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Chromaticity of Complete 6-Partite Graphs with Certain Star or Matching Deleted

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Abstract. Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. We write $[G] = \{H|H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 6-partite graphs with $6n$ vertices according to the number of 7-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 6-partite graphs with certain star or matching deleted are obtained.

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1. Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G,\lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply χ-equivalent), symbolically $G \sim H$, if $P(G,l) = P(H,l)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply χ-unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be χ-*closed*. Many families of χ-unique graphs are known (see [6, 7, 8]).

For a graph G, let $V(G)$, $E(G)$ and $t(G)$ be the vertex set, edge set and number of triangles in G, respectively. Let S be a set of s edges in G. Let $G-S$ (or $G-s$) be the graph obtained from G by deleting all edges in S, and by $\langle S\rangle$ the graph

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induced by S. Let $K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph. We denote by $\mathcal{K}^{-s}(n_1, n_2, \cdots, n_t)$ the family of graphs which are obtained from $K(n_1, n_2, \cdots, n_t)$ by deleting a set S of some s edges.

In [4, 5, 7–10, 12–18], one can find many results on the chromatic uniqueness of certain families of complete t-partite graphs ($t = 2, 3, 4, 5$). However, there are very few 6-partite graphs known to be χ-unique, see [3].

In [3], Chen obtained many families of χ-unique graphs which are obtained by deleting the edges of a star or matching from a complete 6-partite graph with $6n+5$ vertices. A natural extension is to study the chromaticity of the graphs obtained by deleting the edges of a star or matching from a complete partite graph with $6n+i$ vertices, where $0 \le i \le 4$. Thus, the aim of this paper is to study the chromaticity of the graphs which are obtained by deleting the edges of a star or matching from a complete 6-partite graph with $6n$ vertices.

Let G be a complete 6-partite graph with $6n$ vertices. In this paper, we characterize certain complete 6-partite graphs with $6n$ vertices according to the number of 7-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 6-partite graphs with certain star or matching deleted are obtained.

2. Some lemmas and notations

For a graph G and a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is an independent set of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of G. Then, we have $P(G, \lambda) = \sum_{r=1}^{p} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ (see [11]). Therefore, $\alpha(G, r) = \alpha(H, r)$ for each $r = 1, 2, \cdots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G, x) = \sum_{r=1}^{p} \alpha(G, r)x^r$ is called the σ-polynomial of G (see [2]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs G and H.

For disjoint graphs G and H, $G \cup H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer [1].

Lemma 2.1. [2, 7] *Let G and H be two disjoint graphs. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ *and* $\alpha(G, r) = \alpha(H, r)$ *for* $r = 1, 2, 3, \cdots, p$ *if* $G \sim H$;

(2) $\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x)$.

Lemma 2.2. [2] *Let* $G = K(n_1, n_2, n_3, \cdots, n_t)$ *and* $\sigma(G, x) = \sum_{r\ge1} \alpha(G, r)x^r$. *Then* $\alpha(G, r) = 0$ *for* $1 \le r \le t-1$, $\alpha(G, t) = 1$ *and* $\alpha(G, t+1) = \sum_{i=1}^{t} 2^{n_i-1} - t$.

Let $x_1 \le x_2 \le x_3 \le x_4 \le x_5 \le x_6$ be positive integers and $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. If there are two elements x_{i_1} and x_{i_2} in $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ such that $x_{i_2} - x_{i_1} \ge 2$, then $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ is called an *improvement* of $H = K(x_1, x_2, x_3, x_4, x_5, x_6)$.

Lemma 2.3. [3] *Suppose* $x_1 \le x_2 \le x_3 \le x_4 \le x_5 \le x_6$ *and* $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ *is an improvement of* $H = K(x_1, x_2, x_3, x_4, x_5, x_6)$*. Then*

$$\alpha(H,7) - \alpha(H',7) = 2^{x_{i_2}-2} - 2^{x_{i_1}-1} \ge 2^{x_{i_1}-1}.$$

Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H,7) - \alpha(G,7)$. Clearly, $\alpha'(H) \ge 0$.

Lemma 2.4. [3] *Let* $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$*. Suppose that* $\min\{n_i | i = 1,2,3,4,5,6\} \ge s+1 \ge 1$ *and* $H = G - S$*, where* S *is a set of some* s *edges of* G*. Then*

$$s \le \alpha'(H) = \alpha(H,7) - \alpha(G,7) \le 2^s - 1,$$

$\alpha'(H) = s$ *iff the set of end-vertices of any* $r \ge 2$ *edges in* S *is not independent in* H*, and* $\alpha'(H) = 2^s - 1$ *iff* S *induces a star* $K_{1,s}$ *and all vertices of* $K_{1,s}$ *other than its center belong to a same* A_i*.*

Let $K(A_1, A_2)$ be a complete bipartite graph with partite sets A_1 and A_2. We denote by $K^{-K_{1,s}}(A_i, A_j)$ the graph obtained from $K(A_i, A_j)$ by deleting s edges that induce a star with its center in A_i. Note that $K^{-K_{1,s}}(A_i, A_j) \ne K^{-K_{1,s}}(A_j, A_i)$ if $|A_i| \ne |A_j|$ for $i \ne j$ (see [5]).

Lemma 2.5. [4] *Let* $K(n_1, n_2)$ *be a complete bipartite graph with partite sets* A_1 *and* A_2 *such that* $|A_i| = n_i$ *for* $i = 1, 2$*. If* $\min\{n_1, n_2\} \ge s+2$*, then every* $K^{-K_{1,s}}(A_i, A_j)$ *is* χ*-unique, where* $i \ne j$ *and* $i, j = 1, 2$*.*

Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 6-partite graph with partite sets $A_i (i = 1, 2, \cdots, 6)$ such that $|A_i| = n_i$. Let $\langle A_i \cup A_j \rangle$ be the subgraph of G induced by $A_i \cup A_j$, where $i \ne j$ and $i, j \in \{1,2,3,4,5,6\}$. By $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$, we denote the graph obtained from $K(n_1, n_2, n_3, n_4, n_5, n_6)$ by deleting a set of s edges that induce a $K_{1,s}$ with its center in A_i and all its end-vertices are in A_j. Note that

$$K_{i,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6) = K_{j,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$$

and

$$K_{l,i}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6) = K_{l,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$$

for $n_i = n_j$ and $l \ne i, j$.

Lemma 2.6. [3] *If* $i, j \in \{1, 2, 3, \cdots, t\}$*,* $i \ne j$*,* $n_i \ne n_j$*, then*

$$P(K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, \cdots, n_t), \lambda) \ne P(K_{j,i}^{-K_{1,s}}(n_1, n_2, n_3, \cdots, n_t), \lambda).$$

3. Classification

In this section, we shall characterize certain complete 6-partite graphs $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ according to the number of 7-independent partitions of G where $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n, n \ge 1$.

Theorem 3.1. *Let* $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ *be a complete 6-partite graph such that* $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n$*,* $n \ge 1$*. Define*

$$\theta(G) = [\alpha(G,7) - 2^{n+1} - 2^n + 6]/2^{n-2}.$$

Then

(i) $\theta(G) \geq 0$;
(ii) $\theta(G) = 0$ *if and only if* $G = K(n,n,n,n,n,n)$;
(iii) $\theta(G) = 1$ *if and only if* $G = K(n-1,n,n,n,n,n+1)$;
(iv) $\theta(G) = 2$ *if and only if* $G = K(n-1,n-1,n,n,n+1,n+1)$;
(v) $\theta(G) = 5/2$ *if and only if* $G = K(n-2,n,n,n,n+1,n+1)$;
(vi) $\theta(G) = 3$ *if and only if* $G = K(n-1,n-1,n-1,n+1,n+1,n+1)$;
(vii) $\theta(G) = 7/2$ *if and only if* $G = K(n-2,n-1,n,n+1,n+1,n+1)$;
(viii) $\theta(G) = 4$ *if and only if* $G = K(n-1,n-1,n,n,n,n+2)$;
(ix) $\theta(G) = 17/4$ *if and only if* $G = K(n-3,n,n,n+1,n+1,n+1)$;
(x) $\theta(G) \geq 9/2$ *if and only if* G *is not one of the graphs appeared in (ii)–(ix).*

Proof. For a complete 6-partite graph H_1 with $6n$ vertices, we can construct a sequence of complete 6-partite graphs with $6n$ vertices, say $H_1, H_2, \cdots, H_t$, such that H_i is an improvement of H_{i-1} for each $i = 2,3,\cdots,t$, and $H_t = K(n,n,n,n,n,n)$. By Lemma 2.3, $\alpha(H_{i-1},7) - \alpha(H_i,7) > 0$. So $\theta(H_{i-1}) - \theta(H_i) > 0$, which implies that $\theta(G) \geq \theta(H_t) = \theta(K(n,n,n,n,n,n))$. From Lemma 2.2 and by a simple calculation, $\theta(K(n,n,n,n,n,n)) = 0$. Thus, (ii) is true.

Since $H_t = K(n,n,n,n,n,n)$ and H_t is an improvement of H_{t-1}, it is not hard to see that H_{t-1} must be $K(n-1,n,n,n,n,n+1)$. The proof of (iii) is complete.

Note that $H_{t-1} = K(n-1,n,n,n,n,n+1)$ is an improvement of H_{t-2}. Similarly, it is not hard to see that $H_{t-2} \in \{R_i | i = 1,2,3,4\}$, where R_i and $\theta(R_i)$ are shown in Table 1.

To complete the proof of the theorem, we need only determine all complete 6-partite graphs G with $6n$ vertices such that $\theta(G) < 9/2$. By Lemma 2.3, $\theta(H_{t-3}) > 9/2$ for each H_{t-3} if $H_{t-2} \in R_4$. All graphs H_{t-3} and its θ-values are listed in Table 2 when $H_{t-2} \in \{R_i | i = 1,2,3\}$.

Table 1. H_{t-2} and its θ-values

R_i	Graphs H_{t-2}	$\theta(R_i)$
R_1	$K(n-1,n-1,n,n,n+1,n+1)$	2
R_2	$K(n-2,n,n,n,n+1,n+1)$	5/2
R_3	$K(n-1,n-1,n,n,n,n+2)$	4
R_4	$K(n-2,n,n,n,n,n+2)$	9/2

By Lemma 2.3, $\theta(H_{t-4}) > 9/2$ for every H_{t-4} if $H_{t-3} \in \{M_i | 4 \leq i \leq 8\}$. One can easily obtain the following: If $H_{t-3} = M_1$, then $H_{t-4} \in \{M_2, M_4, M_{12}\}$; $H_{t-4} \in \{M_3, M_5, M_9, M_{10}, M_{12}, M_{13}, M_{14}\}$ if $H_{t-3} = M_2$ and $H_{t-4} \in \{M_6, M_{10}, M_{11}, M_{14}, M_{15}\}$ if $H_{t-3} = M_3$, where $M_9 = K(n-2,n-2,n+1,n+1,n+1,n+1)$,$M_{10} = K(n-3,n-1,n+1,n+1,n+1,n+1)$, $M_{11} = K(n-4,n,n+1,n+1,n+1,n+1)$, $M_{12} = K(n-2,n-1,n-1,n+1,n+1,n+2)$, $M_{13} = K(n-2,n-2,n,n+1,n+1,n+2)$, $M_{14} = K(n-3,n-1,n,n+1,n+1,n+2)$ and $M_{15} = K(n-4,n,n,n+1,n+1,n+2)$. From Lemma 2.2 and by a calculation, we have $\theta(M_i) \geq 9/2$ for $9 \leq i \leq 15$. Hence, from Lemma 2.3, Table 1, Table 2 and the above arguments, we conclude that the theorem holds. ∎

Table 2. H_{t-3} and its θ-values

M_i	Graphs H_{t-3}	$\theta(M_i)$
M_1	$K(n-1, n-1, n-1, n+1, n+1, n+1)$	3
M_2	$K(n-2, n-1, n, n+1, n+1, n+1)$	7/2
M_3	$K(n-3, n, n, n+1, n+1, n+1)$	17/4
M_4	$K(n-1, n-1, n-1, n, n+1, n+2)$	5
M_5	$K(n-2, n-1, n, n, n+1, n+2)$	11/2
M_6	$K(n-3, n, n, n, n+1, n+2)$	25/4
M_7	$K(n-1, n-1, n-1, n, n, n+3)$	11
M_8	$K(n-2, n-1, n, n, n, n+3)$	23/2

4. Chromatically closed 6-partite graphs

In this section, we obtain several χ-closed families of graphs $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5, n_6)$.

Theorem 4.1. *If $n \geq s+2$, then the family of graphs $\mathcal{K}^{-s}(n, n, n, n, n, n)$ is χ-closed.*

Proof. Let $G = K(n, n, n, n, n, n)$ and $Z \in \mathcal{K}^{-s}(n, n, n, n, n, n)$. The 6-independent partition of G is a 6-independent partition of Z. So $\alpha(Z, 6) \geq \alpha(G, 6) = 1$. Let $H \sim Z$, then $\alpha(H, 6) = \alpha(Z, 6) \geq \alpha(G, 6) = 1$. Let $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ be a 6-independent partition of H, $|A_i| = t_i$, $i = 1, 2, 3, 4, 5, 6$ and $F = K(t_1, t_2, t_3, t_4, t_5, t_6)$. Then, there exists $S' \in E(F)$ such that $H = F - S'$. Let $q(G)$ be the number of edges in graph G. Since $q(H) = q(Z)$, therefore $s' = |S'| = q(F) - q(G) + s$.

From Lemma 2.4, we have

$$\alpha(Z, 7) = \alpha(G, 7) + \alpha'(Z), s \leq \alpha'(Z) \leq 2^s - 1, \quad \text{and}$$
$$\alpha(H, 7) = \alpha(F, 7) + \alpha'(H), s' \leq \alpha'(H).$$

Thus $\alpha(H, 7) - \alpha(Z, 7) = \alpha(F, 7) - \alpha(G, 7) + \alpha'(H) - \alpha'(Z)$. Since $H \sim Z$, then $\alpha(Z, 7) = \alpha(H, 7)$. So $\alpha(H, 7) - \alpha(Z, 7) = 0$.

Suppose $F \neq G$, we need to show that $\alpha(H, 7) \geq \alpha(Z, 7)$, this leads to a contradiction. Hence, the conclusion of the theorem.

Now, if $F \neq G$, from Theorem 3.1, we have $\theta(F) - \theta(G) \geq 1$. So

$$\alpha(F, 7) - \alpha(G, 7) = (\theta(F) - \theta(G)) \cdot 2^{n-2} \geq 2^{n-2}.$$

Hence

$$\alpha(H, 7) - \alpha(Z, 7) \geq 2^{n-2} + \alpha'(H) - \alpha'(Z) \geq 2^{n-2} + 0 - (2^s - 1) \geq 1.$$

This is a contradiction. So $F = G$, $s = s'$. Thus, $H \in \mathcal{K}^{-s}(n, n, n, n, n, n)$. Therefore, $\mathcal{K}^{-s}(n, n, n, n, n, n)$ is χ-closed if $n \geq s+2$. The proof is now completed. ∎

By using proofs similar to that of Theorem 4.1, we can obtain the following results.

Theorem 4.2. *If $n \geq s+3$, then the family of graphs $\mathcal{K}^{-s}(n-1, n, n, n, n, n+1)$ is χ-closed.*

Theorem 4.3. *If $n \geq s+3$, then the family of graphs $\mathcal{K}^{-s}(n-1, n-1, n, n, n+1, n+1)$ is χ-closed.*

Theorem 4.4. *If $n \geq s+4$, then the family of graphs $\mathcal{K}^{-s}(n-2, n, n, n, n+1, n+1)$ is χ-closed.*

Theorem 4.5. *If $n \geq s+4$, then the family of graphs $\mathcal{K}^{-s}(n-1, n-1, n-1, n+1, n+1, n+1)$ is χ-closed.*

Theorem 4.6. *If $n \geq s+5$, then the family of graphs $\mathcal{K}^{-s}(n-2, n-1, n, n+1, n+1, n+1)$ is χ-closed.*

Theorem 4.7. *If $n \geq s+4$, then the family of graphs $\mathcal{K}^{-s}(n-1, n-1, n, n, n, n+2)$ is χ-closed.*

Theorem 4.8. *If $n \geq s+7$, then the family of graphs $\mathcal{K}^{-s}(n-3, n, n, n+1, n+1, n+1)$ is χ-closed.*

5. Chromatically unique 6-partite graphs

In this section, we first study the chromatically unique 6-partite graphs with $6n$ vertices and a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$.

Theorem 5.1. *If $n \geq s+2$, then the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n, n, n)$ are χ-unique for $(i, j) = (1, 2)$.*

Proof. Suppose that $H \sim K_{1,2}^{-K_{1,s}}(n, n, n, n, n, n)$. From Theorem 4.1, $H \in \mathcal{K}^{-s}(n, n, n, n, n, n)$. Note that $\alpha(H, 7) = \alpha(K_{1,2}^{-K_{1,s}}(n, n, n, n, n, n), 7) = \alpha(K(n, n, n, n, n, n), 7) + 2^s - 1$. By Lemma 2.4, we have

$H \in \{K_{i,j}^{-K_{1,s}}(n, n, n, n, n, n) | i \neq j,\ i, j = 1, 2, 3, 4, 5, 6\} = \{K_{1,2}^{-K_{1,s}}(n, n, n, n, n, n)\}$.

This completes the proof. ∎

Theorem 5.2. *If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1)$ are χ-unique for each $(i, j) \in \{(1, 2), (2, 1), (2, 6), (6, 2)\}$.*

Proof. Let $F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1) | (i, j) = \{(1, 2), (2, 1), (2, 6), (6, 2)\}\}$ and $H \sim F$. By Theorem 4.2, $H \in \mathcal{K}^{-s}(n-1, n, n, n, n, n+1)$. Since

$$\alpha(H, 7) = \alpha(F, 7) = \alpha(K(n-1, n, n, n, n, n+1), 7) + 2^s - 1,$$

from Lemma 2.4, we know that $H \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1) | i \neq j,\ i, j = 1, 2, 3, 4, 5, 6\}$. It is easy to see that $H \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1) | i \neq j,\ i, j = 1, 2, 3, 4, 5, 6\} = \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1) | (i, j) \in \{(1, 2), (2, 1), (1, 6), (6, 1), (2, 3), (2, 6), (6, 2)\}\}$.

Now let's determine the number of triangles in H and F. Let $t(G)$ be the number of triangles in the graphs G. Then we obtain that $t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1)) = t(K(n-1, n, n, n, n, n+1)) - s(4n+1)$ for $(i, j) \in \{(1, 2), (2, 1)\}$, $t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1)) = t(K(n-1, n, n, n, n, n+1)) - 4sn$ for $(i, j) \in \{(1, 6), (6, 1), (2, 3)\}$, $t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1)) = t(K(n-1, n, n, n, n, n+1)) - s(4n-1)$ for $(i, j) \in \{(2, 6), (6, 2)\}$.

Recalling

$$F \in \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)|(i,j) \in \{(1,2),(2,1),(2,6),(6,2)\}\}$$

and $t(H) = t(F)$, thus we have

$$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)|(i,j) \in \{(1,2),(2,1)\}\}$$

or

$$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)|(i,j) \in \{(2,6),(6,2)\}\}.$$

It follows from Lemma 2.6 that

$$P(K_{1,2}^{-K_{1,s}}(n-1,n,n,n,n,n+1),\lambda) \neq P(K_{2,1}^{-K_{1,s}}(n-1,n,n,n,n,n+1),\lambda);$$
$$P(K_{2,6}^{-K_{1,s}}(n-1,n,n,n,n,n+1),\lambda) \neq P(K_{6,2}^{-K_{1,s}}(n-1,n,n,n,n,n+1),\lambda).$$

Hence, by Lemma 2.1, we conclude that the graphs $K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n,n+1)$ are χ-unique where $n \geq s+3$ for each $(i,j) \in \{(1,2),(2,1),(2,6),(6,2)\}$. ∎

Similar to the proof of Theorem 5.2, we can prove Theorems 5.3 and 5.4.

Theorem 5.3. *If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1,n-1,n,n,n+1,n+1)$ are χ-unique for each $(i,j) \in \{(1,2),(1,3),(3,1),(3,5),(5,3),(5,6)\}$.*

Theorem 5.4. *If $n \geq s+5$, then the graphs $K_{i,j}^{-K_{1,s}}(n-2,n-1,n,n+1,n+1,n+1)$ are χ-unique for each $(i,j) \in \{(1,2),(2,1),(1,3),(3,1),(2,4),(4,2),(3,4),(4,3),(4,5)\}$.*

Theorem 5.5. *If $n \geq s+4$, then the graphs $K_{i,j}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1)$ are χ-unique for each $(i,j) \in \{(1,2),(2,1),(1,5),(5,1),(2,3),(2,5),(5,2),(5,6)\}$.*

Proof. From Theorem 4.4, we know that $K^{-s}(n-2,n,n,n,n+1,n+1)$ is χ-closed if $n \geq s+4$. Comparing the number of 7-independent partitions of the graphs in $K^{-s}(n-2,n,n,n,n+1,n+1)$ and by using Lemma 2.4, we have that $K_{i,j}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1) = \{K_{i,j}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1)|(i,j) \in \{(1,2),(2,1),(1,5),(5,1),(2,3),(2,5),(5,2),(5,6)\}$ is χ-closed.

Note that $t(K_{i,j}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1)) = t(K(n-2,n,n,n,n+1,n+1)) - s(4n+2)$ for $(i,j) \in \{(1,2),(2,1)\}$, $t(K_{i,j}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1)) = t(K(n-2,n,n,n,n+1,n+1)) - s(4n+1)$ for $(i,j) \in \{(1,5),(5,1)\}$, $t(K_{i,j}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1)) = t(K(n-2,n,n,n,n+1,n+1)) - s(4n-1)$ for $(i,j) \in \{(2,5),(5,2)\}$, $t(K_{2,3}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1)) = t(K(n-2,n,n,n,n+1,n+1)) - 4sn$, $t(K_{5,6}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1)) = t(K(n-2,n,n,n,n+1,n+1)) - s(4n-2)$.

It follows from Lemma 2.6 that

$$P(K_{1,2}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1),\lambda) \neq P(K_{2,1}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1),\lambda);$$
$$P(K_{1,5}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1),\lambda) \neq P(K_{5,1}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1),\lambda);$$
$$P(K_{2,5}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1),\lambda) \neq P(K_{5,2}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1),\lambda).$$

Hence, by Lemma 2.1, we can conclude that the graphs $K_{i,j}^{-K_{1,s}}(n-2,n,n,n,n+1,n+1)$ are χ-unique where $n \geq s+4$ for each $(i,j) \in \{(1,2),(2,1),(1,5),(5,1),(2,3),(2,5),(5,2),(5,6)\}$. ∎

Similar to the proof of Theorem 5.5, we can prove Theorems 5.6, 5.7 and 5.8.

Theorem 5.6. *If* $n \geq s+4$*, then the graphs* $K_{i,j}^{-K_{1,s}}(n-1, n-1, n-1, n+1, n+1, n+1)$ *are* χ*-unique for each* $(i, j) \in \{(1, 2), (1, 4), (4, 1), (4, 5)\}$.

Theorem 5.7. *If* $n \geq s+4$*, then the graphs* $K_{i,j}^{-K_{1,s}}(n-1, n-1, n, n, n, n+2)$ *are* χ*-unique for each* $(i, j) \in \{(1, 2), (1, 3), (3, 1), (1, 6), (6, 1), (3, 4), (3, 6), (6, 3)\}$.

Theorem 5.8. *If* $n \geq s+7$*, then the graphs* $K_{i,j}^{-K_{1,s}}(n-3, n, n, n+1, n+1, n+1)$ *are* χ*-unique for each* $(i, j) \in \{(1, 2), (2, 1), (1, 4), (4, 1), (2, 3), (2, 4), (4, 2), (4, 5)\}$.

Let $K_{i,j}^{-sK_2}(n_1, n_2, n_3, n_4, n_5, n_6)$ denote the graph obtained from $K(n_1, n_2, n_3, n_4, n_5, n_6)$ by deleting a set of s edges that forms a matching in $\langle A_i \cup A_j \rangle$. We now investigate the chromatically unique 6-partite graphs with $6n$ vertices and a set S of s edges deleted where the deleted edges induce a matching sK_2.

Theorem 5.9. *If* $n \geq s+3$*, then the graphs* $K_{1,2}^{-sK_2}(n-1, n-1, n, n, n+1, n+1)$ *are* χ*-unique.*

Proof. Let $F \sim K_{1,2}^{-sK_2}(n-1, n-1, n, n, n+1, n+1)$. It is sufficient to prove that $F = K_{1,2}^{-sK_2}(n-1, n-1, n, n, n+1, n+1)$. By Theorem 4.3 and Lemma 2.4, we have $F \in \mathcal{K}^{-s}(n-1, n-1, n, n, n+1, n+1)$ and $\alpha'(F) = s$. Let $F = G - S$ where $G = K(n-1, n-1, n, n, n+1, n+1)$. Next we consider the number of triangles in F. Let $e_i \in S$ and $t(e_i)$ be the number of triangles in G containing the edge e_i. It is easy to see that $t(e_i) \leq 4n+2$. As $n-1 \leq n-1 < n \leq n \leq n+1 \leq n+1$, we know that $t(e_i) = 4n+2$ if and only if e_i is an edge in the subgraph $\langle A_1 \cup A_2 \rangle$ in G. So we have

$$t(F) \geq t(G) - \sum_{i=1}^{s} t(e_i) \geq t(G) - s(4n+2);$$

and the equality holds if and only if each edge e_i in S is an edge of the subgraph $\langle A_1 \cup A_2 \rangle$ in G.

Note that $t(F) = t(G) - s(4n+2)$ and $\alpha'(F) = s$. By Lemma 2.4, we know that $F = K_{1,2}^{-sK_2}(n-1, n-1, n, n, n+1, n+1)$. This completes the proof. ∎

Similar to the proof of Theorem 5.9, we can prove Theorems 5.10 and 5.11.

Theorem 5.10. *If* $n \geq s+5$*, then the graphs* $K_{1,2}^{-sK_2}(n-2, n-1, n, n+1, n+1, n+1)$ *are* χ*-unique.*

Theorem 5.11. *If* $n \geq s+4$*, then the graphs* $K_{1,2}^{-sK_2}(n-1, n-1, n, n, n, n+2)$ *are* χ*-unique.*

We end this paper with the following open problems:

(1) Study the chromaticity of the following graphs:
   (i) $K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n, n+1)$ where $n \geq s+3$ for each $(i, j) \in \{(1, 6), (6, 1), (2, 3)\}$,
   (ii) $K_{i,j}^{-K_{1,s}}(n-1, n-1, n, n, n+1, n+1)$ where $n \geq s+3$ for each $(i, j) \in \{(1, 5), (5, 1), (3, 4)\}$ and
   (iii) $K_{i,j}^{-K_{1,s}}(n-2, n-1, n, n+1, n+1, n+1)$ where $n \geq s+5$ for each $(i, j) \in \{(1, 4), (4, 1), (2, 3), (3, 2)\}$.

(2) Study the chromaticity of the following graphs:
  (i) $K_{1,2}^{-sK_2}(n,n,n,n,n,n)$ where $n \geq s+2$,
  (ii) $K_{1,2}^{-sK_2}(n-1,n,n,n,n,n+1)$ where $n \geq s+3$,
  (iii) $K_{1,2}^{-sK_2}(n-2,n,n,n,n+1,n+1)$ where $n \geq s+4$,
  (iv) $K_{1,2}^{-sK_2}(n-1,n-1,n-1,n+1,n+1,n+1)$ where $n \geq s+4$ and
  (v) $K_{1,2}^{-sK_2}(n-3,n,n,n+1,n+1,n+1)$ where $n \geq s+7$.

Remark 5.1. For the detail proofs of Theorems 4.2–4.8, 5.3, 5.4, 5.6–5.8, the reader may refer to [15].

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