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ORIGINAL ARTICLE

Chromaticity of a family of 5-partite graphs



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Abstract Let $P(G, \lambda)$ be the chromatic polynomial of a graph G . Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H \mid H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 5-partite graphs G with $5n$ vertices according to the number of 6-independent partitions of G . Using these results, we investigate the chromaticity of G with certain stars or matching deleted parts. As a by-product, two new families of chromatically unique complete 5-partite graphs G with certain stars or matching deleted parts are obtained.

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1. Introduction

All graphs considered here are simple and finite. For a graph G , let $P(G, \lambda)$ be the chromatic polynomial of G . Two graphs G and H are said to be *chromatically equivalent* (or simply χ -equivalent), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by G under \sim is denoted by $[G]$. A graph G is *chromatically unique* (or simply χ -unique) if $H \cong G$ whenever $H \sim G$, i.e. $[G] = \{G\}$ up to isomorphism. For a set \mathcal{G} of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then \mathcal{G} is said to be χ -closed. Many families of χ -unique graphs are known (Koh and Teo, 1990, 1997).

For a graph G , let $V(G)$, $E(G)$, $t(G)$ and $\chi(G)$ be the vertex set, edge set, number of triangles and chromatic number of G , respectively. Let O_n be an edgeless graph with n vertices. Let $Q(G)$ and $K(G)$ be the number of induced subgraphs isomorphic to C_4 and complete subgraphs K_4 in G . Let S be a set of s edges in G . By $G - S$ (or $G - s$) we denote the graph obtained from G by deleting all edges in S , and $\langle S \rangle$ the graph induced by S . For $t \geq 2$ and $1 \leq n_1 \leq n_2 \leq \dots \leq n_t$, let $K(n_1, n_2, \dots, n_t)$ be a complete t -partite graph with partition sets V_i such that $|V_i| = n_i$ for $i = 1, 2, \dots, t$. In (Dong et al., 2001; Lau and Peng, 2010a,b, Lau et al., 2010; Roslan et al., 2010, 2011a, 2012a; Zhao et al., 2004; Zhao, 2004, 2005), the authors proved that certain families of complete t -partite graphs ($t = 2, 3, 4, 5$) with a matching or a star deleted are χ -unique. The case for the complete 6-partite graphs has been investigated in (2011c; 2012b; 2012c). In particular, Zhao et al. (2004) and Zhao (2005) investigated the chromaticity of complete 5-partite graphs G of $5n$ and $5n + 4$ vertices with certain stars or matching deleted parts. Roslan et al. (2011b) studied the chromaticity of complete 5-partite graphs G with $5n + 1$

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vertices for $i = 1, 2, 3$ with certain stars or matching deleted parts. As a continuation, in this paper, we characterize certain complete 5-partite graphs G with $5n$ vertices according to the number of 6-independent partitions of G . Using these results, we investigate the chromaticity of G with certain stars or matching deleted parts. As a by-product, two new families of chromatically unique complete 5-partite graphs with certain stars or matching deleted parts are obtained. These results generalized Theorems 3 and 4 in (Zhao, 2005).

2. Some lemmas and notations

Let $\mathcal{K}^{-s}(n_1, n_2, \dots, n_t)$ be the family $\{K(n_1, n_2, \dots, n_t) - S \mid S \subset E(K(n_1, n_2, \dots, n_t)) \text{ and } |S| = s\}$. For $n_1 \geq s + 1$, we denote by $K_{i_j}^{-K_{1s}}(n_1, n_2, \dots, n_t)$ (respectively, $K_{i_j}^{-sK_2}(n_1, n_2, \dots, n_t)$) the graph in $\mathcal{K}^{-s}(n_1, n_2, \dots, n_t)$ where the s edges in S induce a $K_{1,s}$ with center in V_i and all the end vertices in V_j (respectively, a matching with end vertices in V_i and V_j).

For a graph G and a positive integer r , a partition $\{A_1, A_2, \dots, A_r\}$ of $V(G)$, where r is a positive integer, is called an r -independent partition of G if every A_i is independent of G . Let $\alpha(G, r)$ denote the number of r -independent partitions in G . Then, we have $P(G, \lambda) = \sum_{r=1}^p \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - r + 1)$ and p is the number of vertices of G (see Read and Tutte, 1988). Therefore, $\alpha(G, r) = \alpha(H, r)$ for each $r = 1, 2, \dots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G, x) = \sum_{r=1}^p \alpha(G, r)x^r$ is called the σ -polynomial of G (see Brenti, 1992). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs G and H .

For disjoint graphs G and H , $G + H$ denotes the disjoint union of G and H . The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer to (West, 2001).

Lemma 2.1. (Koh and Teo, 1990) *Let G and H be two graphs with $H \sim G$, then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G, r) = \alpha(H, r)$ for $r \geq 1$, and $2K(G) - Q(G) = 2K(H) - Q(H)$. Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.*

Lemma 2.2. (Brenti, 1992) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, \dots, n_t), x) = \prod_{i=1}^t \sigma(O_{n_i}, x).$$

Lemma 2.3 Zhao et al., 2004. *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ and S be a set of some s edges of G . If $H \sim G - S$, then there is a complete graph $F = K(p_1, p_2, p_3, p_4, p_5)$ and a subset S' of $E(F)$ of some s' edges of F such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s$.*

Let $n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5$ be positive integers and $\{n_{i_1}, n_{i_2}, n_{i_3}, n_{i_4}, n_{i_5}\} = \{n_1, n_2, n_3, n_4, n_5\}$. If there exist two elements n_{i_1} and n_{i_2} in $\{n_1, n_2, n_3, n_4, n_5\}$ such that $n_{i_2} - n_{i_1} \geq 2$, $H' = K(n_{i_1} + 1, n_{i_2} - 1, n_{i_3}, n_{i_4}, n_{i_5})$ is called an *improvement* of $H = K(n_1, n_2, n_3, n_4, n_5)$.

Lemma 2.4 Zhao et al., 2004. *Suppose $n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5$ and $H' = K(n_{i_1} + 1, n_{i_2} - 1, n_{i_3}, n_{i_4}, n_{i_5})$ is an improvement of $H = K(n_1, n_2, n_3, n_4, n_5)$, then*

$$\alpha(H, 6) - \alpha(H', 6) = 2^{n_{i_2}-2} - 2^{n_{i_1}-1} \geq 2^{n_{i_1}-1}.$$

Let $G = K(n_1, n_2, n_3, n_4, n_5)$. For a graph $H = G - S$, where S is a set of some s edges of G , define $\alpha'(H) = \alpha(H, 6) - \alpha(G, 6)$. Clearly, $\alpha'(H) \geq 0$.

Lemma 2.5 Zhao et al., 2004. *Let $G = K(n_1, n_2, n_3, n_4, n_5)$. Suppose that $\min \{n_i \mid i = 1, 2, 3, 4, 5\} \geq s + 1 \geq 1$ and $H = G - S$, where S is a set of some s edges of G , then*

$$s \leq \alpha'(H) = \alpha(H, 6) - \alpha(G, 6) \leq 2^s - 1,$$

$\alpha'(H) = s$ if and only if the set of end-vertices of any $r \geq 2$ edges in S is not independent in H , and $\alpha'(H) = 2^s - 1$ if and only if S induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same A_i .

Lemma 2.6 Dong et al., 2001. *Let n_1, n_2 and s be positive integers with $3 \leq n_1 \leq n_2$, then*

- (1) $K_{1,2}^{-K_{1s}}(n_1, n_2)$ is χ -unique for $1 \leq s \leq n_2 - 2$,
- (2) $K_{2,1}^{-K_{1s}}(n_1, n_2)$ is χ -unique for $1 \leq s \leq n_1 - 2$, and
- (3) $K^{-sK_2}(n_1, n_2)$ is χ -unique for $1 \leq s \leq n_1 - 1$.

For a graph $G \in \mathcal{K}^{-s}(n_1, n_2, \dots, n_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively Type 2 and Type 3) if the vertices of the induced C_4 are in exactly two (respectively three and four) partite sets of $V(G)$. An example of induced C_4 of Types 1, 2 and 3 is shown in Fig. 1.

Suppose G is a graph in $\mathcal{K}^{-s}(n_1, n_2, \dots, n_t)$. Let $S_{ij} (1 \leq i \leq t, 1 \leq j \leq t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \geq 0$.

Lemma 2.7 Lau and Peng, 2010b. *For integer $t \geq 3$, let $F = K(n_1, n_2, \dots, n_t)$ be a complete t -partite graph and let $G = F - S$, where S is a set of s edges in F . If S induces a matching in F , then*

$$\begin{aligned} Q(G) = & Q(F) - \sum_{1 \leq i < j \leq t} (n_i - 1)(n_j - 1)s_{ij} + \binom{s}{2} - \sum_{1 \leq i < j < l \leq t} s_{ij}s_{il} \\ & - \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq k < l \leq t \\ i < k}} s_{ij}s_{kl} + \sum_{1 \leq i < j \leq t} \left[s_{ij} \sum_{k \notin \{i, j\}} \binom{n_k}{2} \right] \\ & + \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq i < k < l \leq t \\ j \notin \{k, l\}}} s_{ij}s_{kl}, \end{aligned}$$

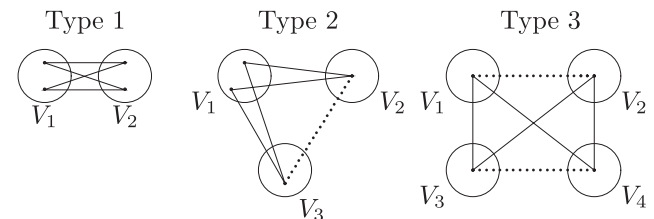


Figure 1 Three types of induced C_4 .

and

$$K(G) = K(F) - \sum_{1 \leq i < j \leq t} \left[s_{ij} \sum_{\substack{1 \leq k < l \leq t \\ \{i,j\} \cap \{k,l\} = \emptyset}} n_k n_l \right] + \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq i < k < l \leq t \\ j \notin \{k,l\}}} s_{ij} s_{kl}.$$

By using Lemma 2.7, we obtain the following.

Lemma 2.8. *Let $F = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph and let $G = F - S$ where S is a set of s edges in F . If S induces a matching in F , then*

$$Q(G) = Q(F) - \sum_{1 \leq i < j \leq 5} (n_i - 1)(n_j - 1)s_{ij} + \binom{s}{2} - s_{12}(s_{13} + s_{14} + s_{15} + s_{23} + s_{24} + s_{25}) - s_{13}(s_{14} + s_{15} + s_{23} + s_{34} + s_{35}) - s_{14}(s_{15} + s_{24} + s_{34} + s_{45}) - s_{15}(s_{25} + s_{35} + s_{45}) - s_{23}(s_{24} + s_{25} + s_{34} + s_{35}) - s_{24}(s_{25} + s_{34} + s_{45}) - s_{25}(s_{35} + s_{45}) - s_{34}(s_{35} + s_{45}) - s_{35}s_{45} + \sum_{1 \leq i < j \leq 5} \left[s_{ij} \sum_{k \notin \{i,j\}} \binom{n_k}{2} \right],$$

and

$$K(G) = K(F) - \sum_{1 \leq i < j \leq 5} \left[s_{ij} \sum_{\substack{1 \leq k < l \leq 5 \\ \{i,j\} \cap \{k,l\} = \emptyset}} n_k n_l \right] + s_{12}(s_{34} + s_{35} + s_{45}) + s_{13}(s_{24} + s_{25} + s_{45}) + s_{14}(s_{23} + s_{25} + s_{35}) + s_{15}(s_{23} + s_{24} + s_{34}) + s_{23}s_{45} + s_{24}s_{35} + s_{25}s_{34}.$$

3. Characterization

In this section, we shall characterize certain complete 5-partite graphs $G = K(n_1, n_2, n_3, n_4, n_5)$ according to the number of 6-independent partitions of G where $n_5 - n_1 \leq 4$.

Theorem 3.1. *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 = 5n$ and $n_5 - n_1 \leq 4$. Define $\theta(G) = [\alpha(G, 6) - 2^{n+1} - 2^{n-1} + 5] / 2^{n-2}$. Then*

- (i) $\theta(G) = 0$ if and only if $G = K(n, n, n, n, n)$;
- (ii) $\theta(G) = 1$ if and only if $G = K(n - 1, n, n, n, n + 1)$;
- (iii) $\theta(G) = 2$ if and only if $G = K(n - 1, n - 1, n, n + 1, n + 1)$;
- (iv) $\theta(G) = 2\frac{1}{2}$ if and only if $G = K(n - 2, n, n, n + 1, n + 1)$;
- (v) $\theta(G) = 3\frac{1}{2}$ if and only if $G = K(n - 2, n - 1, n + 1, n + 1, n + 1)$;
- (vi) $\theta(G) = 4$ if and only if $G = K(n - 1, n - 1, n, n + 2)$;

- (vii) $\theta(G) = 4\frac{1}{4}$ if and only if $G = K(n - 3, n, n + 1, n + 1, n + 1)$;
- (viii) $\theta(G) = 4\frac{1}{2}$ if and only if $G = K(n - 2, n, n, n + 2)$;
- (ix) $\theta(G) = 5$ if and only if $G = K(n - 1, n - 1, n - 1, n + 1, n + 2)$;
- (x) $\theta(G) = 5\frac{1}{2}$ if and only if $G = K(n - 2, n - 1, n, n + 1, n + 2)$;
- (xi) $\theta(G) = 7$ if and only if $G = K(n - 2, n - 2, n + 1, n + 1, n + 2)$;
- (xii) $\theta(G) = 8\frac{1}{2}$ if and only if $G = K(n - 2, n - 1, n - 1, n + 2, n + 2)$;
- (xiii) $\theta(G) = 9$ if and only if $G = K(n - 2, n - 2, n, n + 2, n + 2)$;
- (xiv) $\theta(G) = 11$ if and only if $G = K(n - 1, n - 1, n - 1, n, n + 3)$;

Proof. In order to complete the proof of the theorem, we first give a table for the θ -value of various complete 5-partite graphs with $5n$ vertices as shown in Table 1.

By using Table 1, Lemma 2.4 and the definition of improvement, the proof is complete. \square

4. Chromatically closed 5-partite graphs

In this section, we obtain a χ -closed family of graphs from the graphs in Theorem 3.1.

Theorem 4.1. *The family of graphs $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5)$ where $n_1 + n_2 + n_3 + n_4 + n_5 = 5n$, $n_5 - n_1 \leq 4$ and $n_1 \geq s + 5$ is χ -closed.*

Proof. By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 (i), (ii), ..., (xiv) by G_1, G_2, \dots, G_{14} , respectively. Suppose $H \sim G_i - S$. It suffices to show that $H \in \{G_i - S\}$. By Lemma 2.3, we know there exists a complete 5-partite graph $F = (p_1, p_2, p_3, p_4, p_5)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s \geq 0$.

Case (i). Let $G = G_1$ with $n \geq s + 2$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n, n, n, n, n)$. By Lemma 2.5, we have $\alpha(G - S, 6) = \alpha(G, 6) + \alpha'(G - S)$ with $s \leq \alpha'(G - S) \leq 2^s - 1$, $\alpha(F - S', 6) = \alpha(F, 6) + \alpha'(F - S')$ with $0 \leq s' \leq \alpha'(F - S')$.

Hence,

$$\alpha(F - S', 6) - \alpha(G - S, 6) = \alpha(F, 6) - \alpha(G, 6) + \alpha'(F - S') - \alpha'(G - S).$$

By the definition, $\alpha(F, 6) - \alpha(G, 6) = 2^{n-2}(\theta(F) - \theta(G))$. By Theorem 3.1, $\theta(F) \geq 0$. Suppose $\theta(F) > 0$, then

$$\begin{aligned} \alpha(F - S', 6) - \alpha(G - S, 6) &\geq 2^{n-2} + \alpha'(F - S') - \alpha'(G - S) \\ &\geq 2^s + \alpha'(F - S') - 2^s + 1, \\ &\geq 1, \end{aligned}$$

contradicting $\alpha(F - S', 6) = \alpha(G - S, 6)$. Hence, $\theta(F) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathcal{K}^{-s}(n, n, n, n, n)$.

Case (ii). Let $G = G_2$ with $n \geq s + 3$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n - 1, n, n, n + 1)$. By Lemma 2.5, we have

Table 1 Some complete 5-partite graphs with $5n$ vertices and their θ -values.

$G_i(1 \leq i \leq 23)$	$\theta(G_i)$	$G_i(24 \leq i \leq 46)$	$\theta(G_i)$
$G_1 = K(n,n,n,n,n)$	0	$G_{24} = K(n-4,n,n+1,n+1,n+2)$	$8\frac{1}{8}$
$G_2 = K(n-1,n,n,n,n+1)$	1	$G_{25} = K(n-4,n,n,n+2,n+2)$	$10\frac{1}{8}$
$G_3 = K(n-1,n-1,n,n+1,n+1)$	2	$G_{26} = K(n-4,n,n,n+1,n+3)$	$14\frac{1}{8}$
$G_4 = K(n-2,n,n,n+1,n+1)$	$2\frac{1}{2}$	$G_{27} = K(n-1,n-1,n-1,n-1,n+4)$	26
$G_5 = K(n-1,n-1,n,n,n+2)$	4	$G_{28} = K(n-2,n-1,n-1,n,n+4)$	$26\frac{1}{2}$
$G_6 = K(n-2,n,n,n,n+2)$	$4\frac{1}{2}$	$G_{29} = K(n-2,n-2,n-1,n+2,n+3)$	16
$G_7 = K(n-1,n-1,n-1,n+1,n+2)$	5	$G_{30} = K(n-3,n-1,n-1,n+2,n+3)$	$16\frac{1}{4}$
$G_8 = K(n-2,n-1,n+1,n+1,n+1)$	$3\frac{1}{2}$	$G_{31} = K(n-3,n-2,n+1,n+2,n+2)$	$10\frac{3}{4}$
$G_9 = K(n-2,n-1,n,n+1,n+2)$	$5\frac{1}{2}$	$G_{32} = K(n-3,n-2,n+1,n+1,n+3)$	14
$G_{10} = K(n-3,n,n+1,n+1,n+1)$	$4\frac{1}{4}$	$G_{33} = K(n-4,n-1,n+1,n+2,n+2)$	11
$G_{11} = K(n-3,n,n,n+1,n+2)$	$6\frac{1}{4}$	$G_{34} = K(n-4,n-1,n+1,n+1,n+3)$	$15\frac{1}{8}$
$G_{12} = K(n-1,n-1,n-1,n,n+3)$	11	$G_{35} = K(n-3,n-2,n,n+2,n+3)$	$16\frac{3}{8}$
$G_{13} = K(n-2,n-1,n,n,n+3)$	$11\frac{1}{2}$	$G_{36} = K(n-4,n-1,n,n+2,n+3)$	17
$G_{14} = K(n-3,n,n,n,n+3)$	$12\frac{1}{4}$	$G_{37} = K(n-5,n+1,n+1,n+1,n+2)$	$10\frac{1}{16}$
$G_{15} = K(n-2,n-1,n-1,n+2,n+2)$	$8\frac{1}{2}$	$G_{38} = K(n-5,n,n+1,n+2,n+2)$	$12\frac{1}{16}$
$G_{16} = K(n-2,n-1,n-1,n+1,n+3)$	$12\frac{1}{2}$	$G_{39} = K(n-5,n,n+1,n+1,n+3)$	$16\frac{1}{16}$
$G_{17} = K(n-2,n-2,n+1,n+1,n+2)$	7	$G_{40} = K(n-5,n,n,n+2,n+3)$	$18\frac{1}{16}$
$G_{18} = K(n-3,n-1,n+1,n+1,n+2)$	$7\frac{1}{4}$	$G_{41} = K(n-3,n-3,n+2,n+2,n+2)$	$14\frac{1}{8}$
$G_{19} = K(n-2,n-2,n,n+2,n+2)$	9	$G_{42} = K(n-3,n-3,n+1,n+2,n+3)$	$18\frac{1}{8}$
$G_{20} = K(n-2,n-2,n,n+1,n+3)$	13	$G_{43} = K(n-4,n-2,n+2,n+2,n+2)$	14
$G_{21} = K(n-3,n-1,n,n+2,n+2)$	$9\frac{1}{4}$	$G_{44} = K(n-4,n-2,n+1,n+2,n+3)$	19
$G_{22} = K(n-3,n-1,n,n+1,n+3)$	$13\frac{1}{4}$	$G_{45} = K(n-6,n+1,n+1,n+2,n+2)$	$14\frac{1}{32}$
$G_{23} = K(n-4,n+1,n+1,n+1,n+1)$	$6\frac{1}{8}$	$G_{46} = K(n-6,n+1,n+1,n+1,n+3)$	$18\frac{1}{32}$

$\alpha(G - S, 6) = \alpha(G, 6) + \alpha'(G - S)$ with $s \leq \alpha'(G - S) \leq 2^s - 1$,
 $\alpha(F - S', 6) = \alpha(F, 6) + \alpha'(F - S')$ with $0 \leq s' \leq \alpha'(F - S')$.

Hence,

$$\alpha(F - S', 6) - \alpha(G - S, 6) = \alpha(F, 6) - \alpha(G, 6) + \alpha'(F - S') - \alpha'(G - S).$$

By the definition, $\alpha(F, 6) - \alpha(G, 6) = 2^{n-2}(\theta(F) - \theta(G))$. Suppose $\theta(F) \neq \theta(G)$. Then, we consider two subcases.

Subcase (a). $\theta(F) < \theta(G)$. By Theorem 3.1, $F = G_1$ and $H = G_1 - S' \in \{G_1 - S'\}$. However, $G - S \notin \{G_1 - S'\}$ since by Case (i) above, $\{G_1 - S'\}$ is χ -closed, a contradiction.

Subcase (b). $\theta(F) > \theta(G)$. By Theorem 3.1, $\alpha(F, 6) - \alpha(G, 6) \geq 2^{n-2}$. So,

$$\alpha(F - S', 6) - \alpha(G - S, 6) \geq 2^{n-2} + \alpha'(F - S') - \alpha'(G - S) \geq 2^s + \alpha'(F - S') - 2^s + 1 \geq 1,$$

contradicting $\alpha(F - S', 6) = \alpha(G - S, 6)$. Hence, $\theta(F) = \theta(G) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathcal{H}^{-s}(n-1, n, n, n, n+1)$.

Using Table 1, we can prove (iii) to (xiv) in a similar way. This completes the proof. \square

5. Chromatically unique 5-partite graphs

The following results give two families of chromatically unique complete 5-partite graphs having $5n$ vertices with a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$ and a matching sK_2 , respectively.

Theorem 5.1. *em* The graphs $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5)$ where $n_1 + n_2 + n_3 + n_4 + n_5 = 5n$, $n_5 - n_1 \leq 4$ and $n_1 \geq s + 5$ are χ -unique for $1 \leq i \neq j \leq 5$.

Proof. By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 (i),(ii),..., (xiv) by G_1, G_2, \dots, G_{14} , respectively. The proof for each graph obtained from G_i ($i = 1, 2, \dots, 14$) is similar, so we only give the detailed proof for the graphs obtained from G_2 below.

By Lemma 2.5 and Case 2 of Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1) = \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1) | (i, j) \in \{(1, 2), (2, 1), (1, 5), (5, 1), (2, 3), (4, 5), (5, 4)\}\}$ is χ -closed for $n \geq s + 3$. Note that

$$\begin{aligned} t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1)) &= \\ t(G_2) - s(3n+1) &\text{ for } (i, j) \in \{(1, 2), (2, 1)\}, \\ t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1)) &= \\ t(G_2) - 3sn &\text{ for } (i, j) \in \{(1, 5), (5, 1)\}, \\ t(K_{2,3}^{-K_{1,s}}(n-1, n, n, n, n+1)) &= t(G_2) - 3sn, \\ t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1)) &= \\ t(G_2) - s(3n-1) &\text{ for } (i, j) \in \{(4, 5), (5, 4)\}. \end{aligned}$$

By Lemmas 2.2 and 2.6, we conclude that $\sigma(K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1)) \neq \sigma(K_{i,i}^{-K_{1,s}}(n-1, n, n, n, n+1))$ for each $(i, j) \in \{(1, 2), (1, 5), (4, 5)\}$. We now show that $K_{2,3}^{-K_{1,s}}(n-1, n, n, n, n+1)$ and $K_{i,j}^{-K_{1,s}}(n-1, n, n, n, n+1)$ are not χ -equivalent for $(i, j) \in \{(1, 5), (5, 1)\}$. We have

$$\begin{aligned} Q(K_{2,3}^{-K_{1,s}}(n-1, n, n, n, n+1)) &= Q(G_2) - s(n-1)^2 + \binom{s}{2} \\ &\quad + s \left(\binom{n-1}{2} + \binom{n}{2} + \binom{n+1}{2} \right); \end{aligned}$$

$$\begin{aligned} Q(K_{ij}^{-K_{1,s}}(n-1, n, n, n, n+1)) &= Q(G_2) - sn(n-2) + \binom{s}{2} \\ &\quad + 3s \binom{n}{2} \text{ for } (i, j) \\ &\in \{(1, 5), (5, 1)\}; \end{aligned}$$

with

$$\begin{aligned} Q(K_{2,3}^{-K_{1,s}}(n-1, n, n, n, n+1)) - Q(K_{ij}^{-K_{1,s}}(n-1, n, n, n, n+1)) \\ = 0 \end{aligned}$$

since $s_{ij} = 0$ if $(i, j) \neq \{(2, 3), (1, 5), (5, 1)\}$.

We also obtain

$$\begin{aligned} K(K_{2,3}^{-K_{1,s}}(n-1, n, n, n, n+1)) &= K(G_2) - s(3n^2 - 1); \\ K(K_{ij}^{-K_{1,s}}(n-1, n, n, n, n+1)) &= K(G_2) - 3sn^2 \text{ for } (i, j) \in \{(1, 5), (5, 1)\}, \end{aligned}$$

with

$$K(K_{2,3}^{-K_{1,s}}(n-1, n, n, n, n+1)) - K_{ij}^{-K_{1,s}}(n-1, n, n, n, n+1) = s$$

since $s_{ij} = 0$ if $(i, j) \neq \{(2, 3), (1, 5), (5, 1)\}$.

This means that $2K(K_{ij}^{-K_{1,s}}(n-1, n, n, n, n+1)) - Q(K_{ij}^{-K_{1,s}}(n-1, n, n, n, n+1)) \neq 2K(K_{2,3}^{-K_{1,s}}(n-1, n, n, n, n+1)) - Q(K_{2,3}^{-K_{1,s}}(n-1, n, n, n, n+1))$, contradicting Lemma 2.1. Hence, $K_{ij}^{-K_{1,s}}(n-1, n, n, n, n+1)$ is χ -unique where $n \geq s + 3$ for $1 \leq i \neq j \leq 5$.

The proof is thus complete. \square

Theorem 5.2. *The graphs $K_{1,2}^{-sK_2}(n_1, n_2, n_3, n_4, n_5)$ where $n_1 + n_2 + n_3 + n_4 + n_5 = 5n$, $n_5 - n_1 \leq 4$ and $n_1 \geq s + 5$ are χ -unique.*

Proof. By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 (i), (ii), ..., (xiv) by G_1, G_2, \dots, G_{14} , respectively. For a graph $K(p_1, p_2, p_3, p_4, p_5)$, let $S = \{e_1, e_2, \dots, e_s\}$ be the set of s edges in $E(K(p_1, p_2, p_3, p_4, p_5))$ and let $t(e_i)$ denote the number of triangles containing e_i in $K(p_1, p_2, p_3, p_4, p_5)$. The proofs for each graph obtained from $G_i (i = 1, 2, \dots, 14)$ are similar, so we only give the proof for the graph obtained from G_1 and G_2 as follows.

Suppose $H \sim G = K_{1,2}^{-sK_2}(n, n, n, n, n)$ for $n \geq s + 2$. By Theorem 4.1 and Lemma 2.1, $H \in \mathcal{H}^{-s}(n, n, n, n, n)$ and $\alpha'(H) = \alpha'(G) = s$. Let $H = F - S$ where $F = K(n, n, n, n, n)$. Clearly, $t(e_i) \leq 3n$ for each $e_i \in S$. So,

$$t(H) \geq t(F) - 3ns,$$

with equality holds only if $t(e_i) = 3n$ for all $e_i \in S$. Since $t(H) = t(G) = t(F) - 3ns$, the equality above holds with $t(e_i) = 3n$ for all $e_i \in S$. Therefore each edge in S has an end-vertex in V_i and another end-vertex in $V_j (1 \leq i < j \leq 5)$. Moreover, S must induce a matching in F . Otherwise, equality does not hold or $\alpha'(H) > s$. By Lemma 2.8, we obtain

$$Q(G) = Q(F) - s(n-1)^2 + \binom{s}{2} + 3s \binom{n}{2},$$

whereas

$$\begin{aligned} Q(H) &= Q(F) - s(n-1)^2 + \binom{s}{2} - s_{12}(s_{13} + s_{14} + s_{15} + s_{23} + s_{24} + s_{25}) \\ &\quad - s_{13}(s_{14} + s_{15} + s_{23} + s_{34} + s_{35}) - s_{14}(s_{15} + s_{24} + s_{34} + s_{45}) \\ &\quad - s_{15}(s_{25} + s_{35} + s_{45}) - s_{23}(s_{24} + s_{25} + s_{34} + s_{35}) - s_{24}(s_{25} + s_{34} + s_{45}) \\ &\quad - s_{25}(s_{35} + s_{45}) - s_{34}(s_{35} + s_{45}) - s_{35}s_{45} + 3s \binom{n}{2} \\ &\leq Q(G) \end{aligned}$$

and the equality holds if and only if $s = s_{ij}$ for $1 \leq i < j \leq 5$, or $s = s_{ij} + s_{kl}$ for $1 \leq i < j \leq 5, 1 \leq k < l \leq 5, \{i, j\} \cap \{k, l\} = \emptyset$. Moreover, $K(G) = K(F) - 3sn^2$ whereas

$$\begin{aligned} K(H) &= K(F) - 3sn^2 + s_{12}(s_{34} + s_{35} + s_{45}) + s_{13}(s_{24} + s_{25} + s_{45}) \\ &\quad + s_{14}(s_{23} + s_{25} + s_{35}) + s_{15}(s_{23} + s_{24} + s_{34}) + s_{23}s_{45} + s_{24}s_{35} + s_{25}s_{34} \\ &\geq K(G) \end{aligned}$$

and the equality holds if and only if $s = s_{ij}$ for $1 \leq i < j \leq 5$. Hence,

$$Q(H) - 2K(H) \leq Q(G) - 2K(G)$$

and the equality holds if and only if $s = s_{ij}$ for $1 \leq i < j \leq 5$. Consequently, $\langle S \rangle = sK_2$ with $H \cong G$.

Suppose $H \sim G = K_{1,2}^{-sK_2}(n-1, n, n, n, n+1)$ for $n \geq s + 3$. By Theorem 4.1 and Lemma 2.1, $H \in \mathcal{H}^{-s}(n-1, n, n, n, n+1)$ and $\alpha'(H) = \alpha'(G) = s$. Let $H = F - S$ where $F = K(n-1, n, n, n, n+1)$. Clearly, $t(e_i) \leq 3n + 1$ for each $e_i \in S$. So,

$$t(H) \geq t(F) - s(3n + 1),$$

with equality holds only if $t(e_i) = 3n + 1$ for all $e_i \in S$. Since $t(H) = t(G) = t(F) - s(3n + 1)$, the equality above holds with $t(e_i) = 3n + 1$ for all $e_i \in S$. Therefore each edge in S has an end-vertex in V_1 and another end-vertex in $V_j (2 \leq j \leq 4)$. Moreover, S must induce a matching in F . Otherwise, equality does not hold or $\alpha'(H) > s$. By Lemma 2.8, we obtain

$$Q(G) = Q(F) - s(n-2)(n-1) + \binom{s}{2} + s \left(2 \binom{n}{2} + \binom{n+1}{2} \right),$$

whereas

$$\begin{aligned} Q(H) &= Q(F) - s(n-2)(n-1) + \binom{s}{2} \\ &\quad - (s_{12}s_{13} + s_{12}s_{14} + s_{13}s_{14}) + s \left(2 \binom{n}{2} + \binom{n+1}{2} \right) \\ &\leq Q(G), \end{aligned}$$

and the equality holds if and only if $s = s_{1j} (2 \leq j \leq 4)$. Moreover, $K(G) = K(H) = K(F) - s(3n^2 + 2n)$. Hence, $2K(G) - Q(G) = 2K(H) - Q(H)$ if and only if $\langle S \rangle \cong sK_2$ with $H \cong G$.

Thus the proof is complete. \square

Remark: Our results generalized Theorems 3 and 4 in (Zhao, 2005).

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