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A Remark on Chromatically Unique 5-Partite Graphs

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Abstract

Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H|H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, two new families of chromatically unique complete 5-partite graphs G having $5n+4$ vertices with certain star or matching deleted are obtained.

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1 Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G,\lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G,l) = P(H,l)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. Many families of χ-unique graphs are known (see [3,4]).

For a graph G, let $V(G)$, $E(G)$, $t(G)$ and $\chi(G)$ be the vertex set, edge set, number of triangles and chromatic number of G, respectively. Let O_n be an edgeless graph with n vertices. Let $Q(G)$ and $K(G)$ be the number of induced subgraphs isomorphic to C_4 and complete subgraph K_4 in G. Let S be a set of s edges in G. By $G - S$ (or $G - s$) we denote the graph obtained from G by deleting all edges in S, and $\langle S\rangle$ the graph induced by S. For $t \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_t$, let $K(n_1, n_2, \cdots, n_t)$ be a complete $t-$partite graph with partition sets V_i such that $|V_i| = n_i$ for $i = 1, 2, \cdots, t$. In [2,5-7,9-11,15-17], the authors proved that certain families of complete t-partite graphs ($t = 2, 3, 4, 5$) with a matching or a star deleted are χ-unique. In particular, Zhao et al. [15,16] investigated the chromaticity of complete 5-partite graphs G of $5n$ and $5n + 4$ vertices with certain star or matching deleted. In [12,13], Roslan et al. studied the chromaticity of complete 5-partite graphs G with $5n + i$ vertices for $i = 0, 1, 2, 3$ with certain star or matching deleted. The case for chromaticity of complete 5-partite graphs G with $5n$ vertices in [12] generalized the results obtained in Zhao's paper [15]. As a continuation, in this paper, we characterize certain complete 5-partite graphs G with $5n+4$ vertices according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, two new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

2 Some Lemmas and Notations

Let $\mathcal{K}^{-s}(n_1, n_2, \cdots, n_t)$ be the family $\{K(n_1, n_2, \cdots, n_t)-S|\ S \subset E(K(n_1, n_2, \cdots, n_t))$ and $|S| = s\}$. For $n_1 \geq s + 1$, we denote by $K_{i,j}^{-K_{1,s}}(n_1, n_2, \cdots, n_t)$ (respectively, $K_{i,j}^{-sK_2}(n_1, n_2, \cdots, n_t)$) the graph in $K^{-s}(n_1, n_2, \cdots, n_t)$ where the s edges in S induce a $K_{1,s}$ with center in V_i and all the end vertices in V_j (respectively, a matching with end vertices in V_i and V_j).

For a graph G and a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G, r)$ denote the number of r-independent partitions in G. Then, we have $P(G,\lambda) = \sum_{r=1}^{p} \alpha(G,r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ and p is the number of vertices of G(see [8]). Therefore, $\alpha(G,r) = \alpha(H,r)$ for each $r = 1,2,\cdots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G,x) = \sum_{r=1}^{p} \alpha(G,r)x^r$ is called the σ-polynomial of G (see [1]). Clearly, $P(G,\lambda) = P(H,\lambda)$ implies that $\sigma(G,x) = \sigma(H,x)$ for any graphs G and H.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer to [14].

Lemma 2.1 *(Koh and Teo [3]) Let G and H be two graphs with $H \sim G$, then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G,r) = \alpha(H,r)$ for $r \geq 1$, and $2K(G) - Q(G) = 2K(H) - Q(H)$. Note that if $\chi(G) = 3$, then $G \sim H$ implies that $Q(G) = Q(H)$.*

Lemma 2.2 *(Brenti [1]) Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G,x)\sigma(H,x).$$

In particular,

$$\sigma(K(n_1,n_2,\cdots,n_t),x) = \prod_{i=1}^{t} \sigma(O_{n_i},x)$$

Lemma 2.3 *(Zhao et al. [15]) Let $G = K(n_1,n_2,n_3,n_4,n_5)$ and S be a set of some s edges of G. If $H \sim G - S$, then there is a complete graph $F = K(p_1,p_2,p_3,p_4,p_5)$ and a subset S' of $E(F)$ of some s' edges of F such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s$.*

Let $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ be positive integers and $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}\} = \{x_1,x_2,x_3,x_4,x_5\}$. If there exist two elements x_{i_1} and x_{i_2} in $\{x_1,x_2,x_3,x_4,x_5\}$ such that $x_{i_2} - x_{i_1} \geq 2$, $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5})$ is called an *improvement* of $H = K(x_1,x_2,x_3,x_4,x_5)$.

Lemma 2.4 *(Zhao et al. [15]) Suppose $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ and $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5}\}$ is an improvement of $H = K(x_1,x_2,x_3,x_4,x_5)$, then*

$$\alpha(H,6) - \alpha(H',6) = 2^{x_{i_2}-2} - 2^{x_{i_1}-1} \geq 2^{x_{i_1}-1}.$$

Let $G = K(n_1,n_2,n_3,n_4,n_5)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H,6) - \alpha(G,6)$. Clearly, $\alpha'(H) \geq 0$.

Lemma 2.5 *(Zhao et al. [15]) Let $G = K(n_1, n_2, n_3, n_4, n_5)$. Suppose that min $\{n_i | i = 1, 2, 3, 4, 5\} \geq s + 1 \geq 1$ and $H = G - S$, where S is a set of some s edges of G, then*

$$s \leq \alpha'(H) = \alpha(H, 6) - \alpha(G, 6) \leq 2^s - 1,$$

$\alpha'(H) = s$ if and only if the set of end-vertices of any $r \geq 2$ edges in S is not independent in H, and $\alpha'(H) = 2^s - 1$ if and only if S induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same A_i.

Lemma 2.6 *(Dong et al. [2]) Let n_1, n_2 and s be positive integers with $3 \leq n_1 \leq n_2$, then*

(1) $K_{1,2}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leq s \leq n_2 - 2$,

(2) $K_{2,1}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leq s \leq n_1 - 2$, and

(3) $K^{-sK_2}(n_1, n_2)$ is χ-unique for $1 \leq s \leq n_1 - 1$.

For a graph $G \in K^{-s}(n_1, n_2, \cdots, n_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively Type 2 and Type 3) if the vertices of the induced C_4 are in exactly two (respectively three and four) partite sets of $V(G)$. An example of induced C_4 of Types 1, 2 and 3 are shown in Figure 1.

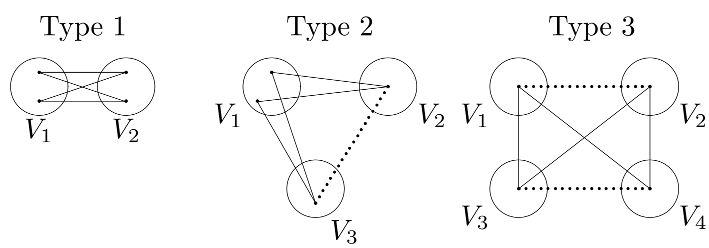

FIGURE 1. Three types of induced C_4

Suppose G is a graph in $K^{-s}(n_1, n_2, \cdots, n_t)$. Let S_{ij} $(1 \leq i \leq t, 1 \leq j \leq t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \geq 0$.

Lemma 2.7 *(Lau and Peng [6]) For integer $t \geq 3$, let $F = K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph and let $G = F - S$, where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) &= Q(F) - \sum_{1\leq i<j\leq t} (n_i - 1)(n_j - 1)s_{ij} + \binom{s}{2} - \sum_{1\leq i<j<l\leq t} s_{ij}s_{il} - \\ & \sum_{\substack{1 \leq i < j \leq t \\ 1 \leq k < l \leq t \\ i < k}} s_{ij}s_{kl} + \sum_{1\leq i<j\leq t} \left[s_{ij} \sum_{k\notin\{i,j\}} \binom{n_k}{2} \right] + \end{aligned}$$

$$\sum_{\substack{1 \le i < j \le t \\ 1 \le i < k < l \le t \\ j \notin \{k,l\}}} s_{ij}s_{kl},$$

and

$$K(G) = K(F) - \sum_{1\le i<j\le t}\Big[s_{ij} \sum_{\substack{1 \le k < l \le t \\ \{i,j\}\cap\{k,l\}=\emptyset}} n_k n_l\Big] + \sum_{\substack{1 \le i < j \le t \\ 1 \le i < k < l \le t \\ j \notin \{k,l\}}} s_{ij}s_{kl}.$$

By using Lemma 2.7, we obtain the following.

Lemma 2.8 *Let $F = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) &= Q(F) - \sum_{1\le i<j\le 5}(n_i-1)(n_j-1)s_{ij} + \binom{s}{2} - s_{12}(s_{13}+s_{14}+s_{15}+s_{23} \\ &\quad +s_{24}+s_{25}) - s_{13}(s_{14}+s_{15}+s_{23}+s_{34}+s_{35}) - s_{14}(s_{15}+s_{24}+s_{34} \\ &\quad +s_{45}) - s_{15}(s_{25}+s_{35}+s_{45}) - s_{23}(s_{24}+s_{25}+s_{34}+s_{35}) - s_{24}(s_{25} \\ &\quad +s_{34}+s_{45}) - s_{25}(s_{35}+s_{45}) - s_{34}(s_{35}+s_{45}) - s_{35}s_{45} \\ &\quad + \sum_{1\le i<j\le 5}\Big[s_{ij}\sum_{k\notin\{i,j\}}\binom{n_k}{2}\Big], \end{aligned}$$

and

$$\begin{aligned} K(G) &= K(F) - \sum_{1\le i<j\le 5}\Big[s_{ij} \sum_{\substack{1 \le k < l \le 5 \\ \{i,j\}\cap\{k,l\}=\emptyset}} n_k n_l\Big] + s_{12}(s_{34}+s_{35}+s_{45}) \\ &\quad +s_{13}(s_{24}+s_{25}+s_{45}) + s_{14}(s_{23}+s_{25}+s_{35}) + s_{15}(s_{23}+s_{24}+s_{34}) \\ &\quad +s_{23}s_{45}+s_{24}s_{35}+s_{25}s_{34}. \end{aligned}$$

3 Characterization

In this section, we shall characterize certain complete 5-partite graphs $G = K(n_1, n_2, n_3, n_4, n_5)$ according to the number of 6-independent partitions of G where $n_5 - n_1 \le 4$.

Theorem 3.1 *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 4$ and $n_5 - n_1 \le 4$. Define $\theta(G) = [\alpha(G, 6) - 2^{n+2} - 2^{n-1} + 5]/2^{n-1}$. Then* *(i)* $\theta(G) = 0$ *if and only if* $G = K(n, n+1, n+1, n+1, n+1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(n, n, n+1, n+1, n+2)$;

(iii) $\theta(G) = 1\frac{1}{2}$ *if and only if* $G = K(n-1, n+1, n+1, n+1, n+2)$;

(iv) $\theta(G) = 2$ *if and only if* $G = K(n, n, n, n+2, n+2)$;

(v) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(n-1, n, n+1, n+2, n+2)$;

(vi) $\theta(G) = 3\frac{1}{4}$ *if and only if* $G = K(n-2, n+1, n+1, n+2, n+2)$;

(vii) $\theta(G) = 4$ *if and only if* $G = K(n, n, n, n+1, n+3)$ *or* $G = K(n-1, n-1, n+2, n+2, n+2)$;

(viii) $\theta(G) = 4\frac{1}{4}$ *if and only if* $G = K(n-2, n, n+2, n+2, n+2)$;

(ix) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(n-1, n, n+1, n+1, n+3)$;

(x) $\theta(G) = 5\frac{1}{2}$ *if and only if* $G = K(n-1, n, n, n+2, n+3)$;

(xi) $\theta(G) = 6$ *if and only if* $G = K(n-1, n-1, n+1, n+2, n+3)$;

(xii) $\theta(G) = 9$ *if and only if* $G = K(n-1, n-1, n, n+3, n+3)$;

(xiii) $\theta(G) = 11$ *if and only if* $G = K(n, n, n, n, n+4)$.

Proof. In order to complete the proof of the theorem, we first give a table for the θ-value of various complete 5-partite graphs with $5n+4$ vertices as shown in Table 1.

By the definition of improvement, we have the following:

(i) G_1 is the improvement of G_2 and G_3 with $\theta(G_2) = 1$ and $\theta(G_3) = 1\frac{1}{2}$;

(ii) G_2 is the improvement of G_3, G_4, G_5, G_6 and G_7 with $\theta(G_3) = 1\frac{1}{2}$, $\theta(G_4) = 2$, $\theta(G_5) = 4$, $\theta(G_6) = 2\frac{1}{2}$ and $\theta(G_7) = 4\frac{1}{2}$;

(iii) G_3 is the improvement of G_6, G_7, G_8 and G_9 with $\theta(G_6) = 2\frac{1}{2}$, $\theta(G_7) = 4\frac{1}{2}$, $\theta(G_8) = 3\frac{1}{4}$ and $\theta(G_9) = 5\frac{1}{4}$;

(iv) G_4 is the improvement of G_5, G_6 and G_{10} with $\theta(G_5) = 4$, $\theta(G_6) = 2\frac{1}{2}$ and $\theta(G_{10}) = 5\frac{1}{2}$;

(v) G_5 is the improvement of G_7, G_{10}, G_{11} and G_{12} with $\theta(G_7) = 4\frac{1}{2}$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{11}) = 11$ and $\theta(G_{12}) = 11\frac{1}{2}$;

(vi) G_6 is the improvement of G_7, G_8, G_{10}, G_{13}, G_{14}, G_{15} and G_{16} with $\theta(G_7) = 4\frac{1}{2}$, $\theta(G_8) = 3\frac{1}{4}$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{13}) = 4$, $\theta(G_{14}) = 6$, $\theta(G_{15}) = 4\frac{1}{4}$ and $\theta(G_{16}) = 6\frac{1}{4}$;

G_i $(1 \le i \le 28)$	$\theta(G_i)$	G_i $(29 \le i \le 56)$	$\theta(G_i)$
$G_1 = K(n, n+1, n+1, n+1, n+1)$	0	$G_{29} = K(n-2, n-1, n+1, n+2, n+4)$	$13\frac{3}{4}$
$G_2 = K(n, n, n+1, n+1, n+2)$	1	$G_{30} = K(n-3, n, n+2, n+2, n+3)$	$8\frac{1}{8}$
$G_3 = K(n-1, n+1, n+1, n+1, n+2)$	$1\frac{1}{2}$	$G_{31} = K(n-3, n, n+1, n+3, n+3)$	$10\frac{1}{8}$
$G_4 = K(n, n, n, n+2, n+2)$	2	$G_{32} = K(n-3, n, n+1, n+2, n+4)$	$13\frac{1}{8}$
$G_5 = K(n, n, n, n+1, n+3)$	4	$G_{33} = K(n-4, n+2, n+2, n+2, n+2)$	$7\frac{1}{16}$
$G_6 = K(n-1, n, n+1, n+2, n+2)$	$2\frac{1}{2}$	$G_{34} = K(n-4, n+1, n+2, n+2, n+3)$	$9\frac{1}{16}$
$G_7 = K(n-1, n, n+1, n+1, n+3)$	$4\frac{1}{2}$	$G_{35} = K(n-4, n+1, n+1, n+3, n+3)$	$11\frac{1}{16}$
$G_8 = K(n-2, n+1, n+1, n+2, n+2)$	$3\frac{1}{4}$	$G_{36} = K(n-4, n+1, n+1, n+2, n+4)$	$15\frac{1}{16}$
$G_9 = K(n-2, n+1, n+1, n+1, n+3)$	$5\frac{1}{4}$	$G_{37} = K(n-1, n-1, n-1, n+3, n+4)$	$16\frac{1}{2}$
$G_{10} = K(n-1, n, n, n+2, n+3)$	$5\frac{1}{2}$	$G_{38} = K(n-2, n-1, n, n+3, n+4)$	$16\frac{3}{4}$
$G_{11} = K(n, n, n, n, n+4)$	11	$G_{39} = K(n-3, n, n, n+3, n+4)$	$8\frac{1}{16}$
$G_{12} = K(n-1, n, n, n+1, n+4)$	$11\frac{1}{2}$	$G_{40} = K(n-2, n-2, n+2, n+3, n+3)$	$11\frac{1}{2}$
$G_{13} = K(n-1, n-1, n+2, n+2, n+2)$	4	$G_{41} = K(n-2, n-2, n+2, n+2, n+4)$	$15\frac{1}{2}$
$G_{14} = K(n-1, n-1, n+1, n+2, n+3)$	6	$G_{42} = K(n-3, n-1, n+2, n+3, n+3)$	$11\frac{5}{8}$
$G_{15} = K(n-2, n, n+2, n+2, n+2)$	$4\frac{1}{4}$	$G_{43} = K(n-3, n-1, n+2, n+2, n+4)$	$15\frac{5}{8}$
$G_{16} = K(n-2, n, n+1, n+2, n+3)$	$6\frac{1}{4}$	$G_{44} = K(n-2, n-2, n+1, n+3, n+4)$	$17\frac{1}{2}$
$G_{17} = K(n-1, n-1, n+1, n+1, n+4)$	12	$G_{45} = K(n-3, n-1, n+1, n+3, n+4)$	$17\frac{5}{8}$
$G_{18} = K(n-2, n, n+1, n+1, n+4)$	$12\frac{1}{4}$	$G_{46} = K(n-4, n, n+2, n+3, n+3)$	$12\frac{1}{16}$
$G_{19} = K(n-3, n+1, n+2, n+2, n+2)$	$10\frac{1}{4}$	$G_{47} = K(n-4, n, n+2, n+2, n+4)$	$16\frac{1}{16}$
$G_{20} = K(n-3, n+1, n+1, n+2, n+3)$	$7\frac{1}{8}$	$G_{48} = K(n-4, n, n+1, n+3, n+4)$	$18\frac{1}{16}$
$G_{21} = K(n-3, n+1, n+1, n+1, n+4)$	$13\frac{1}{8}$	$G_{49} = K(n-5, n+2, n+2, n+2, n+3)$	$11\frac{1}{32}$
$G_{22} = K(n-1, n-1, n, n+3, n+3)$	9	$G_{50} = K(n-5, n+1, n+2, n+3, n+3)$	$13\frac{1}{32}$
$G_{23} = K(n-1, n-1, n, n+2, n+4)$	13	$G_{51} = K(n-5, n+1, n+2, n+2, n+4)$	$17\frac{1}{32}$
$G_{24} = K(n-2, n, n, n+3, n+3)$	$9\frac{1}{4}$	$G_{52} = K(n-3, n, n, n+2, n+5)$	$29\frac{1}{8}$
$G_{25} = K(n-2, n, n, n+2, n+4)$	$13\frac{1}{4}$	$G_{53} = K(n-3, n-1, n, n+4, n+4)$	$24\frac{5}{8}$
$G_{26} = K(n-1, n, n, n, n+5)$	$26\frac{1}{2}$	$G_{54} = K(n-3, n-1, n, n+3, n+5)$	$32\frac{5}{8}$
$G_{27} = K(n-2, n-1, n+2, n+2, n+3)$	$7\frac{3}{4}$	$G_{55} = K(n-4, n, n, n+4, n+4)$	$25\frac{1}{16}$
$G_{28} = K(n-2, n-1, n+1, n+3, n+3)$	$9\frac{3}{4}$	$G_{56} = K(n-4, n, n, n+3, n+5)$	$33\frac{1}{16}$

Table 1: Some complete 5-partite graphs with $5n+4$ vertices and their θ-values.

(vii) G_7 is the improvement of G_9, G_{10}, G_{12}, G_{14}, G_{16}, G_{17} and G_{18} with $\theta(G_9) = 5\frac{1}{4}$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{12}) = 11\frac{1}{2}$, $\theta(G_{14}) = 6$, $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{17}) = 12$ and $\theta(G_{18}) = 12\frac{1}{4}$;

(viii) G_8 is the improvement of G_9, G_{15}, G_{16}, G_{19} and G_{20} with $\theta(G_9) = 5\frac{1}{4}$, $\theta(G_{15}) = 4\frac{1}{4}$, $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{19}) = 10\frac{1}{4}$ and $\theta(G_{20}) = 7\frac{1}{8}$;

(ix) G_9 is the improvement of G_{16}, G_{18}, G_{20} and G_{21} with $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{18}) = 12\frac{1}{4}$, $\theta(G_{20}) = 7\frac{1}{8}$ and $\theta(G_{21}) = 13\frac{1}{8}$;

(x) G_{10} is the improvement of G_{12}, G_{14}, G_{16}, G_{22}, G_{23}, G_{24} and G_{25} with $\theta(G_{12}) = 11\frac{1}{2}$, $\theta(G_{14}) = 6$, $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{22}) = 9$, $\theta(G_{23}) = 13$, $\theta(G_{24}) = 9\frac{1}{4}$ and $\theta(G_{25}) = 13\frac{1}{4}$;

(xi) G_{11} is the improvement of G_{12} and G_{26} with $\theta(G_{12}) = 11\frac{1}{2}$ and $\theta(G_{26}) = 26\frac{1}{2}$;

(xii) G_{13} is the improvement of G_{14}, G_{15} and G_{27} with $\theta(G_{14}) = 6$, $\theta(G_{15}) = 4\frac{1}{4}$ and $\theta(G_{27}) = 7\frac{3}{4}$;

(xiii) G_{14} is the improvement of G_{16}, G_{17}, G_{22}, G_{23}, G_{27}, G_{28} and G_{29} with $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{17}) = 12$, $\theta(G_{22}) = 9$, $\theta(G_{23}) = 13$, $\theta(G_{27}) = 7\frac{3}{4}$, $\theta(G_{28}) = 9\frac{3}{4}$ and $\theta(G_{29}) = 13\frac{3}{4}$;

(xiv) G_{15} is the improvement of G_{16}, G_{19}, G_{27} and G_{30} with $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{19}) = 10\frac{1}{4}$, $\theta(G_{27}) = 7\frac{3}{4}$ and $\theta(G_{30}) = 8\frac{1}{8}$;

(xv) G_{16} is the improvement of G_{18}, G_{20}, G_{24}, G_{25}, G_{27}, G_{28}, G_{29}, G_{30}, G_{31} and G_{32} with $\theta(G_{18}) = 12\frac{1}{4}$, $\theta(G_{20}) = 7\frac{1}{8}$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{25}) = 13\frac{1}{4}$, $\theta(G_{27}) = 7\frac{3}{4}$, $\theta(G_{28}) = 9\frac{3}{4}$, $\theta(G_{29}) = 13\frac{3}{4}$, $\theta(G_{30}) = 8\frac{1}{8}$, $\theta(G_{31}) = 10\frac{1}{8}$ and $\theta(G_{32}) = 13\frac{1}{8}$;

(xvi) G_{19} is the improvement of G_{20}, G_{30}, G_{33} and G_{34} with $\theta(G_{20}) = 7\frac{1}{8}$, $\theta(G_{30}) = 8\frac{1}{8}$, $\theta(G_{33}) = 7\frac{1}{16}$ and $\theta(G_{34}) = 9\frac{1}{16}$;

(xvii) G_{20} is the improvement of G_{21}, G_{30}, G_{31}, G_{32}, G_{34}, G_{35} and G_{36} with $\theta(G_{21}) = 13\frac{1}{8}$, $\theta(G_{30}) = 8\frac{1}{8}$, $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{32}) = 13\frac{1}{8}$, $\theta(G_{34}) = 9\frac{1}{16}$, $\theta(G_{35}) = 11\frac{1}{16}$ and $\theta(G_{36}) = 15\frac{1}{16}$;

(xviii) G_{22} is the improvement of G_{23}, G_{24}, G_{28}, G_{37} and G_{38} with $\theta(G_{23}) = 13$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{28}) = 9\frac{3}{4}$, $\theta(G_{37}) = 16\frac{1}{2}$ and $\theta(G_{38}) = 16\frac{3}{4}$;

(xix) G_{24} is the improvement of G_{25}, G_{28}, G_{31}, G_{38} and G_{39} with $\theta(G_{25}) = 13\frac{1}{4}$, $\theta(G_{28}) = 9\frac{3}{4}$, $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{38}) = 16\frac{3}{4}$ and $\theta(G_{39}) = 8\frac{1}{16}$;

(xx) G_{27} is the improvement of G_{28}, G_{29}, G_{30}, G_{40}, G_{41}, G_{42} and G_{43} with $\theta(G_{28}) = 9\frac{3}{4}$, $\theta(G_{29}) = 16\frac{1}{8}$, $\theta(G_{30}) = 8\frac{1}{8}$, $\theta(G_{40}) = 11\frac{1}{2}$, $\theta(G_{41}) = 15\frac{1}{2}$, $\theta(G_{42}) = 11\frac{5}{8}$ and $\theta(G_{43}) = 15\frac{5}{8}$;

(xxi) G_{28} is the improvement of G_{29}, G_{31}, G_{38}, G_{40}, G_{42}, G_{44} and G_{45} with $\theta(G_{29}) = 16\frac{1}{8}$, $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{38}) = 16\frac{3}{4}$, $\theta(G_{40}) = 11\frac{1}{2}$, $\theta(G_{42}) = 11\frac{5}{8}$, $\theta(G_{44}) = 17\frac{1}{2}$ and $\theta(G_{45}) = 17\frac{5}{8}$.

(xxii) G_{30} is the improvement of G_{31}, G_{32}, G_{34}, G_{42}, G_{43}, G_{46} and G_{47} with $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{32}) = 13\frac{1}{8}$, $\theta(G_{34}) = 9\frac{1}{16}$, $\theta(G_{42}) = 11\frac{5}{8}$, $\theta(G_{43}) = 15\frac{5}{8}$, $\theta(G_{46}) = 12\frac{1}{16}$ and $\theta(G_{47}) = 16\frac{1}{16}$.

(xxiii) G_{31} is the improvement of G_{32}, G_{35}, G_{39}, G_{42}, G_{45}, G_{46} and G_{48} with $\theta(G_{32}) = 13\frac{1}{8}$, $\theta(G_{35}) = 11\frac{1}{16}$, $\theta(G_{39}) = 8\frac{1}{16}$, $\theta(G_{42}) = 11\frac{5}{8}$, $\theta(G_{45}) = 17\frac{5}{8}$, $\theta(G_{46}) = 12\frac{1}{16}$ and $\theta(G_{48}) = 18\frac{1}{16}$.

(xxiv) G_{33} is the improvement of G_{34} and G_{49} with $\theta(G_{34}) = 9\frac{1}{16}$ and $\theta(G_{49}) = 11\frac{1}{32}$.

(xxv) G_{34} is the improvement of G_{35}, G_{36}, G_{46}, G_{47}, G_{49}, G_{50} and G_{51} with $\theta(G_{35}) = 11\frac{1}{16}$, $\theta(G_{36}) = 15\frac{1}{16}$, $\theta(G_{46}) = 12\frac{1}{16}$, $\theta(G_{47}) = 16\frac{1}{16}$, $\theta(G_{49}) = 11\frac{1}{32}$, $\theta(G_{50}) = 13\frac{1}{32}$ and $\theta(G_{51}) = 17\frac{1}{32}$.

(xxvi) G_{39} is the improvement of G_{45}, G_{48}, G_{52}, G_{53}, G_{54}, G_{55} and G_{56} with $\theta(G_{45}) = 17\frac{5}{8}$, $\theta(G_{48}) = 18\frac{1}{16}$, $\theta(G_{52}) = 29\frac{1}{8}$, $\theta(G_{53}) = 24\frac{5}{8}$, $\theta(G_{54}) = 32\frac{5}{8}$, $\theta(G_{55}) = 25\frac{1}{16}$ and $\theta(G_{56}) = 33\frac{1}{16}$.

Hence, by Lemma 2.4 and the above arguments, we know (i) to (xiii) holds. Thus the proof is complete.

4 Chromatically Closed 5-Partite Graphs

In this section, we obtain a χ-closed family of graphs from the graphs in Theorem 3.1.

Theorem 4.1 *The family of graphs $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5)$ where $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 4$, $n_5 - n_1 \leq 4$ and $n_1 \geq s + 5$ is χ-closed except that $\{\mathcal{K}^{-s}(n, n, n, n+1, n+3), \mathcal{K}^{-(s-2)}(n-1, n-1, n+2, n+2, n+2)\}$ is χ-closed.*

Proof. By Theorem 3.1, there are 13 cases to consider. Denote each graph in Theorem 3.1 $(i), \cdots, (vi), (viii), \cdots, (xiii)$ by $G_1, G_2, \cdots, G_6, G_8, \cdots, G_{13}$, respectively, and denote the two graphs in Theorem 3.1(vii) by $G_7' = K(n, n, n, n+1, n+3)$ and $G_7'' = K(n-1, n-1, n+2, n+2, n+2)$. Suppose $H \sim G_i - S$. It suffices to show that $H \in \{G_i - S\}$. By Lemma 2.3, we know there exists a complete 5-partite graph $F = (p_1, p_2, p_3, p_4, p_5)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s \geq 0$.

Case (i). Let $G = G_1$ with $n \geq s + 1$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n, n+1, n+1, n+1, n+1)$. By Lemma 2.5, we have

$$\alpha(G - S, 6) = \alpha(G, 6) + \alpha'(G - S) \text{ with } s \leq \alpha'(G - S) \leq 2^s - 1, \quad (1)$$
$$\alpha(F - S', 6) = \alpha(F, 6) + \alpha'(F - S') \text{ with } 0 \leq s' \leq \alpha'(F - S'). \quad (2)$$

Hence,

$$\alpha(F - S', 6) - \alpha(G - S, 6) = \alpha(F, 6) - \alpha(G, 6) + \alpha'(F - S') - \alpha'(G - S).$$

By the definition, $\alpha(F, 6) - \alpha(G, 6) = 2^{n-1}(\theta(F) - \theta(G))$. By Theorem 3.1, $\theta(F) \geq 0$. Suppose $\theta(F) > 0$, then

$$\begin{aligned}\alpha(F - S', 6) - \alpha(G - S, 6) \geq\ & 2^{n-1} + \alpha'(F - S') - \alpha'(G - S)\\ & \geq 2^s + \alpha'(F - S') - 2^s + 1,\\ & \geq 1,\end{aligned}$$

contradicting $\alpha(F-S',6)=\alpha(G-S,6)$. Hence, $\theta(F)=0$ and so $F=G$ and $s=s'$. Therefore, $H\in\mathcal{K}^{-s}(n,n+1,n+1,n+1,n+1)$.

Case (ii). Let $G=G_2$ with $n\geq s+2$. In this case, $H\sim F-S\in\mathcal{K}^{-s}(n,n,n+1,n+1,n+2)$. By Lemma 2.5, we have

$$\alpha(G-S,6)=\alpha(G,6)+\alpha'(G-S) \text{ with } s\leq\alpha'(G-S)\leq 2^s-1, \quad (3)$$
$$\alpha(F-S',6)=\alpha(F,6)+\alpha'(F-S') \text{ with } 0\leq s'\leq\alpha'(F-S'). \quad (4)$$

Hence,

$$\alpha(F-S',6)-\alpha(G-S,6)=\alpha(F,6)-\alpha(G,6)+\alpha'(F-S')-\alpha'(G-S).$$

By the definition, $\alpha(F,6)-\alpha(G,6)=2^{n-1}(\theta(F)-\theta(G))$. Suppose $\theta(F)\neq\theta(G)$. Then, we consider two subcases.

Subcase (a). $\theta(F)<\theta(G)$. By Theorem 3.1, $F=G_1$ and $H=G_1-S'\in\{G_1-S'\}$. However, $G-S\notin\{G_1-S'\}$ since by Case (i) above, $\{G_1-S'\}$ is χ-closed, a contradiction.

Subcase (b). $\theta(F)>\theta(G)$. By Theorem 3.1, $\alpha(F,6)-\alpha(G,6)\geq 2^{n-1}$. So,

$$\begin{aligned}\alpha(F-S',6)-\alpha(G-S,6) \;\geq\; & 2^{n-1}+\alpha'(F-S')-\alpha'(G-S)\\ & \geq 2^s+\alpha'(F-S')-2^s+1,\\ & \geq 1,\end{aligned}$$

contradicting $\alpha(F-S',6)=\alpha(G-S,6)$. Hence, $\theta(F)-\theta(G)=0$ and so $F=G$ and $s=s'$. Therefore, $H\in\mathcal{K}^{-s}(n,n,n+1,n+1,n+2)$.

By an argument to that in Cases (i)–(vi), we can also prove Cases (viii)–(xiii).

We now prove Case (vii).

Case (iii). Let $G=G_7'=K(n,n,n,n+1,n+3)$ with $n\geq s+3$ or $G=G_7''=K(n-1,n-1,n+2,n+2,n+2)$ with $n\geq s+4$. For $G_7'=K(n,n,n,n+1,n+3)$, $\theta(F)=\theta(G_7)$ implies that (a) $F=G_7'=K(n,n,n,n+1,n+3)$, or (b) $F=G_7''=K(n-1,n-1,n+2,n+2,n+2)$. So, in (a), $s'=s$ and $H\in\{K(n,n,n,n+1,n+3)-s\}$, and in (b), $s'=s-2$ and $H\in\{K(n-1,n-1,n+2,n+2,n+2)-s+2\}$. Therefore, $H\in\{\mathcal{K}^{-s}(n,n,n,n+1,n+3),\mathcal{K}^{-(s-2)}(n-1,n-1,n+2,n+2,n+2)\}$. Hence, $\{\mathcal{K}^{-s}(n,n,n,n+1,n+3),\mathcal{K}^{-(s-2)}(n-1,n-1,n+2,n+2,n+2)\}$ is χ-closed.

This completes the proof.

5 Chromatically Unique 5-Partite Graphs

The following results give two families of chromatically unique complete 5-partite graphs having $5n+4$ vertices with a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$ and a matching sK_2, respectively.

Theorem 5.1 *The graphs* $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5=5n+4$, $n_5-n_1\le 4$ *and* $n_1\ge s+5$ *are* χ*-unique for* $1\le i\ne j\le 5$ *except the graph* $K_{i,j}^{-K_{1,s}}(n-1,n-1,n+2,n+2,n+2)$.

Proof. By Theorem 3.1, there are 13 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\cdots,(vi),(viii),\cdots,(xiii)$ by $G_1,G_2,\cdots,G_6,G_8,\cdots,G_{13}$, respectively, and two graphs in Case (vii) by $G_7'=K(n,n,n,n+1,n+3)$ and $G_7''=K(n-1,n-1,n+2,n+2,n+2)$. The proof for graphs in Cases (i)–(vi) and Cases (viii)–(xiii) are similar, so we only present the detailed proofs of Case (iv) and Case (vii). Now we give the proof of Case (iv).

By Lemma 2.5 and Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n,n,n,n+2,n+2)=$

$\{K_{i,j}^{-K_{1,s}}(n,n,n,n+2,n+2)|(i,j)\in\{(1,2),(1,4),(4,1),(4,5)\}$ is χ-closed for $n\ge s+3$. Note that

$t(K_{1,2}^{-K_{1,s}}(n,n,n,n+2,n+2))=t(K(n,n,n,n+2,n+2))-s(3n+4)$,

$t(K_{i,j}^{-K_{1,s}}(n,n,n,n+2,n+2))=t(K(n,n,n,n+2,n+2))-s(3n+2)$ for $(i,j)\in\{(1,4),(4,1)\}$,

$t(K_{4,5}^{-K_{1,s}}(n,n,n,n+2,n+2))=t(K(n,n,n,n+2,n+2))-3sn$.

By Lemma 2.6, we conclude that $\sigma(K_{1,4}^{-K_{1,s}}(n,n,n,n+2,n+2),\lambda)\ne\sigma(K_{4,1}^{-K_{1,s}}(n,n,n,n+2,n+2),\lambda)$. Hence, by Lemma 2.2, the graphs $K_{i,j}^{-K_{1,s}}(n,n,n,n+2,n+2)$ are χ-unique where $n\ge s+3$ for $1\le i\ne j\le 5$.

We now present the proof of Case (vii).

We first determine the chromatic uniqueness of $G=G_7'-S$ with $\langle S\rangle$ is a star joining vertices in V_i and V_j of G_7'. By Case 7 of Theorem 4.1, if $H\sim G$, then $H=G_7'-S$ or $H=G_7''-S'$ with $s'=s-2$. If $H=G_7''-S'$, then by Lemma 2.5, $\alpha'(H)\le 2^{s-2}-1<2^s-1=\alpha'(G)$, a contradiction. Hence, $H=G_7'-S$. This shows that $G=\mathcal{K}^{-s}(n,n,n,n+1,n+3)$ where $\langle S\rangle$ is a star is χ-closed.

By Lemma 2.5 and Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n,n,n,n+1,n+3)=$

$\{K_{i,j}^{-K_{1,s}}(n,n,n,n+1,n+3)|(i,j)\in\{(1,2),(1,4),(4,1),(1,5),(5,1),(4,5),(5,4)\}$ is χ-closed for $n\ge s+3$. Note that

$t(K_{1,2}^{-K_{1,s}}(n,n,n,n+1,n+3))=t(K(n,n,n,n+1,n+3))-s(3n+4)$,

$t(K_{i,j}^{-K_{1,s}}(n,n,n,n+1,n+3))=t(K(n,n,n,n+1,n+3))-s(3n+3)$ for $(i,j)\in\{(1,4),(4,1)\}$,

$t(K_{i,j}^{-K_{1,s}}(n,n,n,n+1,n+3))=t(K(n,n,n,n+1,n+3))-s(3n+1)$ for $(i,j)\in$

$\{(1,5),(5,1)\}$,
$t(K_{i,j}^{-K_{1,s}}(n,n,n,n+1,n+3)) = t(K(n,n,n,n+1,n+3)) - 3sn$ for $(i,j) \in \{(4,5),(5,4)\}$.

By Lemma 2.6, we conclude that $\sigma(K_{i,j}^{-K_{1,s}}(n,n,n,n+1,n+3),\lambda) \neq \sigma(K_{j,i}^{-K_{1,s}}(n,n,n,n+1,n+3),\lambda)$ for each $(i,j) \in \{(1,4),(1,5),(4,5)\}$. Hence, by Lemma 2.2, the graphs $K_{i,j}^{-K_{1,s}}(n,n,n,n+1,n+3)$ are χ-unique where $n \geq s+3$ for $1 \leq i \neq j \leq 5$.

This completes the proof.

Theorem 5.2 *The graphs $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5 = 5n+4$, $n_5 - n_1 \leq 4$ and $n_1 \geq s+5$ are χ-unique except the graph $K_{1,2}^{-sK_2}(n,n,n,n+1,n+3)$.*

Proof. By Theorem 3.1, there are 13 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\cdots,(vi),(viii),\cdots,(xiii)$ by $G_1,G_2,\cdots,G_6,G_8,\cdots,G_{13}$, respectively, and two graphs in Case (vii) by $G'_7 = K(n,n,n,n+1,n+3)$ and $G''_7 = K(n-1,n-1,n+2,n+2,n+2)$. For a graph $K(p_1,p_2,p_3,p_4,p_5)$, let $S = \{e_1,e_2,\cdots,e_s\}$ be the set of s edges in $E(K(p_1,p_2,p_3,p_4,p_5))$ and let $t(e_i)$ denote the number of triangles containing e_i in $K(p_1,p_2,p_3,p_4,p_5)$. The proof for graphs in Cases (i)–(vi) and Cases (viii)–(xiii) are similar, so we only present the detailed proof for Case (iv) and Case (vii).

Now we give the proof of Case (iv).

Suppose $H \sim G = K_{i,j}^{-sK_2}(n,n,n,n+2,n+2)$ for $n \geq s+3$. By Theorem 4.1 and Lemma 2.1, $H \in \mathcal{K}^{-s}(n,n,n,n+2,n+2)$ and $\alpha'(H) = \alpha'(G) = s$. Let $H = F - S$ where $F = K(n,n,n,n+2,n+2)$. Clearly, $t(e_i) \leq 3n+4$ for each $e_i \in S$. So,

$$t(H) \geq t(F) - s(3n+4),$$

with equality holds only if $t(e_i) = 3n+4$ for all $e_i \in S$. Since $t(H) = t(G) = t(F) - s(3n+4)$, the equality above holds with $t(e_i) = 3n+4$ for all $e_i \in S$. Therefore each edge in S has an end-vertex in V_i and another end-vertex in V_j $(1 \leq i < j \leq 3)$. Moreover, S must induce a matching in F. Otherwise, equality does not hold or $\alpha'(H) > s$.

Clearly, $H \cong G$ if S is ideal. Otherwise, there exists i,j,k and l such that $S_{ij}(1 \leq i < j \leq 3)$ and $S_{kl}(1 \leq k < l \leq 3)$ are two disjoint non-empty subsets of S. Observe that each induced C_4 in G (respectively H) is of Type 1 or 2. By Lemma 2.8, we obtain

$$Q(G) = Q(F) - s(n-1)^2 + \binom{s}{2} + s\Big[\binom{n}{2} + 2\binom{n+2}{2}\Big]$$

whereas

$$\begin{aligned} Q(G) &= Q(F) - s(n-1)^2 + \binom{s}{2} - s_{12}(s_{13}+s_{23}) - s_{13}s_{23} + \\ & s\Big[\binom{n}{2} + 2\binom{n+2}{2}\Big] \\ &\le Q(G). \end{aligned}$$

Moreover, $K(G) = K(H) = K(F) - s(3n^2 + 8n + 4)$.

Hence, $2K(G) - Q(G) \geq 2K(H) - Q(H)$ and the equality holds if and only if $s = s_{ij}$ $(1 \leq i < j \leq 3)$. Hence $\langle S\rangle \cong sK_2$ with $H \cong G$.

We now present the proof of Case (vii).

We first determine the chromatic uniqueness of $G = G_7'' - S$ with $\langle S\rangle$ is a matching joining vertices in V_1 and V_2 of G_7''. By Case 7 of Theorem 4.1, if $H \sim G$, then $H = G_7'' - S$ or $H = G_7' - S'$ with $s' = s + 2$. If $H = G_7' - S'$, then by Lemma 2.5, $\alpha'(H) \geq s + 2 > s = \alpha'(G)$, a contradiction. Hence, $H = G_7'' - S$. This shows that $G = \mathcal{K}^{-s}(n-1, n-1, n+2, n+2, n+2)$ where $\langle S\rangle$ is a matching is χ-closed.

Suppose $H \sim G = K_{i,j}^{-sK_2}(n-1, n-1, n+2, n+2, n+2)$ for $n \geq s+4$. By Theorem 4.1 and Lemma 2.1, $H \in \mathcal{K}^{-s}(n-1, n-1, n+2, n+2, n+2)$ and $\alpha'(H) = \alpha'(G) = s$. Let $H = F - S$ where $F = K(n-1, n-1, n+2, n+2, n+2)$. Clearly, $t(e_i) \leq 3n + 6$ for each $e_i \in S$. So,

$$t(H) \geq t(F) - s(3n + 6),$$

with equality holds only if $t(e_i) = 3n+6$ for all $e_i \in S$. Since $t(H) = t(G) = t(F) - s(3n + 6)$, the equality above holds with $t(e_i) = 3n + 6$ for all $e_i \in S$. Therefore each edge in S has an end-vertex in V_1 and another end-vertex in V_2. Moreover, S must induce a matching in F. Otherwise, $\alpha'(H) > s$. Hence $\langle S\rangle \cong sK_2$ with $H \cong G$.

Thus the proof is complete.

Remark. This paper generalized some results obtained in paper [16].

We end this paper with the following open problem.

Problem. Study the chromaticity of the graphs $K_{i,j}^{-K_{1,s}}(n-1, n-1, n+2, n+2, n+2)$ and $K_{1,2}^{-sK_2}(n, n, n, n+1, n+3)$.

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