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Chromaticity of Complete 5-Partite Graphs with Certain Star or Matching Deleted

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Abstract : Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. We write $[G] = \{H | H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 5-partite graphs with $5n+1$ vertices according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

Keywords : Chromatic polynomial; Chromatically closed; Chromatic uniqueness.
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1 Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G,\lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. Many families of χ-unique graphs are known (see [1–3]).

For a graph G, let $V(G)$, $E(G)$ and $t(G)$ be the vertex set, edge set and number of triangles in G, respectively. Let S be a set of s edges in G. Let $G-S$ (or $G-s$) be the graph obtained from G by deleting all edges in S, and by $\langle S\rangle$ the graph induced by S. Let $K(n_1,n_2,...,n_t)$ be a complete t-partite graph. We denote by $\mathcal{K}^{-s}(n_1,n_2,...,n_t)$ the family of graphs which are obtained from $K(n_1,n_2,...,n_t)$ by deleting a set S of some s edges.

In [2–5], one can find many results on the chromatic uniqueness of bipartite and tripartite graphs. Also there are some results on the chromaticity of 4-partite graphs. However, there are very few 5-partite graphs known to be χ-unique, see [6, 7].

Let G be a complete 5-partite graph with $5n+1$ vertices. In this paper, we characterize certain complete 5-partite graphs with $5n+1$ vertices according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

2 Some Lemmas and Notations

For a graph G and a positive integer r, a partition $\{A_1,A_2,...,A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G,r)$ denote the number of r-independent partitions of G. Then, we have $P(G,\lambda) = \sum_{r=1}^{p}\alpha(G,r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ (see [8]). Therefore, $\alpha(G,k) = \alpha(H,k)$ for each $k = 1,2,...$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G,x) = \sum_{r=1}^{p}\alpha(G,r)x^r$ is called the σ-polynomial of G (see [9]). Clearly, $P(G,\lambda) = P(H,\lambda)$ implies that $\sigma(G,x) = \sigma(H,x)$ for any graphs G and H.

For disjoint graphs G and H, $G\cup H$ denotes the disjoint union of G and H. The join of G and H denoted by $G\vee H$ is defined as follows: $V(G\vee H) = V(G)\cup V(H)$; $E(G\vee H) = E(G)\cup E(H)\cup\{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer [10].

Lemma 2.1 (Koh et al. [2], Brenti [9]). *Let G and H be two disjoint graphs. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ *and* $\alpha(G, r) = \alpha(H, r)$ *for* $r = 1, 2, 3, ...,$ *if* $G \sim H$*;*

(2) $\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x)$.

Lemma 2.2 (Brenti [9])**.** *Let* $G = K(n_1, n_2, n_3, ..., n_t)$ *and* $\sigma(G, x) = \sum_{r \geq 1} \alpha(G, r)x^r$*, then* $\alpha(G, r) = 0$ *for* $1 \leq r \leq t - 1$*,* $\alpha(G, t) = 1$ *and* $\alpha(G, t + 1) = \sum_{i=1}^{t} 2^{n_i - 1} - t$*.*

Let $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ be positive integers and $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}\} = \{x_1, x_2, x_3, x_4, x_5\}$. If there are two elements x_{i_1} and x_{i_2} in $\{x_1, x_2, x_3, x_4, x_5\}$ such that $x_{i_2} - x_{i_1} \geq 2$, then $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5}\}$ is called an *improvement* of $H = K(x_1, x_2, x_3, x_4, x_5)$.

Lemma 2.3 (Zhao et al. [6])**.** *Suppose* $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ *and* $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5}\}$ *is an improvement of* $H = K(x_1, x_2, x_3, x_4, x_5)$*, then*

$$\alpha(H, 6) - \alpha(H', 6) = 2^{x_{i_2} - 2} - 2^{x_{i_1} - 1} \geq 2^{x_{i_1} - 1}.$$

For a graph G, let $q(G)$ be the number of edges in G.

Lemma 2.4 (Zhao et al. [6])**.** *Let* $G = K(n_1, n_2, n_3, n_4, n_5)$ *and* S *be a set of some* s *edges of* G*. If* $H \sim G - S$*, then there is a graph* $F = K(y_1, y_2, y_3, y_4, y_5)$ *and a subset* S' *of* $E(F)$ *of some* s' *edges of* F *such that* $H = F - S'$ *and* $|S'| = s' = q(F) - q(G) + s$*.*

Let $G = K(n_1, n_2, n_3, n_4, n_5)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H, 6) - \alpha(G, 6)$. Clearly, $\alpha'(H) \geq 0$.

Lemma 2.5 (Zhao [7])**.** *Let* $G = K(n_1, n_2, n_3, n_4, n_5)$*. Suppose that min* $\{n_i | i = 1, 2, 3, 4, 5\} \geq s + 1 \geq 1$ *and* $H = G - S$*, where* S *is a set of some* s *edges of* G*, then*

$$s \leq \alpha'(H) = \alpha(H, 6) - \alpha(G, 6) \leq 2^s - 1,$$

$\alpha'(H) = s$ *iff the set of end-vertices of any* $r \geq 2$ *edges in* S *is not independent in* H*, and* $\alpha'(H) = 2^s - 1$ *iff* S *induces a star* $K_{1,s}$ *and all vertices of* $K_{1,s}$ *other than its center belong to a same* A_i*.*

Let $K(A_1, A_2)$ be a complete bipartite graph with partite sets A_1 and A_2. We denote by $K^{-K_{1,s}}(A_i, A_j)$ the graph obtained from $K(A_i, A_j)$ by deleting s edges that induce a star with its center in A_i. Note that $K^{-K_{1,s}}(A_i, A_j) \neq K^{-K_{1,s}}(A_j, A_i)$ if $|A_i| \neq |A_j|$ for $i \neq j$ (see [5]).

Lemma 2.6 (Dong et al. [5])**.** *Let* $K(n_1, n_2)$ *be a complete bipartite graph with partite sets* A_1 *and* A_2 *such that* $|A_i| = n_i$ *for* $i = 1, 2$*. If min* $\{n_1, n_2\} \geq s + 2$*, then every* $K^{-K_{1,s}}(A_i, A_j)$ *is* χ*-unique, where* $i \neq j$ *and* $i, j = 1, 2$*.*

Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph with partite sets $A_i (i = 1, 2, ..., 5)$ such that $|A_i| = n_i$. Let $\langle A_i \cup A_j \rangle$ be the subgraph of G induced by $A_i \cup A_j$, where $i \neq j$ and $i, j \in \{1, 2, 3, 4, 5\}$. By $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5)$, we denote the graph obtained from $K(n_1, n_2, n_3, n_4, n_5)$ by deleting a set of s edges that induce a $K_{1,s}$ with its center in A_i and all its end vertices are in A_j. Note that $K_{i,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5) = K_{j,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5)$ and $K_{l,i}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5) = K_{l,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5)$ for $n_i = n_j$ and $l \neq i, j$.

Lemma 2.7 (Zhao et al. [6]). *Suppose that min* $\{n_1, n_2, n_3, n_4, n_5\} \geq s + 2$ *and* $n_i \neq n_j$ *for* $i \neq j$, $i, j = 1, 2, 3, 4, 5$, *then* $P(K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5), \lambda) \neq P(K_{j,i}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5), \lambda)$.

3 Classification

In this section, we shall characterize certain complete 5-partite graph $G = K(n_1, n_2, n_3, n_4, n_5)$ according to the number of 6-independent partitions of G where $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 1, n \geq 1$.

Theorem 3.1. *Let* $G = K(n_1, n_2, n_3, n_4, n_5)$ *be a complete 5-partite graph such that* $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 1, n \geq 1$. *Define* $\theta(G) = [\alpha(G, 6) - 2^{n+1} - 2^n + 5]/2^{n-2}$. *Then*

(i) $\theta(G) \geq 0$;

(ii) $\theta(G) = 0$ *if and only if* $G = K(n, n, n, n, n + 1)$;

(iii) $\theta(G) = 1$ *if and only if* $G = K(n - 1, n, n, n + 1, n + 1)$;

(iv) $\theta(G) = 2$ *if and only if* $G = K(n - 1, n - 1, n + 1, n + 1, n + 1)$;

(v) $\theta(G) = 5/2$ *if and only if* $G = K(n - 2, n, n + 1, n + 1, n + 1)$;

(vi) $\theta(G) = 3$ *if and only if* $G = K(n - 1, n, n, n, n + 2)$;

(vii) $\theta(G) \geq 4$ *if and only if* G *is not a graph appeared in (ii)–(vi);*

Proof. For a complete 5-partite graph H_1 with $5n + 1$ vertices, we can construct a sequence of complete 5-partite graphs with $5n+1$ vertices, say $H_1, H_2, ..., H_t$, such that H_i is an improvement of H_{i-1} for each $i = 2, ..., t$, and $H_t = K(n, n, n, n, n + 1)$. By Lemma 2.3, $\alpha(H_{i-1}, 6) - \alpha(H_i, 6) > 0$. So $\theta(H_{i-1}) - \theta(H_i) > 0$, which implies $\theta(G) \geq \theta(H_t) = \theta(K(n, n, n, n, n + 1))$. From Lemma 2.2 and by a simple calculation, we have $\theta(K(n, n, n, n, n + 1)) = 0$. Thus, (ii) is true.

Since $H_t = K(n, n, n, n, n + 1)$ and H_t is an improvement of H_{t-1}, it is not hard to see that $H_{t-1} \in \{M_0, M_3\}$, where $M_0 = K(n - 1, n, n, n + 1, n + 1)$ and $M_3 = K(n - 1, n, n, n, n + 2)$. Hence, by Lemma 2.2, we have $\theta(M_0) = 1$, $\theta(M_3) = 3$. Note that H_{t-1} is an improvement of H_{t-2}, one can see that $H_{t-2} \in \{M_i | i = 1, 2, ..., 7\}$, where M_i and $\theta(M_i)$ are shown in Table 1.

M_i	Graphs H_{t-2}	$\theta(M_i)$
M_1	$K(n-1,n-1,n+1,n+1,n+1)$	2
M_2	$K(n-2,n,n+1,n+1,n+1)$	5/2
M_3	$K(n-1,n,n,n,n+2)$	3
M_4	$K(n-1,n-1,n,n+1,n+2)$	4
M_5	$K(n-2,n,n,n+1,n+2)$	9/2
M_6	$K(n-1,n-1,n,n,n+3)$	10
M_7	$K(n-2,n,n,n,n+3)$	21/2

Table 1: H_{t-2} and its θ-values

R_i	Graphs H_{t-3}	$\theta(R_i)$
R_1	$K(n-3,n+1,n+1,n+1,n+1)$	17/4
R_2	$K(n-2,n-1,n+1,n+1,n+2)$	11/2
R_3	$K(n-3,n,n+1,n+1,n+2)$	25/4

Table 2: H_{t-3} and its θ-values

To complete the proof of the theorem, we need only determine all complete 5-partite graph G with $5n+1$ vertices such that $\theta(G) < 4$. By Lemma 2.3, $\theta(H_{t-3}) > 4$ for each H_{t-3} if $H_{t-2} \in \{M_i | i = 4,5,6,7\}$. All graphs H_{t-3} and its θ-values are listed in Table 2 when $H_{t-2} \in \{M_i | i = 1,2,3\}$.

It is easy to obtain the following: If $H_{t-2} = M_1$, then $H_{t-3} \in \{M_2, M_4, R_2\}$; $H_{t-3} \in \{M_5, R_1, R_2, R_3\}$ if $H_{t-2} = M_2$ and $H_{t-3} \in \{M_i | i = 4,5,6,7\}$ if $H_{t-2} = M_3$. Thus, from Lemma 2.2, Table 1, Table 2 and the above arguments, we conclude that the theorem holds. □

4 Chromatically Closed 5-Partite Graphs

In this section, we obtained several χ-closed families of graphs in $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5)$.

Theorem 4.1.

(i) If $n \geq s+2$, then the family of graphs $\mathcal{K}^{-s}(n,n,n,n,n+1)$ is χ-closed;

(ii) If $n \geq s+3$, then the family of graphs $\mathcal{K}^{-s}(n-1,n,n,n+1,n+1)$ is χ-closed;

(iii) If $n \geq s+3$, then the family of graphs $\mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1)$ is χ-closed;

(iv) If $n \geq s+4$, then the family of graphs $\mathcal{K}^{-s}(n-2, n, n+1, n+1, n+1)$ is χ-closed;

(v) If $n \geq s+3$, then the family of graphs $\mathcal{K}^{-s}(n-1, n, n, n, n+2)$ is χ-closed.

Proof. The proof of each statement of the theorem is similar. So, we only give a proof for (iii) and omit the proofs of the others. For convenience, let $G_1 = K(n, n, n, n, n+1)$, $G_2 = K(n-1, n, n, n+1, n+1)$ and $G_3 = K(n-1, n-1, n+1, n+1, n+1)$. Suppose that $H \sim G_3 - S$. Then it suffices to show that $H \in \mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1)$. By Lemma 2.4, there is a complete 5-partite graph $F = K(y_1, y_2, y_3, y_4, y_5)$ and a set S' for some s' edges in F such that $H = F - S'$ and $|S'| = s' = q(F) - q(G_3) + s \geq 0$. Clearly, $\alpha(F - S', 6) = \alpha(G_3 - S, 6)$.

By definition, we have

$$\alpha(G_3 - S, 6) = \alpha(G_3, 6) + \alpha'(G_3 - S) \quad \text{with} \quad s \leq \alpha'(G_3 - S) \leq 2^s - 1,$$

and

$$\alpha(F - S', 6) = \alpha(F, 6) + \alpha'(F - S').$$

So

$$\alpha(F - S', 6) - \alpha(G_3 - S, 6) = \alpha(F, 6) - \alpha(G_3, 6) + \alpha'(F - S') - \alpha'(G_3 - S) \quad (4.1)$$

By Theorem 3.1, $\alpha(F, 6) - \alpha(G_3, 6) = 2^{n-2}(\theta(F) - \theta(G_3))$. We distinguish the following two cases.

Case 1: $\alpha(F, 6) < \alpha(G_3, 6)$. By Theorem 3.1, then $F \in \{G_1, G_2\}$. If $F = G_1$, we have $\alpha(G_1, 6) - \alpha(G_3, 6) = -2^{n-1}$, and $q(G_1) - q(G_3) = 2$. From Equation (4.1) above, we have

$$\alpha(G_1 - S', 6) - \alpha(G_3 - S, 6) = -2^{n-1} + \alpha'(F - S') - \alpha'(G_3 - S).$$

Note that $n \geq s+3$ and $s' = q(G_1) - q(G_3) + s = s+2 \leq n-1$. By Lemma 2.5, $0 \leq s' \leq \alpha'(F - S') \leq 2^{s'} - 1 \leq 2^{n-1} - 1$, since $0 \leq s \leq \alpha'(G_3 - S) \leq 2^s - 1$, we have

$$\alpha(G_1 - S', 6) - \alpha(G_3 - S, 6) \leq -2^{n-1} + \alpha'(F - S') - \alpha'(G_3 - S) \leq -1,$$

which contradicts $\alpha(F - S', 6) = \alpha(G_3 - S, 6)$.

If $F = G_2$, by Theorem 3.1, we have $\alpha(G_2, 6) - \alpha(G_3, 6) = -2^{n-2}$, and $q(G_2) - q(G_3) = 1$. From Equation (4.1) above, we have

$$\alpha(G_2 - S', 6) - \alpha(G_3 - S, 6) = -2^{n-2} + \alpha'(F - S') - \alpha'(G_3 - S).$$

Note that $n \geq s+3$ and $s' = q(G_2) - q(G_3) + s = s+1 \leq n-2$. By Lemma 2.5, $0 \leq s' \leq \alpha'(F - S') \leq 2^{s'} - 1 \leq 2^{n-2} - 1$, since $0 \leq s \leq \alpha'(G_3 - S) \leq 2^s - 1$, we have

$$\alpha(G_2 - S', 6) - \alpha(G_3 - S, 6) \leq -2^{n-2} + \alpha'(F - S') - \alpha'(G_3 - S) \leq -1,$$

which contradicts $\alpha(F - S', 6) = \alpha(G_3 - S, 6)$.

Case 2: $\alpha(F, 6) > \alpha(G_3, 6)$. By Theorem 3.1, $F \neq G_i$, where $i = 1, 2, 3$ and we have $\alpha(F, 6) - \alpha(G_3, 6) \geq 2^{n-3}$. Hence we have $\alpha(F - S', 6) - \alpha(G_3 - S, 6) \geq 2^{n-3} + \alpha'(F - S') - \alpha'(G_3 - S)$.

Since $n - 3 \geq s$, $0 \leq \alpha'(F - S')$ and $0 \leq s \leq \alpha'(G_3 - S) \leq 2^s - 1$, we have $\alpha(F - S', 6) - \alpha(G_3 - S, 6) \geq 1$, contradicting the fact that $\alpha(F - S', 6) = \alpha(G_3 - S, 6)$. So, from the above two cases, we conclude that $\theta(F) - \theta(G_3) = 0$. Thus $F = G_3$ and $S = S'$. Therefore, $H \in \mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1)$. □

5 Chromatically Unique 5-Partite Graphs

In this section, we first study the chromatically unique 5-partite graphs with $5n+1$ vertices and a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$.

Theorem 5.1. *If $n \geq s+2$, then the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1)$ are χ-unique for each $(i, j) \in \{(1, 2), (1, 5), (5, 1)\}$.*

Proof. By Lemma 2.5 and Theorem 4.1(i), we know that $K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1) = \{K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1) | (i, j) \in \{(1, 2), (1, 5), (5, 1)\}\}$ is χ-closed for $n \geq s+2$. Note that

$$t(K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1)) = t(K(n, n, n, n, n+1)) - 3sn \text{ for } (i, j) \in \{(1, 5), (5, 1)\},$$

$$t(K_{1,2}^{-K_{1,s}}(n, n, n, n, n+1)) = t(K(n, n, n, n, n+1)) - s(3n+1).$$

By Lemma 2.1, we have $K_{1,2}^{-K_{1,s}}(n, n, n, n, n+1)$ is chromatically unique. From Lemma 2.7, we find that $P(K_{1,5}^{-K_{1,s}}(n, n, n, n, n+1), \lambda) \neq P(K_{5,1}^{-K_{1,s}}(n, n, n, n, n+1), \lambda)$. Hence, the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1)$ is χ-unique where $n \geq s+2$ for each $(i, j) \in \{(1, 2), (1, 5), (5, 1)\}$. □

Theorem 5.2. *If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1)$ are χ-unique for each $(i, j) \in \{(1, 2), (2, 1), (2, 4), (4, 2), (4, 5)\}$.*

Proof. Let $F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1) \mid (i, j) = \{(1, 2), (2, 1), (2, 4), (4, 2), (4, 5)\}\}$ and $H \sim F$. By Theorem 4.1(ii), $H \in \mathcal{K}^{-s}(n-1, n, n, n+1, n+1)$.

Without loss of generality, we assume $H \sim K_{1,2}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$, where $(i,j)=(1,2)$. Since

$$\begin{aligned}\alpha(H,6) &= \alpha(K_{1,2}^{-K_{1,s}}(n-1,n,n,n+1,n+1),6)\\ &= \alpha(K(n-1,n,n,n+1,n+1),6)+2^s-1,\end{aligned}$$

from Lemma 2.5, we know that $H \in \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1) \mid i \neq j,\ i,j=1,2,3,4,5\}$. It easy to see that $H \in \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1) \mid i \neq j,\ i,j=1,2,3,4,5\} = \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1) \mid (i,j) \in \{(1,2),(2,1),(1,4),(4,1),(2,3),(2,4),(4,2),(4,5)\}\}$.

Now let's determine the numbers of triangles in H and F. Denote by $t_{i,j}$ the number of triangles in $K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$. Then we obtain that

$$\begin{aligned} t_{1,2}=t_{2,1} &= t(K(n-1,n,n,n+1,n+1))-s(3n+2),\\ t_{1,4}=t_{4,1}=t_{2,3} &= t(K(n-1,n,n,n+1,n+1))-s(3n+1),\\ t_{2,4}=t_{4,2} &= t(K(n-1,n,n,n+1,n+1))-3ns,\\ t_{4,5} &= t(K(n-1,n,n,n+1,n+1))-s(3n-1).\end{aligned}$$

Recalling $F \in \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1) \mid (i,j) \in \{(1,2),(2,1),(2,4),(4,2),(4,5)\}\}$ and $t(H)=t(F)$, we have

$$H,F \in \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1) \mid (i,j) \in \{(1,2),(2,1)\}\}$$

or

$$H,F \in \{K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1) \mid (i,j) \in \{(2,4),(4,2)\}\}.$$

It follows from Lemma 2.7 that

$$\begin{aligned} P(K_{1,2}^{-K_{1,s}}(n-1,n,n,n+1,n+1),\lambda) &\neq P(K_{2,1}^{-K_{1,s}}(n-1,n,n,n+1,n+1),\lambda);\\ P(K_{2,4}^{-K_{1,s}}(n-1,n,n,n+1,n+1),\lambda) &\neq P(K_{4,2}^{-K_{1,s}}(n-1,n,n,n+1,n+1),\lambda).\end{aligned}$$

Hence, the graphs $K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ are χ-unique where $n \geq s+3$ for each $(i,j) \in \{(1,2),(2,1),(2,4),(4,2),(4,5)\}$. □

Similarly to the proofs of Theorems 5.1 and 5.2, we can prove Theorems 5.3, 5.4 and 5.5.

Theorem 5.3. *If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1,n-1,n+1,n+1,n+1)$ are χ-unique for each $(i,j) \in \{(1,2),(1,3),(3,1),(3,4)\}$.*

Theorem 5.4. *If $n \geq s+4$, then the graphs $K_{i,j}^{-K_{1,s}}(n-2,n,n+1,n+1,n+1)$ are χ-unique for each $(i,j) \in \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(3,4)\}$.*

Theorem 5.5. *If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1,n,n,n,n+2)$ are χ-unique for each $(i,j) \in \{(1,2),(2,1),(1,5),(5,1),(2,5),(5,2),(2,3)\}$.*

Let $K_{i,j}^{-sK_2}(n_1, n_2, n_3, n_4, n_5)$ denotes the graph obtained from $K(n_1, n_2, n_3, n_4, n_5)$ by deleting a set of s edges that forms a matching in $\langle A_i \cup A_j \rangle$. We now investigate the chromatically unique 5-partite graphs with $5n+1$ vertices and a set S of s edges deleted where the deleted edges induce a matching sK_2.

Theorem 5.6. *If $n \geq s+3$, then the graphs $K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1)$ are χ-unique.*

Proof. Let $F \sim K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1)$. It is sufficient to prove that $F = K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1)$. By Theorem 4.1(iii) and Lemma 2.5, we have $F \in \mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1)$ and $\alpha'(F) = s$. Let $F = G - S$ where $G = K(n-1, n-1, n+1, n+1, n+1)$. Next we consider the number of triangles of F. Let $e_i \in S$ and $t(e_i)$ be the number of triangles in G containing the edge e_i. Then one can see that $t(e_i) \leq 3n+3$. As $n-1 \leq n-1 < n+1 \leq n+1 \leq n+1$, we know that $t(e_i) = 3n+3$ if and only if e_i is an edge of the subgraph $\langle A_1 \cup A_2 \rangle$ in G. So,

$$t(F) \geq t(G) - s(3n+3);$$

where the equality holds if and only if each edge e_i in S is an edge of the subgraph $\langle A_1 \cup A_2 \rangle$ in G. Note that $t(F) = t(G) - s(3n+3)$ and $\alpha'(F) = s$. By Lemma 2.5, we know that $F = K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1)$. This completes the proof. □

Similarly to the proof of Theorem 5.6, we can prove Theorem 5.7.

Theorem 5.7. *If $n \geq s+4$, then the graphs $K_{1,2}^{-sK_2}(n-2, n, n+1, n+1, n+1)$ are χ-unique.*

We end this paper with the following two open problems.

1. Study the chromaticity of the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1)$ for each $(i,j) \in \{(1,4), (4,1), (2,3)\}$.
2. Study the chromaticity of the graphs $K_{1,2}^{-sK_2}(n, n, n, n, n+1)$, $K_{1,2}^{-sK_2}(n-1, n, n, n+1, n+1)$ and $K_{1,2}^{-sK_2}(n-1, n, n, n, n+2)$.

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