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CLASSIFICATION OF COMPLETE 5-PARTITE GRAPHS AND CHROMATICITY OF 5-PARTITE GRAPHS WITH $5n + 2$ VERTICES

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Abstract

Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. Then two graphs G and H are said to be chromatically equivalent, denoted as $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H \mid H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 5-partite graphs with $5n + 2$ vertices according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As

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a by-product, many new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

1. Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G, \lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply χ-*equivalent*), symbolically, $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply χ-*unique*) if $H \cong G$ whenever $H \sim G$, i.e., $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be χ-*closed*. Many families of χ-unique graphs are known (see [5, 6 and 7]).

For a graph G, let $V(G)$, $E(G)$ and $t(G)$ be the vertex set, edge set and number of triangles in G, respectively. Let S be a set of s edges in G. Let $G - S$ (or $G - s$) be the graph obtained from G by deleting all edges in S, and by $\langle S \rangle$ the graph induced by S. Let $K(n_1, n_2, ..., n_t)$ be a complete t-partite graph. Then we denote by $\mathcal{K}^{-s}(n_1, n_2, ..., n_t)$ the family of graphs which is obtained from $K(n_1, n_2, ..., n_t)$ by deleting a set S of some s edges.

In [3, 4, 6, 7, 12], we can find many results on the chromatic uniqueness of certain families of complete t-partite graphs $(t = 2, 3, 4)$. In [10, 11], Zhao et al. obtained many families of χ-unique graphs by deleting the edges of a star or matching from a complete 5-partite graph with $5n$ and $5n + 4$ vertices. By using similar approach, Roslan et al. [9] obtained many families of χ-unique graphs by deleting the edges of a star or matching from a complete 5-partite graph with $5n + 1$ vertices. As a continuation, this paper studies the chromaticity of the graphs which are obtained by deleting the edges of a star or matching from complete 5-partite graphs with $5n + 2$ vertices.

Let G be a complete 5-partite graph with $5n + 2$ vertices. In this paper, we characterize certain complete 5-partite graphs with $5n + 2$ vertices according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

2. Some Lemmas and Notations

For a graph G and a positive integer k, a partition $\{A_1, A_2, ..., A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of G. Then we have $P(G, \lambda) = \sum_{r=1}^{p} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - r + 1)$ (see [8]). Therefore, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, ...,$ if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G, x) = \sum_{r=1}^{p} \alpha(G, r)x^r$ is called the *σ-polynomial* of G (see [2]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs G and H.

For disjoint graphs G and H, $G \cup H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer to [1].

Lemma 2.1 (Brenti [2], Koh and Teo [6])**.** *Let G and H be two disjoint graphs. Then*

(1) $|V(G)| = |V(H)|, |E(G)| = |E(H)|, t(G) = t(H)$ *and* $\alpha(G, r) = \alpha(H, r)$ *for* $r = 1, 2, 3, ..., p$ *if* $G \sim H$;

(2) $\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x)$.

Lemma 2.2 (Brenti [2])**.** *Let* $G = K(n_1, n_2, n_3, ..., n_t)$ *and* $\sigma(G, x) = \sum_{r \geq 1} \alpha(G, r)x^r$. *Then* $\alpha(G, r) = 0$ *for* $1 \leq r \leq t - 1$, $\alpha(G, t) = 1$ *and* $\alpha(G, t + 1) = \sum_{i=1}^{t} 2^{n_i - 1} - t$.

Let $n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5$ be positive integers and $H = K(n_1, n_2, n_3, n_4, n_5)$. If there exist $i, j \in \{1, 2, 3, 4, 5\}$ such that $i < j$, $n_j - n_i \geq 2$, let $k_i = n_i + 1$, $k_j = n_j - 1$, $k_l = n_l$, $l \in \{1, 2, 3, 4, 5\} - \{i, j\}$ and $H' = K(k_1, k_2, k_3, k_4, k_5)$, then H' is called an *improvement* of H and H is called the *withdrawing* of H'.

Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph with $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 2$ vertices. Then we define θ-value of G as $\theta(G) = [\alpha(G, 6) - 3 \cdot 2^n - 2^{n-1} + 5]/2^{n-2}$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H, 6) - \alpha(G, 6)$. Clearly, $\alpha'(H) \geq 0$.

Lemma 2.3. $\alpha(H, 6) - \alpha(H', 6) \geq 2^{n_i - 1}$.

Proof.

$$\begin{aligned}\alpha(H, 6) - \alpha(H', 6) &= 2^{n_i - 1} + 2^{n_j - 1} - 2^{k_i - 1} - 2^{k_j - 1} \\ &= 2^{n_i - 1} + 2^{n_j - 1} - 2^{n_i} - 2^{n_j - 2} \\ &\geq 2^{n_i - 1}.\end{aligned}$$ □

Lemma 2.4. $\theta(H) - \theta(H') > 0$.

Proof. It follows directly from Lemma 2.3 and the definition of $\theta(G)$. □

Lemma 2.5 (Zhao et al. [10])**.** *Let* $G = K(n_1, n_2, n_3, n_4, n_5)$. *Suppose that* $-\{n_i \mid i = 1, 2, 3, 4, 5\} \geq s + 1 \geq 1$ *and* $H = G - S$, *where S is a set of some s edges of G, then*

$$s \leq \alpha'(H) = \alpha(H, 6) - \alpha(G, 6) \leq 2^s - 1,$$

$\alpha'(H) = s$ *iff the set of end-vertices of any* $r \geq 2$ *edges in S is not independent in H, and* $\alpha'(H) = 2^s - 1$ *iff S induces a star* $K_{1,s}$ *and all vertices of* $K_{1,s}$ *other than its center belong to the same* A_i.

Let $K(A_1, A_2)$ be a complete bipartite graph with partite sets A_1 and A_2. Then we denote by $K^{-K_{1,s}}(A_i, A_j)$ the graph obtained from $K(A_i, A_j)$ by deleting s edges that induce a star with its center in A_i. Note that $K^{-K_{1,s}}(A_i, A_j) \neq K^{-K_{1,s}}(A_j, A_i)$ if $|A_i| \neq |A_j|$ for $i \neq j$ (see [4]).

Lemma 2.6 (Dong et al. [4])**.** *Let* $K(n_1, n_2)$ *be a complete bipartite graph with partite sets* A_1 *and* A_2 *such that* $|A_i| = n_i$ *for* $i = 1, 2$. *If* $\min\{n_1, n_2\} \geq s + 2$, *then every* $K^{-K_{1,s}}(A_i, A_j)$ *is χ-unique, where* $i \neq j$ *and* $i, j = 1, 2$.

Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph with partite sets A_i $(i = 1, 2, ..., 5)$ such that $|A_i| = n_i$. Let $\langle A_i \cup A_j \rangle$ be the subgraph of G induced by $A_i \cup A_j$, where $i \neq j$ and $i, j \in \{1, 2, 3, 4, 5\}$. By $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5)$, we denote the graph obtained from $K(n_1, n_2, n_3, n_4, n_5)$ by deleting a set of s edges that induces a $K_{1,s}$ with its center in A_i and all its end-vertices are in A_j. Note that

$$K_{i,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5) = K_{j,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5)$$

and $K_{l,i}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5) = K_{l,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5)$ for $n_i = n_j$ and $l \neq i, j$.

Lemma 2.7 (Zhao et al. [10])**.** *Suppose that* $\min\{n_1, n_2, n_3, n_4, n_5\} \geq s + 2$ *and* $n_i \neq n_j$ *for* $i \neq j$, $i, j = 1, 2, 3, 4, 5$, *then*

$$P(K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5), \lambda) \neq P(K_{j,i}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5), \lambda).$$

3. Classification

In this section, we shall characterize certain complete 5-partite graph $G = K(n_1, n_2, n_3, n_4, n_5)$ according to the number of 6-independent partitions of G where $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 2$, $n \geq 1$.

Theorem 3.1. *Let* $G = K(n_1, n_2, n_3, n_4, n_5)$ *be a complete* 5*-partite graph such that* $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 2$, $n \geq 1$. *Define*

$$\theta(G) = [\alpha(G, 6) - 3 \cdot 2^n - 2^{n-1} + 5]/2^{n-2}.$$

Then

(i) $\theta(G) \geq 0$;

(ii) $\theta(G) = 0$ *if and only if* $G = K(n, n, n, n+1, n+1)$;

(iii) $\theta(G) = 1$ *if and only if* $G = K(n-1, n, n+1, n+1, n+1)$;

(iv) $\theta(G) = 2$ *if and only if* $G = K(n, n, n, n, n+2)$;

(v) $\theta(G) = 5/2$ *if and only if* $G = K(n-2, n+1, n+1, n+1, n+1)$;

(vi) $\theta(G) = 3$ *if and only if* $G = K(n-1, n, n, n+1, n+2)$;

(vii) $\theta(G) = 4$ *if and only if* $G = K(n-1, n-1, n+1, n+1, n+2)$;

(viii) $\theta(G) \geq 9/2$ *if and only if G is not a graph appeared in* (ii)-(vii).

Proof. We construct a table, namely, Table 1, for the θ-values of various complete 5-partite graphs with $5n+2$ vertices in order to complete the proof of this theorem.

For every complete 5-partite graph G with $5n+2$ vertices, if $G \neq K(n, n, n, n+1, n+1)$, then we obtain $K(n, n, n, n+1, n+1)$ by using several improving operations from G. Thus (i) and (ii) are valid from Lemma 2.4 and Table 1.

The withdrawing of G_1 is G_2, G_3 or G_5. From Table 1, we know that $\theta(G_2) = 1$, $\theta(G_3) = 2$ and $\theta(G_5) = 3$, so (iii), (iv) and (vi) are valid from Lemma 2.4.

The withdrawing of G_2 is G_4, G_6 or G_7 and $\theta(G_4) = 5/2$, $\theta(G_6) = 4$, $\theta(G_7) = 9/2$, so (v) and (vii) are valid.

Table 1. G_i and its θ-values

G_i	$\theta(G_i)$
$G_1 = K(n, n, n, n+1, n+1)$	0
$G_2 = K(n-1, n, n+1, n+1, n+1)$	1
$G_3 = K(n, n, n, n, n+2)$	2
$G_4 = K(n-2, n+1, n+1, n+1, n+1)$	5/2
$G_5 = K(n-1, n, n, n+1, n+2)$	3
$G_6 = K(n-1, n-1, n+1, n+1, n+2)$	4
$G_7 = K(n-2, n, n+1, n+1, n+2)$	9/2
$G_8 = K(n-1, n, n, n, n+3)$	9
$G_9 = K(n-3, n+1, n+1, n+1, n+2)$	25/4
$G_{10} = K(n-1, n-1, n, n+2, n+2)$	6

$G_{11} = K(n-2, n, n, n+2, n+2)$	$13/2$
$G_{12} = K(n-1, n-1, n, n+1, n+3)$	10
$G_{13} = K(n-2, n, n, n+1, n+3)$	$21/2$
$G_{14} = K(n-2, n-1, n+1, n+2, n+2)$	$15/2$
$G_{15} = K(n-2, n-1, n+1, n+1, n+3)$	$23/2$

The withdrawing of G_3, G_4, G_5, G_6 are G_5 or G_8; G_7 or G_9; G_6, G_7, G_{10}, G_{11}, G_{12} or G_{13}; G_7, G_{10}, G_{12}, G_{14} or G_{15}, respectively, and the θ-values of $G_5, G_6, G_7, G_8, G_9, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}$ are $3, 4, 9/2, 9, 25/4, 6, 13/2, 10, 21/2, 15/2$ and $23/2$, respectively, so (viii) is valid. This completes the proof. □

4. Chromatically Closed 5-partite Graphs

In this section, we obtain several χ-closed families of graphs in $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5)$.

Theorem 4.1. *If* $n \geq s+2$, *then the family of graphs* $\mathcal{K}^{-s}(n, n, n, n+1, n+1)$ *is* χ*-closed*.

Proof. Let $G = K(n, n, n, n+1, n+1)$ and $Z \in \mathcal{K}^{-s}(n, n, n, n+1, n+1)$. Then the 5-independent partition of G is certainly 5-independent partition of Z. So $\alpha(Z, 5) \geq \alpha(G, 5) = 1$. Let $H \sim Z$. Then $\alpha(H, 5) = \alpha(Z, 5) \geq \alpha(G, 5) = 1$. Let $\{A_1, A_2, A_3, A_4, A_5\}$ be 5-independent partition of H, $|A_i| = t_i$, $i = 1, 2, 3, 4, 5$ and $F = K(t_1, t_2, t_3, t_4, t_5)$. Then there exists $S' \in E(F)$ such that $H = F - S'$. Let $q(G)$ be the number of edges in graph G. Since $q(H) = q(Z)$, therefore $s' = |S'| = q(F) - q(G) + s$.

From Lemma 2.5, we have

$$\alpha(Z, 6) = \alpha(G, 6) + \alpha'(Z), \quad s \leq \alpha'(Z) \leq 2^s - 1,$$

and

$$\alpha(H, 6) = \alpha(F, 6) + \alpha'(H), \quad s' \leq \alpha'(H) \leq 2^{s'} - 1.$$

Thus $\alpha(H, 6) - \alpha(Z, 6) = \alpha(F, 6) - \alpha(G, 6) + \alpha'(H) - \alpha'(Z)$ and $\alpha(Z, 6) = \alpha(H, 6)$, so $\alpha(H, 6) - \alpha(Z, 6) = 0$.

If $F \neq G$, from Theorem 3.1, then we have $\theta(F) - \theta(G) \geq 1$. So

$$\alpha(F, 6) - \alpha(G, 6) = (\theta(F) - \theta(G)) \cdot 2^{n-2} \geq 2^{n-2}.$$

Hence

$$\alpha(H, 6) - \alpha(Z, 6) \geq 2^{n-2} + \alpha'(H) - \alpha'(Z) \geq 2^{n-2} + 0 - (2^{s} - 1) \geq 1.$$

This is a contradiction. So $F = G$, $s = s'$. Thus, $H \in \mathcal{K}^{-s}(n, n, n, n+1, n+1)$. Therefore, $\mathcal{K}^{-s}(n, n, n, n+1, n+1)$ is χ-closed if $n \geq s+2$. The proof is now completed. □

By using the similar proof of Theorem 4.1, we can obtain the following results.

Theorem 4.2. *If* $n \geq s+3$, *then the family of graphs* $\mathcal{K}^{-s}(n-1, n, n+1, n+1, n+1)$ *is* χ*-closed*.

Theorem 4.3. *If* $n \geq s+2$, *then the family of graphs* $\mathcal{K}^{-s}(n, n, n, n, n+2)$ *is* χ*-closed*.

Theorem 4.4. *If* $n \geq s+4$, *then the family of graphs* $\mathcal{K}^{-s}(n-2, n+1, n+1, n+1, n+1)$ *is* χ*-closed*.

Theorem 4.5. *If* $n \geq s+3$, *then the family of graphs* $\mathcal{K}^{-s}(n-1, n, n, n+1, n+2)$ *is* χ*-closed*.

Theorem 4.6. *If* $n \geq s+4$, *then the family of graphs* $\mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+2)$ *is* χ*-closed*.

5. Chromatically Unique 5-partite Graphs

In this section, we first study the chromatically unique 5-partite graphs with $5n+2$ vertices and a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$.

Theorem 5.1. *If* $n \geq s+2$, *then the graphs* $K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1)$ *are* χ*-unique for each* $(i, j) \in \{(1, 2), (1, 4), (4, 1), (4, 5)\}$.

Proof. From Theorem 4.1, we know that $K^{-s}(n, n, n, n+1, n+1)$ is χ-closed if $n \geq s+2$. Comparing the number of 6-independent partitions of the graphs in $K^{-s}(n, n, n, n+1, n+1)$ and by using Lemma 2.5, we have that $K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1) = \{K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1) | (i, j) \in \{(1, 2), (1, 4), (4, 1), (4, 5)\}\}$ is χ-closed.

Note that

$$t(K_{1,2}^{-K_{1,s}}(n, n, n, n+1, n+1)) = t(K(n, n, n, n+1, n+1)) - s(3n+2),$$

$$t(K_{4,5}^{-K_{1,s}}(n, n, n, n+1, n+1)) = t(K(n, n, n, n+1, n+1)) - 3sn,$$

$$t(K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1)) = t(K(n, n, n, n+1, n+1)) - s(3n+1) \text{ for}$$

$$(i, j) \in \{(1, 4), (4, 1)\}.$$

From Lemma 2.7, we have

$$P(K_{1,4}^{-K_{1,s}}(n, n, n, n+1, n+1), \lambda) \neq P(K_{4,1}^{-K_{1,s}}(n, n, n, n+1, n+1), \lambda).$$

Hence, by Lemma 2.1, we conclude that the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1)$ are χ-unique where $n \geq s+2$ for each $(i, j) \in \{(1, 2), (1, 4), (4, 1), (4, 5)\}$. □

Theorem 5.2. *If* $n \geq s+3$, *then the graphs* $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2)$ *are* χ*-unique for each* $(i, j) \in \{(1, 2), (2, 1), (2, 5), (5, 2), (4, 5), (5, 4)\}$.

Proof. Let

$$F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2) | (i, j)$$
$$= \{(1, 2), (2, 1), (2, 5), (5, 2), (4, 5), (5, 4)\}\}$$

and $H \sim F$. Then by Theorem 4.5, $H \in \mathcal{K}^{-s}(n-1, n, n, n+1, n+2)$. Since

$$\alpha(H, 6) = \alpha(F, 6) = \alpha(K(n-1, n, n, n+1, n+2), 6) + 2^s - 1,$$

from Lemma 4.1, we know that

$$H \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2) \mid i \neq j, i, j = 1, 2, 3, 4, 5\}.$$

It is easy to see that

$$H \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2) \mid i \neq j, i, j = 1, 2, 3, 4, 5\}$$

$$= \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2) \mid (i, j)$$

$$\in \{(1, 2), (2, 1), (1, 4), (4, 1), (1, 5), (5, 1),$$

$$(2, 3), (2, 4), (4, 2), (2, 5), (5, 2), (4, 5), (5, 4)\}\}.$$

Now let us determine the numbers of triangles in H and F. Then we obtain that

$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2)) = t(K(n-1, n, n, n+1, n+2)) - s(3n+3)$ for $(i, j) \in \{(1, 2), (2, 1)\}$,

$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2)) = t(K(n-1, n, n, n+1, n+2)) - s(3n+2)$ for $(i, j) \in \{(1, 4), (4, 1), (2, 3)\}$,

$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2)) = t(K(n-1, n, n, n+1, n+2)) - s(3n+1)$ for $(i, j) \in \{(1, 5), (5, 1), (2, 4), (4, 2)\}$,

$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2)) = t(K(n-1, n, n, n+1, n+2)) - 3sn$ for $(i, j) \in \{(2, 5), (5, 2)\}$,

$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2)) = t(K(n-1, n, n, n+1, n+2)) - s(3n-1)$ for $(i, j) \in \{(4, 5), (5, 4)\}$.

Recalling

$$F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2) \mid (i, j)$$

$$\in \{(1, 2), (2, 1), (2, 5), (5, 2), (4, 5), (5, 4)\}\}$$

and $t(H) = t(F)$, thus we have

$$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2) \mid (i, j) \in \{(1, 2), (2, 1)\}\}$$

or

$$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2) \mid (i, j) \in \{(2, 5), (5, 2)\}\}$$

or

$$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2) \mid (i, j) \in \{(4, 5), (5, 4)\}\}.$$

It follows from Lemma 2.7 that

$$P(K_{1,2}^{-K_{1,s}}(n-1, n, n, n+1, n+2), \lambda) \neq P(K_{2,1}^{-K_{1,s}}(n-1, n, n, n+1, n+2), \lambda);$$

$$P(K_{2,5}^{-K_{1,s}}(n-1, n, n, n+1, n+2), \lambda) \neq P(K_{5,2}^{-K_{1,s}}(n-1, n, n, n+1, n+2), \lambda);$$

$$P(K_{4,5}^{-K_{1,s}}(n-1, n, n, n+1, n+2), \lambda) \neq P(K_{5,4}^{-K_{1,s}}(n-1, n, n, n+1, n+2), \lambda).$$

Hence, by Lemma 2.1, we conclude that the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2)$ are χ-unique, where $n \geq s+3$ for each $(i, j) \in \{(1, 2), (2, 1), (2, 5), (5, 2), (4, 5), (5, 4)\}$. □

Similar to the proofs of Theorems 5.1 and 5.2, we can prove Theorems 5.3, 5.4, 5.5 and 5.6.

Theorem 5.3. *If* $n \geq s+3$, *then the graphs* $K_{i,j}^{-K_{1,s}}(n-1, n, n+1, n+1, n+1)$ *are* χ*-unique for each* $(i, j) \in \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (3, 4)\}$.

Theorem 5.4. *If* $n \geq s+2$, *then the graphs* $K_{i,j}^{-K_{1,s}}(n, n, n, n, n+2)$ *are* χ*-unique for each* $(i, j) \in \{(1, 2), (1, 5), (5, 1)\}$.

Theorem 5.5. *If* $n \geq s+4$, *then the graphs*

$$K_{i,j}^{-K_{1,s}}(n-2, n+1, n+1, n+1, n+1)$$

are χ*-unique for each* $(i, j) \in \{(1, 2), (2, 1), (2, 3)\}$.

Theorem 5.6. *If* $n \geq s + 3,$ *then the graphs*

$$K_{i,\, j}^{-K_{1,\, s}}(n-1,\ n-1,\ n+1,\ n+1,\ n+2)$$

are χ*-unique for each*

$$(i,\ j) \in \{(1,\ 2),\ (1,\ 3),\ (3,\ 1),\ (1,\ 5),\ (5,\ 1),\ (3,\ 4),\ (3,\ 5),\ (5,\ 3)\}.$$

Let $K_{i,\, j}^{-sK_2}(n_1,\ n_2,\ n_3,\ n_4,\ n_5)$ denote the graph obtained from $K(n_1,\ n_2,\ n_3,\ n_4,\ n_5)$ by deleting a set of s edges that forms a matching in $\langle A_i \cup A_j \rangle$. We now investigate the chromatically unique 5-partite graphs with $5n + 2$ vertices and a set S of s edges deleted where the deleted edges induce a matching sK_2.

Theorem 5.7. *If* $n \geq s + 3,$ *then the graphs* $K_{1,\, 2}^{-sK_2}(n-1,\ n,\ n+1,\ n+1,\ n+1)$ *are* χ*-unique*.

Proof. Let $F \sim K_{1,\, 2}^{-sK_2}(n-1,\ n,\ n+1,\ n+1,\ n+1)$. Then it is sufficient to prove that

$$F = K_{1,\, 2}^{-sK_2}(n-1,\ n,\ n+1,\ n+1,\ n+1).$$

By Theorem 4.2 and Lemma 2.5,

$$F \in \mathcal{K}^{-s}(n-1,\ n,\ n+1,\ n+1,\ n+1)$$

and $\alpha'(F) = s.$ Let $F = G - S,$ where $G = K(n-1,\ n,\ n+1,\ n+1,\ n+1).$ Next, we consider the number of triangles in F. Let $e_i \in S$ and $t(e_i)$ be the number of triangles in G containing the edge e_i for each $i = 1,\ 2,\ \ldots,\ s.$ Then it is easy to see that $t(e_i) \leq 3n + 3.$ As $n - 1 < n < n + 1 \leq n + 1 \leq n + 1,$ we know that $t(e_i) = 3n + 3$ if and only if e_i is an edge in the subgraph $\langle A_1 \cup A_2 \rangle$ in G. So we have

$$t(F) \geq t - \sum_{i=1}^{s} t(e_i) \geq t - s(3n + 3);$$

and the equality holds if and only if each edge in S is an edge of the subgraph $\langle A_1 \cup A_2 \rangle$ in G. It follows from Lemmas 2.1 and 2.5 that

$$\alpha(F,\ 6) = \alpha(K(n-1,\ n,\ n+1,\ n+1,\ n+1),\ 6) + s$$

and the set of end-vertices of any $r \geq 2$ edges in S is not independent in F. Therefore, S induces a matching of s edges in $\langle A_1 \cup A_2 \rangle$ and

$$F = K_{1,2}^{-sK_2}(n-1, n, n+1, n+1, n+1).$$

This completes the proof. □

Similar to the proof of Theorem 5.7, we can prove Theorem 5.8.

Theorem 5.8. *If* $n \geq s+4$, *then the graphs*

$$K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+2)$$

are χ*-unique*.

We end this paper with the following two problems:

1. Study the chromaticity of the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+2)$ for each

$$(i, j) \in \{(1, 4), (4, 1), (2, 3), (1, 5), (5, 1), (2, 4), (4, 2)\}.$$

2. Study the chromaticity of the graphs

$$K_{1,2}^{-sK_2}(n, n, n, n+1, n+1), \quad K_{1,2}^{-sK_2}(n, n, n, n, n+2),$$

$$K_{1,2}^{-sK_2}(n-2, n+1, n+1, n+1, n+1)$$

and

$$K_{1,2}^{-sK_2}(n-1, n, n, n+1, n+2).$$

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