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A Note on Chromaticity of Certain 6-Partite Graphs

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Abstract—**Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H|H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 6-partite graphs with $6n+3$ vertices according to the number of 7-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 6-partite graphs with certain star or matching deleted are obtained.**

Keywords: ***Chromatic Polynomial, Chromatically Closed, Chromatic Uniqueness***

1 Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G, \lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G, l) = P(H, l)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. Many families of χ-unique graphs are known (see [6,7,8]).

For a graph G, let $V(G)$, $E(G)$ and $t(G)$ be the vertex set, edge set and number of triangles in G, respectively. Let S be a set of s edges in G. Let $G - S$ (or $G - s$) be the graph obtained from G by deleting all edges in S, and by $\langle S \rangle$ the graph induced by S. Let $K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph. We denote by $\mathcal{K}^{-s}(n_1, n_2, \cdots, n_t)$ the family of graphs which are obtained from $K(n_1, n_2, \cdots, n_t)$ by deleting a set S of some s edges.

In [4,5,7–10,12–14,18–20], one can find many results on the chromatic uniqueness of certain families of complete t-partite graphs ($t = 2, 3, 4, 5$). There are several families complete 6-partite graphs known to be χ-unique, see [3,15,16,17].

Let G be a complete 6-partite graph with $6n+3$ vertices. In this paper, we characterize certain complete 6-partite graphs with $6n + 3$ vertices according to the number of 7-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 6-partite graphs with certain star or matching deleted are obtained.

2 Some Lemmas and Notations

For a graph G and a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of G. Then, we have $P(G, \lambda) = \sum_{r=1}^{p} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ (see [11]). Therefore, $\alpha(G, r) = \alpha(H, r)$ for each $r = 1, 2, \cdots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G, x) = \sum_{r=1}^{p} \alpha(G, r)x^r$ is called the σ-polynomial of G (see [2]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs G and H.

For disjoint graphs G and H, $G \cup H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer [1].

Lemma 2.1 (Brenti [2], Koh and Teo [7]) Let G and H be two disjoint graphs. Then

(i) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ and $\alpha(G, r) = \alpha(H, r)$ for $r = 1, 2, 3, \cdots, p$, if $G \sim H$;

(ii) $\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x)$.

Lemm 2.2 (Brenti [2]) Let $G = K(n_1, n_2, n_3, \cdots, n_t)$ and $\sigma(G, x) = \sum_{r\geq 1} \alpha(G, r)x^r$, then $\alpha(G, r) = 0$ for $1 \leq r \leq t-1$, $\alpha(G, t) = 1$ and $\alpha(G, t+1) = \sum_{i=1}^{t} 2^{n_i-1} - t$.

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Let $x_1 \le x_2 \le x_3 \le x_4 \le x_5 \le x_6$ be positive integers and $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. If there are two elements x_{i_1} and x_{i_2} in $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ such that $x_{i_2} - x_{i_1} \ge 2$, then $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ is called an *improvement* of $H = K(x_1, x_2, x_3, x_4, x_5, x_6)$.

Lemma 2.3 (Chen [3]) Suppose $x_1 \le x_2 \le x_3 \le x_4 \le x_5 \le x_6$ and $H' = K(x_{i_1}+1, x_{i_2}-1, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}\}$ is an improvement of $H = K(x_1, x_2, x_3, x_4, x_5, x_6)$, then

$$\alpha(H,7) - \alpha(H',7) = 2^{x_{i_2}-2} - 2^{x_{i_1}-1} \ge 2^{x_{i_1}-1}.$$

Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H,7) - \alpha(G,7)$. Clearly, $\alpha'(H) \ge 0$.

Lemma 2.4 (Chen [3]) Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$. Suppose that $\min\{n_i | i = 1,2,3,4,5,6\} \ge s+1 \ge 1$ and $H = G - S$, where S is a set of some s edges of G, then

$$s \le \alpha'(H) = \alpha(H,7) - \alpha(G,7) \le 2^s - 1,$$

$\alpha'(H) = s$ iff the set of end-vertices of any $r \ge 2$ edges in S is not independent in H, and $\alpha'(H) = 2^s - 1$ iff S induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same A_i.

Let $K(A_1, A_2)$ be a complete bipartite graph with partite sets A_1 and A_2. We denote by $K^{-K_{1,s}}(A_i, A_j)$ the graph obtained from $K(A_i, A_j)$ by deleting s edges that induce a star with its center in A_i. Note that $K^{-K_{1,s}}(A_i, A_j) \ne K^{-K_{1,s}}(A_j, A_i)$ if $|A_i| \ne |A_j|$ for $i \ne j$ (see [5]).

Lemma 2.5 (Dong et al. [4]) Let $K(n_1, n_2)$ be a complete bipartite graph with partite sets A_1 and A_2 such that $|A_i| = n_i$ for $i = 1,2$. If $\min\{n_1, n_2\} \ge s+2$, then every $K^{-K_{1,s}}(A_i, A_j)$ is χ-unique, where $i \ne j$ and $i, j = 1, 2$.

Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 5-partite graph with partite sets $A_i (i = 1, 2, \cdots, 6)$ such that $|A_i| = n_i$. Let $\langle A_i \cup A_j \rangle$ be the subgraph of G induced by $A_i \cup A_j$, where $i \ne j$ and $i, j \in \{1,2,3,4,5,6\}$. By $K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$, we denote the graph obtained from $K(n_1, n_2, n_3, n_4, n_5, n_6)$ by deleting a set of s edges that induce a $K_{1,s}$ with its center in A_i and all it end vertices are in A_j. Note that $K_{i,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6) = K_{j,l}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$ and $K_{l,i}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6) = K_{l,j}^{-K_{1,s}}(n_1, n_2, n_3, n_4, n_5, n_6)$ for $n_i = n_j$ and $l \ne i, j$.

Lemma 2.6 (Chen [3]) If $i, j \in \{1, 2, 3, \cdots, t\}$, $i \ne j$, $n_i \ne n_j$, then $P(K_{i,j}^{-K_{1,s}}(n_1, n_2, n_3, \cdots, n_t), \lambda) \ne P(K_{j,i}^{-K_{1,s}}(n_1, n_2, n_3, \cdots, n_t), \lambda)$.

3 Classification

In this section, we shall characterize certain complete 6-partite graph $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ according to the number of 7-independent partitions of G where $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n+3, n \ge 1$.

Theorem 3.1 Let $G = K(n_1, n_2, n_3, n_4, n_5, n_6)$ be a complete 6-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = 6n+3$, $n \ge 1$. Define $\theta(G) = [\alpha(G,7) - 9 \cdot 2^{n-1} + 6]/2^{n-2}$. Then

(i) $\theta(G) \ge 0$;

(ii) $\theta(G) = 0$ if and only if $G = K(n, n, n, n+1, n+1, n+1)$;

(iii) $\theta(G) = 1$ if and only if $G = K(n-1, n, n+1, n+1, n+1, n+1)$;

(iv) $\theta(G) = 2$ if and only if $G = K(n, n, n, n, n+1, n+2)$;

(v) $\theta(G) = 5/2$ if and only if $G = K(n-2, n+1, n+1, n+1, n+1, n+1)$;

(vi) $\theta(G) = 3$ if and only if $G = K(n-1, n, n, n+1, n+1, n+2)$;

(vii) $\theta(G) = 4$ if and only if $G = K(n-1, n-1, n+1, n+1, n+1, n+2)$;

(viii) $\theta(G) \ge 9/2$ if and only if G is not a graph appeared in (ii)–(vii).

Proof. For a complete 6-partite graph H_1 with $6n+3$ vertices, we can construct a sequence of complete 6-partite graphs with $6n+3$ vertices, say $H_1, H_2, \cdots, H_t$, such that H_i is an improvement of H_{i-1} for each $i = 2, 3, \cdots, t$, and $H_t = K(n, n, n, n+1, n+1, n+1)$. By Lemma 2.3, $\alpha(H_{i-1}, 7) - \alpha(H_i, 7) > 0$. So $\theta(H_{i-1}) - \theta(H_i) > 0$, which implies that $\theta(G) \ge \theta(H_t) = \theta(K(n, n, n, n+1, n+1, n+1))$. From Lemma 2.2 and by a simple calculation, $\theta(K(n, n, n, n+1, n+1, n+1)) = 0$. Thus, (ii) is true.

Since $H_t = K(n, n, n, n+1, n+1, n+1)$ and H_t is an improvement of H_{t-1}, it is not hard to see that $H_{t-1} \in \{R, R_0, R_2\}$, where $R = K(n-1, n, n+1, n+1, n+1, n+1)$, $R_0 = K(n, n, n, n, n+1, n+2)$ and $R_2 = K(n-1, n, n, n+1, n+1, n+2)$. Hence, by Lemma 2.2, we have $\theta(R) = 1$, $\theta(R_0) = 2$ and $\theta(R_3) = 3$. Note that H_{t-1} is an improvement of H_{t-2} and it is not hard to see that $H_{t-2} \in \{R_i | i = 1, 2, \cdots, 11\}$, where R_i and $\theta(R_i)$ are shown in Table 1.

To complete the proof of the theorem, we need only determine all complete 6-partite graph G with $6n+3$ vertices such that $\theta(G) < 9/2$. By Lemma 2.3, $\theta(H_{t-3}) > 9/2$ for each H_{t-3} if $H_{t-2} \in \{R_i | i = 4, 5, 6, \cdots, 11\}$. All

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R_i	**Graphs** H_{t-2}	$\theta(R_i)$
R_1	$K(n-2, n+1, n+1, n+1, n+1, n+1)$	5/2
R_2	$K(n-1, n, n, n+1, n+1, n+2)$	3
R_3	$K(n-1, n-1, n+1, n+1, n+1, n+2)$	4
R_4	$K(n-2, n, n+1, n+1, n+1, n+2)$	9/2
R_5	$K(n-1, n, n, n, n+2, n+2)$	5
R_6	$K(n-1, n-1, n, n+1, n+2, n+2)$	6
R_7	$K(n-2, n, n, n+1, n+2, n+2)$	13/2
R_8	$K(n, n, n, n, n, n+3)$	8
R_9	$K(n-1, n, n, n, n+1, n+3)$	9
R_{10}	$K(n-1, n-1, n, n+1, n+1, n+3)$	10
R_{11}	$K(n-2, n, n, n+1, n+1, n+3)$	53/2

Table 1: H_{t-2} and its θ-values

M_i	**Graphs** H_{t-3}	$\theta(M_i)$
M_1	$K(n-3, n+1, n+1, n+1, n+1, n+2)$	25/4
M_2	$K(n-2, n-1, n+1, n+1, n+2, n+2)$	15/2
M_3	$K(n-2, n-1, n+1, n+1, n+1, n+3)$	23/2

Table 2: H_{t-3} and its θ-values

graphs H_{t-3} and its θ-values are listed in Table 2 when $H_{t-2} \in \{R_i | i = 1, 2, 3\}$.

By Lemma 2.3, $\theta(H_{t-4}) > 9/2$ for every H_{t-4} if $H_{t-3} \in \{M_i | i = 1, 2, 3\}$.So, from Lemma 2.3, Tables 1 and 2, and the above arguments, we conclude that the theorem holds.

4 Chromatically closed 6-partite graphs

In this section, we obtained the $\chi-$closed of the families in $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5, n_6)$.

Theorem 4.1 If $n \geq s + 2$, then the family of graphs $\mathcal{K}^{-s}(n, n, n, n+1, n+1, n+1)$ is χ-closed.

Proof. Let $G = K(n, n, n, n+1, n+1, n+1)$ and $Z \in \mathcal{K}^{-s}(n, n, n, n+1, n+1, n+1)$. The 6-independent partition of G is certainly 6-independent partition of Z. So $\alpha(Z, 6) \geq \alpha(G, 6) = 1$. Let $H \sim Z$, then $\alpha(H, 6) = \alpha(Z, 6) \geq \alpha(G, 6) = 1$. Let $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ be a 6-independent partition of H, $|A_i| = t_i$, $i = 1, 2, 3, 4, 5, 6$ and $F = K(t_1, t_2, t_3, t_4, t_5, t_6)$. Then there exist $S' \in E(F)$ such that $H = F - S'$. Let $q(G)$ be the number of edges in graph G. Since $q(H) = q(Z)$, therefore $s' = |S'| = q(F) - q(G) + s$.

From Lemma 2.4, we have

$$\alpha(Z, 7) = \alpha(G, 7) + \alpha'(Z), s \leq \alpha'(Z) \leq 2^s - 1, \text{ and}$$
$$\alpha(H, 7) = \alpha(F, 7) + \alpha'(H), s' \leq \alpha'(H).$$

Thus $\alpha(H, 7) - \alpha(Z, 7) = \alpha(F, 7) - \alpha(G, 7) + \alpha'(H) - \alpha'(Z)$ and $\alpha(Z, 7) = \alpha(H, 7)$, so $\alpha(H, 7) - \alpha(Z, 7) = 0$.

If $F \neq G$, from Theorem 3.1, we have $\theta(F) - \theta(G) \geq 1$. So, we have

$$\alpha(F, 7) - \alpha(G, 7) = (\theta(F) - \theta(G)) \cdot 2^{n-2} \geq 2^{n-2}.$$

Hence

$$\begin{aligned} \alpha(H, 7) - \alpha(Z, 7) &\geq 2^{n-2} + \alpha'(H) - \alpha'(Z) \\ &\geq 2^{n-2} + 0 - (2^s - 1) \geq 1. \end{aligned}$$

This is a contradiction. So $F = G$, $s = s'$. Thus, $H \in \mathcal{K}^{-s}(n, n, n, n+1, n+1, n+1)$. Therefore, $\mathcal{K}^{-s}(n, n, n, n+1, n+1, n+1)$ is χ-closed if $n \geq s + 2$. The proof is now completed.

By using the similar proof of Theorem 4.1, we can obtain the following results.

Theorem 4.2 If $n \geq s + 3$, then the family of graphs $\mathcal{K}^{-s}(n-1, n, n+1, n+1, n+1, n+1)$, $\mathcal{K}^{-s}(n, n, n, n, n+1, n+2)$, $\mathcal{K}^{-s}(n-1, n, n, n+1, n+1, n+2)$ and $\mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1, n+2)$ are χ-closed.

Theorem 4.3 If $n \geq s + 5$, then the family of graphs $\mathcal{K}^{-s}(n-2, n+1, n+1, n+1, n+1, n+1)$ is χ-closed.

5 Chromatically unique 6-partite graphs

In this section, we first study the chromatically unique 6-partite graphs with $6n + 3$ vertices and a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$.

Theorem 5.1 If $n \geq s + 2$, then the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1)$ are χ-unique for $(i, j) \in \{(1, 2), (1, 4), (4, 1), (4, 5)\}$.

Proof. From Lemma 2.4 and Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1) = \{K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1) | (i, j) \in \{(1, 2), (1, 4), (4, 1), (4, 5)\}$ is χ-closed if $n \geq s + 2$. Note that

$t(K_{1,2}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1)) = t(K(n, n, n, n+1, n+1, n+1)) - s(4n+3)$;
$t(K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1)) = t(K(n, n, n, n+1, n+1, n+1)) - s(4n+2)$ for $(i, j) \in \{(1, 4), (4, 1)\}$;
$t(K_{4,5}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1)) = t(K(n, n, n, n+1, n+1, n+1)) - s(4n+1)$.

By Lemmas 2.1 and 2.6, we conclude that $\sigma(K_{1,4}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1)) \neq \sigma(K_{4,1}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1))$. Hence, by Lemma 2.1, the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n+1, n+1, n+1)$ are χ-unique where $n \geq s + 2$ for $(i, j) \in \{(1, 2), (1, 4), (4, 1), (4, 5)\}$. The proof is now completed.

Similarly to the proof of Theorem 5.1, we can prove Theorems 5.2–5.5.

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Theorem 5.2 If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n+1, n+1, n+1, n+1)$ are χ-unique for $(i,j) \in \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(3,4)\}$.

Theorem 5.3 If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1, n+2)$ are χ-unique for $(i,j) \in \{(1,2),(1,5),(5,1),(1,6),(6,1),(5,6),(6,5)\}$.

Theorem 5.4 If $n \geq s+5$, then the graphs $K_{i,j}^{-K_{1,s}}(n-2, n+1, n+1, n+1, n+1, n+1)$ are χ-unique for $(i,j) \in \{(1,2),(2,1),(2,3)\}$.

Theorem 5.5 If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n-1, n+1, n+1, n+1, n+2)$ are χ-unique for $(i,j) \in \{(1,2),(1,3),(3,1),(1,6),(6,1),(3,4),(3,6),(6,3)\}$.

Theorem 5.6 If $n \geq s+3$, then the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2)$ are χ-unique for $(i,j) \in \{(1,2),(2,1),(4,6),(6,4)\}$.

Proof. Let $F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2) | (i,j) \in \{(1,2),(2,1),(4,6),(6,4)\}\}$ and $H \sim F$. By Theorem 4.2, $H \in \mathcal{K}^{-s}(n-1, n, n, n+1, n+1, n+2)$. Since

$\alpha(H,7) = \alpha(F,7) = \alpha(K(n-1, n, n, n+1, n+1, n+2), 7) + 2^s - 1$,

from Lemma 2.4, we know that $H \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2) | i \neq j, \ i,j = 1,2,3,4,5,6\}$. It easy to see that $H \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2) | i \neq j, \ i,j = 1,2,3,4,5,6\} = \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2) | (i,j) \in \{(1,2),(2,1),(1,4),(4,1),(1,6),(6,1),(2,3),(2,4),(4,2),(2,6),(6,2),(4,5),(4,6),(6,4)\}\}$.

Now let's determine the number of triangles in H and F. Then we obtain that

$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2)) = t(K(n-1, n, n, n+1, n+1, n+2)) - s(4n+4)$ for $(i,j) \in \{(1,2),(2,1)\}$,
$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2)) = t(K(n-1, n, n, n+1, n+1, n+2)) - s(4n+3)$ for $(i,j) \in \{(1,4),(4,1),(2,3)\}$,
$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2)) = t(K(n-1, n, n, n+1, n+1, n+2)) - s(4n+2)$ for $(i,j) \in \{(1,6),(6,1),(2,4),(4,2)\}$,
$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2)) = t(K(n-1, n, n, n+1, n+1, n+2)) - s(4n+1)$ for $(i,j) \in \{(2,6),(6,2),(4,5)\}$,
$t(K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2)) = t(K(n-1, n, n, n+1, n+1, n+2)) - 4sn$ for $(i,j) \in \{(4,6),(6,4)\}$.

Recalling

$F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2) | (i,j) \in \{(1,2),(2,1),(4,6),(6,4)\}\}$

and $t(H) = t(F)$, thus we have

$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2) | (i,j) \in \{(1,2),(2,1)\}\}$

or

$H, F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2) | (i,j) \in \{(4,6),(6,4)\}\}$.

It follows from Lemmas 2.1 and 2.6 that

$P(K_{1,2}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2), \lambda) \neq P(K_{2,1}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2), \lambda)$;
$P(K_{4,6}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2), \lambda) \neq P(K_{6,4}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2), \lambda)$.

Hence, by Lemma 2.1, we conclude that the graphs $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2)$ are χ-unique where $n \geq s+3$ for each $(i,j) \in 1,2),(2,1),(4,6),(6,4)\}$.

Let $K_{i,j}^{-sK_2}(n_1, n_2, n_3, n_4, n_5, n_6)$ denotes the graph obtained from $K(n_1, n_2, n_3, n_4, n_5, n_6)$ by deleting a set of s edges that forms a matching in $\langle A_i \cup A_j \rangle$. We now investigate the chromatically unique 6-partite graphs with $6n+3$ vertices and a set S of s edges deleted where the deleted edges induce a matching sK_2.

Theorem 5.7 If $n \geq s+3$, then the graphs $K_{1,2}^{-sK_2}(n-1, n, n+1, n+1, n+1, n+1)$ are χ-unique.

Proof. Let $F \sim K_{1,2}^{-sK_2}(n-1, n, n+1, n+1, n+1, n+1)$. It is sufficient to prove that $F = K_{1,2}^{-sK_2}(n-1, n, n+1, n+1, n+1, n+1)$. By Theorem 4.2 and Lemma 2.4, we have $F \in \mathcal{K}^{-s}(n-1, n, n+1, n+1, n+1, n+1)$ and $\alpha'(F) = s$. Let $F = G - S$ where $G = K(n-1, n, n+1, n+1, n+1, n+1)$. Next we consider the number of triangles in F. Let $e_i \in S$ and $t(e_i)$ be the number of triangles in G containing the edge e_i. It is easy to see that $t(e_i) \leq 4n+4$. As $n-1 \leq n < n+1 \leq n+1 \leq n+1 \leq n+1$, we know that $t(e_i) = 4n+4$ if and only if e_i is an edge in the subgraph $\langle A_1 \cup A_2 \rangle$ in G. So we have

$$t(F) \geq t(G) - \sum_{i=1}^{s} t(e_i) \geq t(G) - s(4n+4);$$

and the equality holds if and only if each edge e_i in S is an edge of the subgraph $\langle A_1 \cup A_2 \rangle$ in G.

Note that $t(F) = t(G) - s(4n+4)$ and $\alpha'(F) = s$. By Lemma 2.4, we know that $F = K_{1,2}^{-sK_2}(n-1, n, n+1, n+1, n+1, n+1)$. This completes the proof.

Similarly to the proof of Theorem 5.7, we can prove Theorem 5.8.

Theorem 5.8 If $n \geq s+3$, then the graphs $K_{1,2}^{-sK_2}(n-$

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$1, n-1, n+1, n+1, n+1, n+2)$ are χ-unique.

We end this paper with the following problems:

[1.] Study the chromaticity of the graph $K_{i,j}^{-K_{1,s}}(n-1, n, n, n+1, n+1, n+2)$ where $n \geq s+3$ for each $(i,j) \in \{(1,4),(4,1),(2,3),(1,6),(6,1),(2,4),(4,2),(2,6),(6,2),(4,5)\}$.

[2.] Study the chromaticity of the following graphs: (i) $K_{1,2}^{-sK_2}(n,n,n,n+1,n+1,n+1)$ where $n \geq s+2$, (ii) $K_{1,2}^{-sK_2}(n,n,n,n,n+1,n+2)$ where $n \geq s+3$, (iii) $K_{1,2}^{-sK_2}(n-2,n+1,n+1,n+1,n+1,n+1)$ where $n \geq s+5$, and (iv) $K_{1,2}^{-sK_2}(n-1,n,n,n+1,n+1,n+2)$ where $n \geq s+3$.

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