Graphs

H. Roslan [1], **A. Sh. Ameen**

School of Mathematical Sciences
University Sains Malaysia, 11800 Penang, Malaysia

Y. H. Peng

Department of Mathematics, and
Institute for Mathematical Research
University Putra Malaysia
43400UPM Serdang, Malaysia

Abstract

Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H|H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 6-partite graphs with $6n + 2$ vertices according to the number of 7-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 6-partite graphs with certain star or matching deleted are obtained.

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[1] hroslan@cs.usm.my

1 Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G, \lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. Many families of χ-unique graphs are known (see [6,7,8]).

For a graph G, let $V(G)$, $E(G)$ and $t(G)$ be the vertex set, edge set and number of triangles in G, respectively. Let S be a set of s edges in G. Let $G - S$ (or $G - s$) be the graph obtained from G by deleting all edges in S, and by $\langle S \rangle$ the graph induced by S. Let $K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph. We denote by $\mathcal{K}^{-s}(n_1, n_2, \cdots, n_t)$ the family of graphs which are obtained from $K(n_1, n_2, \cdots, n_t)$ by deleting a set S of some s edges.

In [4,5,7–10,12,13,14,17,18,19], one can find many results on the chromatic uniqueness of certain families of complete t-partite graphs ($t = 2, 3, 4, 5$). However, there are very few 6-partite graphs known to be χ-unique, see [3,15,16].

In [3,15,16], Chen and Roslan et al. have obtained many families of χ-unique graphs which are obtained by deleting the edges of a star or matching from a complete 6-partite graph with $6n + i$ vertices where $i = 0, 1, 5$. Thus, the aim of this paper is to study the chromaticity of the graphs which are obtained by deleting the edges of a star or matching from a complete 6-partite graph with $6n + 2$ vertices.

Let G be a complete 6-partite graph with $6n + 2$ vertices. In this paper, we characterize certain complete 6-partite graphs with $6n + 2$ vertices according to the number of 7-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 6-partite graphs with certain star or matching deleted are obtained.

2 Some lemmas and notations

For a graph G and a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of G. Then, we have $P(G, \lambda) = \sum_{r=1}^{p} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ (see [11]). Therefore, $\alpha(G, r) = \alpha(H, r)$ for each $r = 1, 2, \cdots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G, x) = \sum_{r=1}^{p} \alpha(G, r)x^r$ is called the σ-polynomial of G (see [2]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs G and H.

For disjoint graphs G and H, $G \cup H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer [1].

Lemma 2.1 *(Brenti [2], Koh and Teo [7]) Let G and H be two disjoint graphs. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ *and* $\alpha(G, r) = \alpha(H, r)$ *for* $r = 1, 2, 3, \cdots, p$, *if* $G \sim H$;

(2) $\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x)$.

Lemma 2.2 *(Brenti [2]) Let* $G = K(n_1, n_2, n_3, \cdots, n_t)$ *and* $\sigma(G, x) = \sum_{r \geq 1} \alpha(G, r)x^r$, *then* $\alpha(G, r) = 0$ *for* $1 \leq r \leq t-1$, $\alpha(G, t) = 1$ *and* $\alpha(G, t+1) = \sum_{i=1}^{t} 2^{n_i - 1} - t$.

Let $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6$ be positive integers and $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. If there are two elements x_{i_1} and x_{i_2} in $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ such that $x_{i_2} - x_{i_1} \geq 2$, then $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ is called an *improvement* of $H = K(x_1, x_2, x_3, x_4, x_5, x_6)$.

Lemma 2.3 *(Chen [3]) Suppose* $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6$ *and* $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ *is an improvement of* $H = K(x_1, x_2, x_3, x_4, x_5, x_6)$, *then*

$$\alpha(H, 7) - \alpha(H', 7) = 2^{x_{i_2}-2} - 2^{x_{i_1}-1} \geq 2^{x_{i_1}-1}.$$

Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H, 7) - \alpha(G, 7)$. Clearly, $\alpha'(H) \geq 0$.

Lemma 2.4 *(Chen [3]) Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$. Suppose that min $\{n_i | i = 1, 2, 3, 4, 5, 6\} \geq s + 1 \geq 1$ and $H = G - S$, where S is a set of some s edges of G, then*

$$s \leq \alpha'(H) = \alpha(H, 7) - \alpha(G, 7) \leq 2^s - 1,$$

$\alpha'(H) = s$ iff the set of end-vertices of any $r \geq 2$ edges in S is not independent in H, and $\alpha'(H) = 2^s - 1$ iff S induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same A_i.

Let $K(A_1, A_2)$ be a complete bipartite graph with partite sets A_1 and A_2. We denote by $K^{-K_{1,s}}(A_i, A_j)$ the graph obtained from $K(A_i, A_j)$ by deleting s edges that induce a star with its center in A_i. Note that $K^{-K_{1,s}}(A_i, A_j) \neq K^{-K_{1,s}}(A_j, A_i)$ if $|A_i| \neq |A_j|$ for $i \neq j$ (see [5]).

Lemma 2.5 *(Dong et al. [4]) Let $K(n_1, n_2)$ be a complete bipartite graph with partite sets A_1 and A_2 such that $|A_i| = n_i$ for $i = 1, 2$. If min $\{n_1, n_2\} \geq s+2$, then every $K^{-K_{1,s}}(A_i, A_j)$ is χ-unique, where $i \neq j$ and $i, j = 1, 2$.*

Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 5-partite graph with partite sets $A_i (i = 1, 2, \cdots, 6)$ such that $|A_i| = n_i$. Let $\langle A_i \cup A_j \rangle$ be the subgraph of G induced by $A_i \cup A_j$, where $i \neq j$ and $i, j \in \{1, 2, 3, 4, 5, 6\}$. By $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$, we denote the graph obtained from $K(n_1, n_2, n_3, n_4, n_5, n_6)$ by deleting a set of s edges that induce a $K_{1,s}$ with its center in A_i and all it end vertices are in A_j. Note that $K_{i,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6) = K_{j,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$ and $K_{l,i}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6) = K_{l,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$ for $n_i = n_j$ and $l \neq i, j$.

Lemma 2.6 *(Chen [3]) If $i, j \in \{1, 2, 3, \cdots, t\}$, $i \neq j$, $n_i \neq n_j$, then $P(K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, \cdots, n_t), \lambda) \neq P(K_{j,i}^{-K_{1,s}}(n_1, n_2, n_3, \cdots, n_t), \lambda)$.*

3 Classification

In this section, we shall characterize certain complete 6-partite graph $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ according to the number of 7-independent partitions of G where $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n + 2, n \geq 1$.

Theorem 3.1 *Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 6-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n + 2$, $n \geq 1$. Define $\theta(G) = [\alpha(G, 7) - 2^{n+2} + 6]/2^{n-2}$. Then*

(i) $\theta(G) \geq 0$;

(ii) $\theta(G) = 0$ *if and only if* $G = K(n, n, n, n, n + 1, n + 1)$;

(iii) $\theta(G) = 1$ *if and only if* $G = K(n - 1, n, n, n + 1, n + 1, n + 1)$;

(iv) $\theta(G) = 2$ *if and only if* $G = K(n, n, n, n, n, n + 2)$ *or* $G = K(n - 1, n - 1, n + 1, n + 1, n + 1, n + 1)$;

(v) $\theta(G) = 5/2$ *if and only if* $G = K(n - 2, n, n + 1, n + 1, n + 1, n + 1)$;

(vi) $\theta(G) = 3$ *if and only if* $G = K(n - 1, n, n, n, n + 1, n + 2)$;

(vii) $\theta(G) = 4$ *if and only if* $G = K(n - 1, n - 1, n, n + 1, n + 1, n + 2)$;

(viii) $\theta(G) = 17/4$ *if and only if* $G = K(n-3, n+1, n+1, n+1, n+1, n+1)$;

(ix) $\theta(G) \geq 9/2$ *if and only if G is not a graph appeared in (ii)–(viii).*

Proof. For a complete 6-partite graph H_1 with $6n + 2$ vertices, we can construct a sequence of complete 6-partite graphs with $6n + 2$ vertices, say $H_1, H_2, \cdots, H_t$, such that H_i is an improvement of H_{i-1} for each $i = 2, 3, \cdots, t$, and $H_t = K(n, n, n, n, n+1, n+1)$. By Lemma 2.3, $\alpha(H_{i-1}, 7) - \alpha(H_i, 7) > 0$. So $\theta(H_{i-1}) - \theta(H_i) > 0$, which implies that $\theta(G) \geq \theta(H_t) = \theta(K(n, n, n, n, n + 1, n + 1))$. From Lemma 2.2 and by a simple calculation, $\theta(K(n, n, n, n, n + 1, n + 1)) = 0$. Thus, (ii) is true.

Since $H_t = K(n, n, n, n, n+1, n+1)$ and H_t is an improvement of H_{t-1}, it is not hard to see that $H_{t-1} \in \{R, R_0, R_3\}$, where $R = K(n-1, n, n, n+1, n+1, n+1)$,

R_i	**Graphs** H_{t-2}	$\theta(R_i)$
R_1	$K(n-1, n-1, n+1, n+1, n+1, n+1)$	2
R_2	$K(n-2, n, n+1, n+1, n+1, n+1)$	5/2
R_3	$K(n-1, n, n, n, n+1, n+2)$	3
R_4	$K(n-1, n-1, n, n+1, n+1, n+2)$	4
R_5	$K(n-2, n, n, n+1, n+1, n+2)$	9/2
R_6	$K(n-1, n-1, n, n, n+2, n+2)$	6
R_7	$K(n-2, n, n, n, n+2, n+2)$	13/2
R_8	$K(n-1, n, n, n, n, n+3)$	9
R_9	$K(n-1, n-1, n, n, n+1, n+3)$	10
R_{10}	$K(n-2, n, n, n, n+1, n+3)$	21/2

Table 1: H_{t-2} and its θ-values

$R_0 = K(n, n, n, n, n, n+2)$ and $R_3 = K(n-1, n, n, n, n+1, n+2)$. Hence, by Lemma 2.2, we have $\theta(R) = 1$, $\theta(R_0) = 2$ and $\theta(R_3) = 3$.

Note that H_{t-1} is an improvement of H_{t-2} and it is not hard to see that $H_{t-2} \in \{R_i | i = 1, 2, \cdots, 10\}$, where R_i and $\theta(R_i)$ are shown in Table 1.

To complete the proof of the theorem, we need only determine all complete 6-partite graph G with $6n+2$ vertices such that $\theta(G) < 9/2$. By Lemma 2.3, $\theta(H_{t-3}) > 9/2$ for each H_{t-3} if $H_{t-2} \in \{R_i | i = 5, 6, \cdots, 10\}$. All graphs H_{t-3} and its θ-values are listed in Table 2 when $H_{t-2} \in \{R_i | i = 1, 2, 3, 4\}$.

By Lemma 2.3, $\theta(H_{t-4}) > 9/2$ for every H_{t-4} if $H_{t-3} \in \{M_i | 2 \le i \le 7\}$, one can easily obtain the following: If $H_{t-3} = M_1$, then $H_{t-4} \in \{M_8, M_9\}$, where $M_8 = K(n-3, n, n+1, n+1, n+1, n+2)$, $M_9 = K(n-4, n+1, n+1, n+1, n+1, n+2)$, and $\theta(M_8) = 25/4$, $\theta(M_9) = 65/8$. Hence, by Lemma 2.3, Table 1, Table 2 and the above arguments, we conclude that the theorem holds. $\square$

4 Chromatically closed 6-partite graphs

In this section, we obtained the $\chi-$closed of the families in $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5, n_6)$.

M_i	**Graphs** H_{t-3}	$\theta(M_i)$
M_1	$K(n-3,n+1,n+1,n+1,n+1,n+1)$	17/4
M_2	$K(n-2,n-1,n+1,n+1,n+1,n+2)$	11/2
M_3	$K(n-3,n,n+1,n+1,n+1,n+2)$	25/4
M_4	$K(n-1,n-1,n-1,n+1,n+2,n+2)$	7
M_5	$K(n-2,n-1,n,n+1,n+2,n+2)$	15/2
M_6	$K(n-1,n-1,n-1,n+1,n+1,n+3)$	11
M_7	$K(n-2,n-1,n,n+1,n+1,n+3)$	23/2

Table 2: H_{t-3} and its θ-values

Theorem 4.1 *If $n \geq s+2$, then the family of graphs $\mathcal{K}^{-s}(n,n,n,n,n+1,n+1)$ is χ-closed.*

Proof. Let $G = K(n,n,n,n,n+1,n+1)$ and $Z \in \mathcal{K}^{-s}(n,n,n,n,n+1,n+1)$. The 6-independent partition of G is certainly 6-independent partition of Z. So $\alpha(Z,6) \geq \alpha(G,6) = 1$. Let $H \sim Z$, then $\alpha(H,6) = \alpha(Z,6) \geq \alpha(G,6) = 1$. Let $\{A_1,A_2,A_3,A_4,A_5,A_6\}$ be a 6-independent partition of H, $|A_i| = t_i$, $i = 1,2,3,4,5,6$ and $F = K(t_1,t_2,t_3,t_4,t_5,t_6)$. Then there exist $S' \in E(F)$ such that $H = F - S'$. Let $q(G)$ be the number of edges in graph G. Since $q(H) = q(Z)$, therefore $s' = |S'| = q(F) - q(G) + s$.

From Lemma 2.4, we have

$$\alpha(Z,7) = \alpha(G,7) + \alpha'(Z), s \leq \alpha'(Z) \leq 2^s - 1, \qquad \text{and}$$
$$\alpha(H,7) = \alpha(F,7) + \alpha'(H), s' \leq \alpha'(H).$$

Thus $\alpha(H,7) - \alpha(Z,7) = \alpha(F,7) - \alpha(G,7) + \alpha'(H) - \alpha'(Z)$ and $\alpha(Z,7) = \alpha(H,7)$, so $\alpha(H,7) - \alpha(Z,7) = 0$.

If $F \neq G$, from Theorem 3.1, we have $\theta(F) - \theta(G) \geq 1$. So

$$\alpha(F,7) - \alpha(G,7) = (\theta(F) - \theta(G)) \cdot 2^{n-2} \geq 2^{n-2}.$$

Hence

$$\alpha(H,7) - \alpha(Z,7) \geq 2^{n-2} + \alpha'(H) - \alpha'(Z) \geq 2^{n-2} + 0 - (2^s - 1) \geq 1.$$

This is a contradiction. So $F = G$, $s = s'$. Thus, $H \in \mathcal{K}^{-s}(n, n, n, n, n+1, n+1)$. Therefore, $\mathcal{K}^{-s}(n, n, n, n, n+1, n+1)$ is χ-closed if $n \geq s+2$. The proof is now completed. $\square$

By using the similar proof of Theorem 4.1, we can obtain the following results.

Theorem 4.2 *If $n \geq s+5$, then the family of graphs $\mathcal{K}^{-s}(n-1, n, n, n+1, n+1, n+1)$ is χ-closed.*

Theorem 4.3 *If $n \geq s+3$, then the families of graphs $\mathcal{K}^{-s}(n, n, n, n, n, n+2)$ and $\mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1, n+1)$ are χ-closed.*

Theorem 4.4 *If $n \geq s+5$, then the family of graphs $\mathcal{K}^{-s}(n-2, n, n+1, n+1, n+1, n+1)$ is χ-closed.*

Theorem 4.5 *If $n \geq s+4$, then the family of graphs $\mathcal{K}^{-s}(n-1, n, n, n, n+1, n+2)$ is χ-closed.*

Theorem 4.6 *If $n \geq s+4$, then the family of graphs $\mathcal{K}^{-s}(n-1, n-1, n, n+1, n+1, n+2)$ is χ-closed.*

Theorem 4.7 *If $n \geq s+7$, then the family of graphs $\mathcal{K}^{-s}(n-3, n+1, n+1, n+1, n+1, n+1)$ is χ-closed.*

5 Chromatically unique 6-partite graphs

In this section, we first study the chromatically unique 6-partite graphs with $6n+2$ vertices and a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$.

Theorem 5.1 *If $n \geq s+2$, then the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1, n+1)$ are χ-unique for $(i, j) \in \{(1, 2), (1, 5), (5, 1), (5, 6)\}$.*

Proof. From Lemma 2.4 and Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1, n+1) = \{K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1, n+1) | (i, j) \in \{(1, 2), (1, 5), (5, 1), (5, 6)\}$ is χ-closed if $n \geq s + 2$. Note that

$t(K_{1,2}^{-K_{1,s}}(n, n, n, n, n+1, n+1)) = t(K(n, n, n, n, n+1, n+1)) - s(4n+2)$;
$t(K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1, n+1)) = t(K(n, n, n, n, n+1, n+1)) - s(4n+1)$ for $(i, j) \in \{(1, 5), (5, 1)\}$;
$t(K_{5,6}^{-K_{1,s}}(n, n, n, n, n+1, n+1)) = t(K(n, n, n, n, n+1, n+1)) - 4sn$.

By Lemmas 2.1 and 2.6, we conclude that $\sigma(K_{1,5}^{-K_{1,s}}(n, n, n, n, n+1, n+1)) \neq \sigma(K_{5,1}^{-K_{1,s}}(n, n, n, n, n+1, n+1))$. Hence, by Lemma 2.1, the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1, n+1)$ are χ-unique where $n \geq s + 2$ for $(i, j) \in \{(1, 2), (1, 5), (5, 1), (5, 6)\}$. The proof is now completed. $\square$

Similarly to the proof of Theorem 5.1, we can prove Theorems 5.2–5.5.

Theorem 5.2 *If $n \geq s + 3$, then the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n, n, n+2)$ are χ-unique for $(i, j) \in \{(1, 2), (1, 6), (6, 1)\}$.*

Theorem 5.3 *If $n \geq s + 3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n-1, n+1, n+1, n+1, n+1)$ are χ-unique for $(i, j) \in \{(1, 2), (1, 3), (3, 1), (3, 4)\}$.*

Theorem 5.4 *If $n \geq s + 5$, then the graphs $K_{i,j}^{-K_{1,s}}(n-2, n, n+1, n+1, n+1, n+1)$ are χ-unique for $(i, j) \in \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (3, 4)\}$.*

Theorem 5.5 *If $n \geq s + 7$, then the graphs $K_{i,j}^{-K_{1,s}}(n-3, n+1, n+1, n+1, n+1, n+1)$ are χ-unique for $(i, j) \in \{(1, 2), (2, 1), (2, 3)\}$.*

Theorem 5.6 *If $n \geq s+5$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)$ are χ-unique for $(i, j) \in \{(1, 2), (2, 1), (2, 4), (4, 2), (4, 5)\}$.*

Proof. Let $F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1) | (i, j) \in \{(1, 2), (2, 1), (2, 4), (4, 2), (4, 5)\}\}$ and $H \sim F$. By Theorem 4.2, $H \in \mathcal{K}^{-s}(n-1, n, n, n+1, n+1, n+1)$. Since

$$\alpha(H, 7) = \alpha(F, 7) = \alpha(K(n-1, n, n, n+1, n+1, n+1), 7) + 2^s - 1,$$

from Lemma 2.4, we know that $H \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1) | i \neq j,\ i, j = 1, 2, 3, 4, 5, 6\}$. It easy to see that $H \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+

$1, n+1)|i \neq j,\ i,j = 1,2,3,4,5,6\} = \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)|(i,j) \in \{(1,2),(2,1),(1,4),(4,1),(2,3),(2,4),(4,2),(4,5)\}\}$.

Now let's determine the number of triangles in H and F. Then we obtain that

$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)) = t(K(n-1, n, n, n+1, n+1, n+1)) - s(4n+3)$ for $(i,j) \in \{(1,2),(2,1)\}$,
$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)) = t(K(n-1, n, n, n+1, n+1, n+1)) - s(4n+2)$ for $(i,j) \in \{(1,4),(4,1),(2,3)\}$,
$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)) = t(K(n-1, n, n, n+1, n+1, n+1)) - s(4n+1)$ for $(i,j) \in \{(2,4),(4,2)\}$,
$t(K_{4,5}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)) = t(K(n-1, n, n, n+1, n+1, n+1)) - 4sn$.
Recalling

$$F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)|(i,j) \in \{(1,2),(2,1),(2,4),(4,2),(4,5)\}\}$$

and $t(H) = t(F)$, thus we have

$$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)|(i,j) \in \{(1,2),(2,1)\}\}$$

or

$$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)|(i,j) \in \{(2,4),(4,2)\}\}.$$

It follows from Lemmas 2.1 and 2.6 that

$$P(K_{1,2}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1), \lambda) \neq P(K_{2,1}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1), \lambda);$$
$$P(K_{2,4}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1), \lambda) \neq P(K_{4,2}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1), \lambda).$$

Hence, by Lemma 2.1, we conclude that the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)$ are χ-unique where $n \geq s+5$ for each $(i,j) \in \{(1,2),(2,1),(2,4),(4,2),(4,5)\}$.
□

Similarly to the proof of Theorem 5.6, we can prove Theorems 5.7 and 5.8.

Theorem 5.7 *If $n \geq s+4$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1, n+2)$ are χ-unique for $(i,j) \in \{(1,2),(2,1),(2,6),(6,2),(5,6),(6,5)\}$.*

Theorem 5.8 *If $n \geq s+4$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n-1, n, n+1, n+1, n+2)$ are χ-unique for $(i,j) \in \{(1,2),(1,3),(3,1),(1,4),(4,1),(4,5)\}$.*

Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 6-partite graph with partite sets $A_i (i = 1, 2, \cdots, 6)$ such that $|A_i| = n_i$. Denote $\langle A_i \cup A_j \rangle$ be the subgraph of G induced by $A_i \cup A_j$, where $i \neq j$ and $i, j \in \{1,2,3,4,5,6\}$. Let $K_{i,j}^{-sK_2}(n_1, n_2, n_3, n_4, n_5, n_6)$ denotes the graph obtained from $K(n_1, n_2, n_3, n_4, n_5, n_6)$ by deleting a set of s edges that forms a matching in $\langle A_i \cup A_j \rangle$.

We now investigate the chromatically unique 6-partite graphs with $6n+2$ vertices and a set S of s edges deleted where the deleted edges induce a matching sK_2.

Theorem 5.9 *If $n \geq s+3$, then the graphs $K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1, n+1)$ are χ-unique.*

Proof. Let $F \sim K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1, n+1)$. It is sufficient to prove that $F = K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1, n+1)$. By Theorem 4.3 and Lemma 2.4, we have $F \in \mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1, n+1)$ and $\alpha'(F) = s$. Let $F = G - S$ where $G = K(n-1, n-1, n+1, n+1, n+1, n+1)$. Next we consider the number of triangles in F. Let $e \in S$ and $t(e)$ be the number of triangles in G containing the edge e. It is easy to see that $t(e) \leq 4n+4$. As $n-1 \leq n-1 < n+1 \leq n+1 \leq n+1 \leq n+1$, we know that $t(e) = 4n+4$ if and only if e is an edge in the subgraph $\langle A_1 \cup A_2 \rangle$ in G. So we have

$$t(F) \geq t(G) - \sum_{i=1}^{s} t(e) \geq t(G) - s(4n+4);$$

and the equality holds if and only if each e edge in S is an edge of the subgraph $\langle A_1 \cup A_2 \rangle$ in G.

Note that $t(F) = t(G) - s(4n+4)$ and $\alpha'(F) = s$. By Lemma 2.4, we know that $F = K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1, n+1)$. This completes the proof. $\square$

Similarly to the proof of Theorem 5.9, we can prove Theorems 5.10 and 5.11.

Theorem 5.10 *If $n \geq s+5$, then the graphs $K_{1,2}^{-sK_2}(n-2, n, n+1, n+1, n+1, n+1)$ are χ-unique.*

Theorem 5.11 *If $n \geq s+4$, then the graphs $K_{1,2}^{-sK_2}(n-1, n-1, n, n+1, n+1, n+2)$ are χ-unique.*

We end this paper with the following problems:

[1.] Study the chromaticity of the following graphs: (i) $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+1)$ where $n \geq s+5$ for each $(i,j) \in \{(1,4), (4,1), (2,3)\}$, (ii) $K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1, n+2)$ where $n \geq s+4$ for each $(i,j) \in \{(1,5), (5,1), (1,6), (6,1), (2,3), (2,5), (5,2)\}$ and (iii) $K_{i,j}^{-K_{1,s}}(n-1, n-1, n, n+1, n+1, n+2)$ where $n \geq s+4$ for each $(i,j) \in \{(1,6), (6,1), (3,4), (4,3), (3,6), (6,3), (4,6), (6,4)\}$.

[2.] Study the chromaticity of the following graphs: (i) $K_{1,2}^{-sK_2}(n, n, n, n, n+1, n+1)$ where $n \geq s+2$, (ii) $K_{1,2}^{-sK_2}(n-1, n, n, n+1, n+1, n+1)$ where $n \geq s+5$, (iii) $K_{1,2}^{-sK_2}(n, n, n, n, n, n+2)$ where $n \geq s+3$, (iv) $K_{1,2}^{-sK_2}(n-1, n, n, n, n+1, n+2)$ where $n \geq s+4$ and (v) $K_{1,2}^{-sK_2}(n-3, n+1, n+1, n+1, n+1, n+1)$ where $n \geq s+7$.

References

1. J.A. Bondy and U.S.R. Murty, Graph Theory with Application, London: Macmillan, 1976.

2. F. Brenti, Expansions of chromatic polynomials and log-concavity, Trans. Amer. Math. Soc. 332(2) (1992), 729-756.

3. X.E. Chen, Chromaticity on 6-partite graphs with 6n+5 vertices, Pure and Applied Math. 21(2) (2005), 134–141.

4. G.L. Chia, B.H. Goh and K.M. Koh, The chromaticity of some families of complete tripartite graphs, Scientia, Series A: Math.Sci. 2 (1988), 27–37.

5. F.M. Dong, K.M. Koh and K.L. Teo, Sharp bounds for the number of 3-independent partition and chromaticity of bipartite graphs, J. Graph Theory 37 (2001), 48–77.

6. F.M. Dong, K.M. Koh and K.L. Teo, Chromatic Polynomials and Chromaticity of Graphs, Word Scientific, 2005.

7. K.M. Koh and K.L. Teo, The search for chromatically unique graphs, Graphs Combin. 6 (1990), 259–285.

8. K.M. Koh and K.L. Teo, The search for chromatically unique graphs II, Discrete Math. 172 (1997), 59–78.

9. G.C. Lau and Y.H. Peng, Chromaticity of complete tripartite graphs with certain star and matching deleted, Ars Comb., accepted.

10. G.C. Lau and Y.H. Peng, Chromaticity of complete 4-partite graphs with certain star and matching deleted, Appl. Anal. Discrete Math. 4 (2010), 253–268.

11. R.C. Read and W.T. Tutte, Chromatic Polynomials, In: L.W. Beineke and R.J. Wilson, eds. Selected Topics in Graph Theory (II), New York: Academic Press, (1988),15-42.

12. H. Roslan, A. Sh. Ameen, Y. H. Peng and H.X. Zhao, Chromaticity of complete 5-partite graphs with certain star and matching deleted, submitted.

13. H. Roslan, A. Sh. Ameen, Y. H. Peng and H.X. Zhao, Classification of complete 5-partite graphs and chromaticity of 5-partite graphs with 5n+2 vertices, Far East Journal of Mathematical Sciences, Vol. 42, No. 1 (2010), 59–72.

14. H. Roslan, A. Sh. Ameen, Y. H. Peng and H.X. Zhao, Chromaticity of complete 5-partite graphs with certain edges deleted, Journal of Applied Mathematics and Computing 35 (2011), 507-516.

15. H. Roslan, A. Sh. Ameen, Y. H. Peng and H.X. Zhao, Chromaticity of complete 6-partite graphs with certain star and matching deleted deleted, Bulletin of Malaysian Math. Society, accepted.

16. H. Roslan, A. Sh. Ameen, Y. H. Peng and H.X. Zhao, Chromaticity of complete 6-partite graphs with certain star and matching deleted deleted II, Acta Mathematicae Appl. Sinica (English Series), accepted.

17. H.X. Zhao, R.Y. Liu and S.G. Zhang, Classification of Complete 5-Partite Graphs and Chromaticity of 5-Partite Graphs With 5n Vertices, Appl. Math. J.Chinese Univ. Ser. B. 19(1) (2004), 116–124.

18. H.X. Zhao, On the chromaticity of 5-partite graphs with 5n+4 vertices, J. of Lanzhou Univ. (Natural Sciences), 40(3) (2004), 12–16 (in Chinese).

19. H.X. Zhao, Chromaticity and adjoint polynomials of graphs, Ph.D. Thesis University of Twente, (2005) Netherland.

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