Periodicity of a time-delay system of fractional order joining *n*-deviating arguments

Rabha W. Ibrahim, M. Zaini Ahmad & M. Jasim Mohammed

Published online: 07 Sep 2016.

Submit your article to this journal ↗

Article views: 421

View related articles ↗

View Crossmark data ↗

Citing articles: 1 View citing articles ↗

Taylor & Francis
Taylor & Francis Group

Periodicity of a time-delay system of fractional order joining *n*-deviating arguments

Rabha W. Ibrahim[a] , M. Zaini Ahmad[b] and M. Jasim Mohammed[b]

[a]Faculty of Computer Science and Information Technology, University of Malaya, Kuala Lumpur, Malaysia; [b]Institute of Engineering Mathematics, Universiti Malaysia Perlis, Arau, Malaysia

ABSTRACT

In this study, we generalize a time-delay system joining *n*-deviating arguments by utilizing the concept of the Riemann–Liouville fractional calculus. The class of equations is taken in view of the Rayleigh-type equation. Our tool is based on the fixed-point theorems. The existence of the outcomes is delivered under some certain conditions. The application is illustrated in the sequel.

1. Introduction

Time-delay schemes or dead-time schemes, transmissible schemes are categories of differential equations (ordinary, partial and fractional) in which the derivative of the unidentified function at a positive time is given in the relations of the values of the function at former times. Usually such equations are imposed with deviating arguments. The applications of this problem are increasing and appearing in many fields such as dynamic systems and its relatives (Luo, Wang, Liang, Wei, and Alsaadi, 2016; Luo, Wang, Wei, and Alsaadi, 2016).

The most important study of time-delay systems is the periodicity of the solution. The periodic solutions of Rayleigh equations with two deviating arguments have been studied by many authors (Huang, He, Huang, and Tan, 2007; Peng, Liu, Zhou, and Huang, 2006; Sirma, Tunç, and Özlem, 2010). These studies mentioned that periodic solutions have desired properties in differential equations, and are among the most vital research directions in the theory of differential equations. Such solutions are also vital in dynamical systems, with applications ranging from celestial mechanics to biology and finance. The class of fractional differential equations (FDEs) is the most important modifications of the field of ODEs to an arbitrary (non-integer) order to model complex phenomena (Ibrahim, 2012; Ibrahim and Jahangiri, 2015; Ibrahim, Ahmad, and Mohammed, 2016; Lizama and Poblete, 2011). Investigations in physics, engineering, biological sciences and other fields have recently demonstrated the application of FDEs in the dynamics of many systems. FDEs with delay can also describe natural phenomena more accurately than those without delay.

$$u''(\tau) + \varphi(u'(\tau)) + \vartheta(\tau, u(\tau)) = \varrho(\tau).$$

The periodic solutions have also been studied for the different generalizations of the Rayleigh equation. Some studies are introduced to discuss the existence of solutions of certain ordinary differential equations with deviating arguments (see Chaddha and Pandey, 2016; Feng, 2013; Wang and Zhang, 2009). In particular, a specific example has been provided in on how *T*-periodic solutions can be acquired via these theorems with $\varrho(t) = 0$. The present study follows this direction. The active performance of the Rayleigh equation has been generally examined because of its request in many practical fields, such as mechanics, physics and the engineering method grounds. A similar method is observed in the case of an improved additional voltage. The Rayleigh system with *n*-deviating arguments is in the form of

$$D^{2\mu}u(\tau) + \varphi(u'(\tau)) + \sum_{k=1}^{n} \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau))) = \varrho(\tau),$$

(1)

where

$$(\mu \in (0,1], \ u \in \mathbb{R}^n, \ \varphi \in C(\mathbb{R}^n, \mathbb{R}), \ \vartheta_k \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}),$$
$$\varepsilon_k, \varrho \in C(\mathbb{R}, \mathbb{R}), \ k = 1, \ldots, n);$$

CONTACT Rabha W. Ibrahim ✉ rabhaibrahim@yahoo.com

$$(\vartheta_k(\tau + T, u) = \vartheta_k(\tau, u), \; \varepsilon_k(\tau + T) = \varepsilon_k(\tau),$$

$$\varrho(\tau + T) = \varrho(\tau)),$$

and $D^{2\mu}$ denotes the Riemann–Liouville differential operator. System (1) can be represented analogy voltage transmission. In a powered problem, φ characterizes a damping or friction time, ϑ_ι signifies the returning force, ϱ is a superficially useful force and ε_k is the time pause of the returning force. The periodicity of solutions trends in vibration and noise engineering. Therefore, we aim to establish the existence of the periodic solutions of Equation (1) under some certain conditions. These results are supported by an example.

2. Setting

In this section, we provide background definitions and existing results that are essential to our arguments to establish the periodic solutions of Equation (1). We let χ be a real Banach space and $\Lambda : \chi \to \chi$ be a completely continuous operator. The following preliminaries can be found in Burton (1985).

Definition 2.1: If u_0 is an isolated fixed point of Λ, then the fixed-point index at u_0 of Λ is defined by

$$\iota(\Lambda, u_0) = \deg(I - \Lambda, \mathcal{N}, \phi), \tag{2}$$

where $\mathcal{N}$ is a neighbourhood of u_0, which satisfies that u_0 is the unique fixed point in $\mathcal{N}$ of Λ.

Definition 2.2: If there exits $u_0 \in \chi$ with $u_0 \neq \phi$ such that $\Lambda u_0 = \alpha u_0, \alpha \in \mathbb{R}$ then α is called an eigenvalue of operator Λ and u_0 is called the eigenfunction of operator Λ corresponding to α.

Definition 2.3: Let $u : \mathbb{R} \to \mathbb{R}$ be continuous. The function $u(\tau)$ is periodic on $\mathbb{R}$ if

$$u(\tau + T) = u(\tau), \quad \forall \tau \in \mathbb{R}. \tag{3}$$

Definition 2.4: The Riemann–Liouville fractional integral is defined as follows:

$$I^\mu u(\tau) = \frac{1}{\Gamma(\mu)} \int_0^\tau (\tau - \varsigma)^{\mu-1} u(\varsigma) \, d\varsigma.$$

where Γ denotes the gamma function. The Riemann–Liouville fractional derivative is defined as follows:

$$D^\mu u(\tau) = \frac{1}{\Gamma(1 - \mu)} \frac{d}{d\tau} \int_0^\tau (\tau - \varsigma)^{-\mu} u(\varsigma) \, d\varsigma,$$

$$0 < \tau < \infty.$$

where Γ denotes the Gamma function (see Jumarie, 2013; Kilbas, Srivastava, and Trujillo, 2006; Podlubny, 1999; Tarasov, 2010).

Lemma 2.5 (Schauder fixed-point theorem): *Every continuous function from a convex compact subset of a Banach space to itself has at least one fixed point.*

Lemma 2.6 (Leray–Schauder fixed-point theorem): *Suppose that χ is a real Banach space and $\Lambda : \chi \to \chi$ is a completely continuous operator. If*

$$\{u : u \in \chi, u = \alpha \Lambda u, 0 < \alpha < 1\} \tag{4}$$

is bounded, then Λ has a fixed point $u^* \in \mathcal{N}$, where

$$\mathcal{N} := \{u, u \in \chi, \|u\| \leq \flat\},$$
$$\flat = \sup\{u : u \in \chi, \; u = \alpha \Lambda u, \; 0 < \alpha < 1\}. \tag{5}$$

Lemma 2.7: *Suppose $u(\tau) \in C^1[0, T]$ and $u(0) = u(T) = 0$. Then*

$$\int_0^T |u(\tau)|^2 \, d\tau \leq \frac{T^2}{\pi^2} \int_0^T |u'(\tau)|^2 \, d\tau.$$

Lemma 2.8: *Assume that $0 \leq \lambda \leq T$ is a constant, $s \in C(\mathbb{R}, \mathbb{R})$ is periodic with period T, and $\max_{\tau \in [0,T]} |s(\tau)| \leq \lambda$. Then, for any $\upsilon \in C^1(\mathbb{R}, \mathbb{R})$ that is periodic with period T, then*

$$\int_0^T |\upsilon(\tau) - \upsilon(\tau - s(\tau))|^2 \, d\tau \leq 2\lambda^2 \int_0^T |\upsilon'(\tau)|^2 \, d\tau.$$

3. Findings

The following assumptions are used in this study:

(H_1) $\varphi(0) = 0$, and there exists $\epsilon > 0$ such that $u\varphi(u) \geq \epsilon |u|^2$, for all $u \in \mathbb{R}^n$ (or $u\varphi(u) \leq -\epsilon |u|^2$, for all $u \in \mathbb{R}^n$).

(H_2) ϑ_k is differentiable with respect to τ, and $a_k \geq 0$, $b_k > 0$, $k = 1, \ldots, n$ exists such that

$$|\vartheta_k'(\tau, u)| \leq a_k + b_k|u|, \quad \forall (\tau, u) \in \mathbb{R} \times \mathbb{R}^n,$$
$$k = 1 \ldots, n. \tag{6}$$

(H_3) There exist $J_k > 0$ such that $|\vartheta_k(\tau, u_1) - \vartheta_k(\tau, u_2)| \leq J_k|u_1 - u_2|$, for all $\tau \in \mathbb{R}$, $u_1, u_2 \in \mathbb{R}^n$, $k = 1, \ldots, n$.

(H_4) There exists $d > 0$, such that for $|u| > d$, we have

$$\left(\sum_{k=1}^n \vartheta_k(\tau, u) - \varrho(\tau) \right) \text{sign } u > 0, \quad \forall \tau \in \mathbb{R}. \tag{7}$$

(H_5) There exist integers n_k such that $S_k \triangleq \max_{\tau \in [0,T]} |\varepsilon_k(\tau) - n_k T| \leq T$, $k = 1, \ldots, n$.

(H_6) $\vartheta'_k(\tau, u)$, $k = 1, \ldots, n$ exist and are continuous, $\sum_{k=1}^{n} \vartheta'_k(\tau, 0) \equiv c < 0$, where c is a negative constant.

(H_7) $\lim_{u \to 0}(\varphi(u')/u') = r \geq \epsilon > 0$.

Theorem 3.1: Let (H_1)–(H_5) hold. If $1 - \alpha = \Gamma(2\mu + 1)$, $\alpha \in (0, 1)$, $\mu \in (0, 1]$, $\sum_{k=1}^{n} \vartheta_k(\tau, 0) \neq \varrho(\tau)$ and

$$\left(\sum_{k=1}^{n} J_k \sqrt{2S_k} + \frac{T^2}{\pi^2} \sum_{k=1}^{n} b_k + \frac{T^{3/2}}{\pi^2} \sum_{k=1}^{n} a_k \right) < \epsilon, \quad \epsilon > 0,$$

then (1) has at least one periodic solution.

Proof: Our aim is to apply Lemma 2.6. Let

$$\chi = \{u : u \in C(\mathbb{R}^n), u(\tau + T) = u(\tau)\},$$
$$\Upsilon = \{u : u \in C^1(\mathbb{R}^n), u(\tau + T) = u(\tau)\}.$$

Then χ and Υ are real Banach spaces with the norm

$$\|u\|_\infty = \max_{\tau \in [0,T]} |u(\tau)|, \quad \|u\| = \|u\|_\infty + \|u'\|_\infty.$$

We define an operator $G_\mu : \Upsilon \to \chi$ by

$$(G_\mu u)(\tau) := \varphi(u'(\tau)) + \sum_{k=1}^{n} \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau)))$$
$$- u(\tau) - \varrho(\tau), \quad u \in \Upsilon. \tag{8}$$

It is clear that G_μ is continuous and bounded. By utilizing G_μ, we define an extended operator $\Lambda := \kappa G_\mu : \Upsilon \to \Upsilon$. Thus, Λ is completely continuous. According to Lemma 2.6, the set

$$\{u : u \in \Upsilon, u = \alpha \Lambda u, 0 < \alpha < 1\}$$

is bounded in Υ, and hence, Λ has a fixed point in Υ. Therefore, Equation (1) has a periodic solution u such that $u(0) = u(T)$. To prove that the periodic solution is bounded, we aim to apply Lemmas 2.7 and 2.8. Consider $u \in \Upsilon$, $0 < \alpha < 1$ achieving $u = \alpha \Lambda u$. Consequently, $u(\tau)$ is a solution for

$$D^{2\mu} u(\tau) + \alpha \varphi(u'(\tau)) + \alpha \sum_{k=1}^{n} \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau)))$$
$$+ (1 - \alpha)u(\tau) = \alpha \varrho(\tau). \tag{9}$$

Assume that $\bar{\tau}$ and $\underline{\tau}$ are the maximum point and minimum point of $u(\tau)$ on $[0, T]$. Then, we have

$$\alpha \sum_{k=1}^{n} \vartheta_k(\underline{\tau}, u(\underline{\tau} - \varepsilon_k(\underline{\tau}))) + (1 - \alpha)u(\underline{\tau}) - \alpha \varrho(\underline{\tau}) \geq 0,$$

$$\alpha \sum_{k=1}^{n} \vartheta_k(\bar{\tau}, u(\bar{\tau} - \varepsilon_k(\bar{\tau}))) + (1 - \alpha)u(\bar{\tau}) - \alpha \varrho(\bar{\tau}) \leq 0,$$

and thus, there exists $\Im \in [\underline{\tau}, \bar{\tau}]$ with

$$\alpha \sum_{k=1}^{n} \vartheta_k(\Im, u(\Im - \varepsilon_k(\Im))) - \alpha \varrho(\Im) + (1 - \alpha)u(\Im) = 0, \tag{10}$$

which yields

$$\alpha \left[\sum_{k=1}^{n} \vartheta_k(\Im, u(\Im - \varepsilon_k(\Im))) \right] \times \text{sign } u(\Im)$$
$$+ (1 - \alpha)|u(\Im)| = 0. \tag{11}$$

By (H_4), it is clear that $|u(\Im)| \leq d$. Therefore, we obtain

$$|u(\tau)| = |u(\Im) + \int_{\Im}^{\tau} u'(\varsigma) \, d\varsigma| \leq d + \sqrt{T}\|u'\|_{L^2}; \tag{12}$$

consequently, we attain

$$\|u\|_\infty \leq d + \sqrt{T}\|u'\|_{L^2}, \tag{13}$$

where $\| \bullet \|_{L^2}$ is the norm of $L^2[0, T]$. By using the fact (see Kilbas et al., 2006),

$$D^{2\mu} u(\tau) \approx (2\mu)! u''(\tau) = \Gamma(2\mu + 1)u''(\tau),$$

Equation (9) can be approximated as follows:

$$u''(\tau) + \frac{\alpha}{\Gamma(2\mu + 1)} \varphi(u'(\tau)) + \frac{\alpha}{\Gamma(2\mu + 1)}$$
$$\times \sum_{k=1}^{n} \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau))) + \frac{(1 - \alpha)}{\Gamma(2\mu + 1)} u(\tau)$$
$$= \frac{\alpha}{\Gamma(2\mu + 1)} \varrho(\tau). \tag{14}$$

If $1 - \alpha = \Gamma(2\mu + 1)$, then $u''(\tau) + u(\tau) = 0$ has a periodic solution. Therefore, by multiplying Equation (14) with $u'(\tau)$, integrating from 0 to T and taking into account that u is a periodic solution of Equation (14), we obtain

$$\int_0^T \varphi(u'(\tau))u'(\tau) \, d\tau$$
$$= -\sum_{k=1}^{n} \int_0^T \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau)))u'(\tau) \, d\tau$$
$$+ \int_0^T \varrho(\tau)u'(\tau) \, d\tau. \tag{15}$$

Using (H_1), we obtain that

$$\int_0^T \varphi(u'(\tau))u'(\tau) \, d\tau \geq \epsilon \int_0^T |u'(\tau)|^2 \, d\tau. \tag{16}$$

Holder's inequality implies

$$\epsilon \int_0^T (u'(\tau))(u')^2(\tau)\, d\tau$$

$$\leq \left| \sum_{k=1}^n \int_0^T \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau))) u'(\tau)\, d\tau \right| + \|\varrho\|_{L^2} \|u'\|_{L^2}$$

$$\leq \sum_{k=1}^n \int_0^T |\vartheta_k(\tau, u(\tau - \varepsilon_k(\tau))) - \vartheta_k(\tau, u(\tau))| |u'(\tau)|\, d\tau$$

$$+ \sum_{k=1}^n \left| \int_0^T \vartheta_k(\tau, u(\tau)) u'(\tau)\, d\tau \right| + \|\varrho\|_{L^2} \|u'\|_{L^2}. \tag{17}$$

Since $\int_0^{u(\tau)} \vartheta_k(\tau, v)\, dv$, $k = 1, \ldots, n$ are T-periodic, differentiable, and

$$\frac{d}{d\tau} \int_0^{u(\tau)} \vartheta_k(\tau, v)\, dv$$

$$= \vartheta_k(\tau, u(\tau)) u'(\tau) + \int_0^{u(\tau)} \vartheta'_k(\tau, v)\, dv, \quad k = 1, \ldots, n, \tag{18}$$

then, we obtain

$$\int_0^T \vartheta_k(\tau, u(\tau)) u'(\tau)\, d\tau = - \int_0^T d\tau \int_0^{u(\tau)} \vartheta'_k(\tau, v)\, dv,$$

$$k = 1, \ldots, n. \tag{19}$$

Joining Equations (17) and (19) with (H_2) and (H_3), we impose

$$\epsilon \int_0^T u'^2(\tau)\, d\tau$$

$$\leq \sum_{k=1}^n J_k \int_0^T |u(\tau) - u(\tau - \varepsilon_k(\tau))| |u'(\tau)|\, d\tau$$

$$+ \sum_{k=1}^n \int_0^T d\tau \int_0^{|u(\tau)|} (a_k + b_k|v|)\, dv + \|\varrho\|_{L^2} \|u'\|_{L^2}$$

$$\leq \|u'\|_{L^2} \sum_{k=1}^n J_k \left(\int_0^T |u(\tau) - u(\tau - \varepsilon_k(\tau) - n_k T)|^2\, d\tau \right)^{1/2}$$

$$+ \left(\sum_{k=1}^n a_k \right) \int_0^T |u(\tau)|\, d\tau + \left(\sum_{k=1}^n b_k \right) \int_0^T |u(\tau)|^2\, d\tau$$

$$+ \|\varrho\|_{L^2} \|u'\|_{L^2}. \tag{20}$$

In view of Lemma 2.7, we conclude

$$\int_0^T |u(\tau)|^2\, d\tau \leq \frac{T^2}{\pi^2} \int_0^T |u'(\tau)|^2\, d\tau. \tag{21}$$

Consequently, we receive

$$\int_0^T |u(\tau)|\, d\tau \leq \sqrt{T} \left(\int_0^T |u(\tau)|^2\, d\tau \right)^{1/2}$$

$$\leq \sqrt{T} \left(\frac{T^2}{\pi^2} \int_0^T |u'(\tau)|^2\, d\tau \right)^{1/2}$$

$$\leq \frac{T^{3/2}}{\pi^2} \left(\int_0^T |u'(\tau)|^2\, d\tau \right)^{1/2}. \tag{22}$$

By (H_5) and Lemma 2.8, we observe

$$\left(\int_0^T |u(\tau) - u(\tau - \varepsilon_k(\tau) - n_k T|^2\, d\tau \right)^{1/2}$$

$$\leq \sqrt{2S_k} \left(\int_0^T |u'(\tau)|^2\, d\tau \right)^{1/2}, \quad k = 1, \ldots, n. \tag{23}$$

Thus, applying Equations (21)–(23) in Equation (20), we arrive at

$$\epsilon \|u'\|_{L^2}^2 \leq \left(\sum_{k=1}^n J_k \sqrt{2S_k} + \frac{T^2}{\pi^2} \sum_{k=1}^n b_k + \frac{T^{3/2}}{\pi^2} \sum_{k=1}^n a_k \right)$$

$$\times \|u'\|_{L^2}^2 + \|\varrho\|_{L^2}] \|u'\|_{L^2}$$

$$:= \Xi \|u'\|_{L^2}^2 + \|\varrho\|_{L^2}] \|u'\|_{L^2}, \tag{24}$$

which is equivalent to

$$\|u'\|_{L^2}^2 \leq \frac{\|\varrho\|_{L^2}] \|u'\|_{L^2}}{\epsilon - \Xi}$$

for sufficient $\epsilon > \Xi$; thus, we get

$$\|u\|_\infty \leq M_1, \quad M_1 > 0. \tag{25}$$

This showed that the periodic solution of Equation (1) is bounded. This completes the proof. ∎

Theorem 3.2: Assume that (H_1)–(H_7) hold. If $1 - \alpha = \Gamma(2\mu + 1)$, $\alpha \in (0, 1)$, $\mu \in (0, 1]$, $\sum_{k=1}^n \vartheta_k(\tau, 0) = \varrho(\tau)$ and

$$\left(\sum_{k=1}^n J_k \sqrt{2S_k} + \frac{T^2}{\pi^2} \sum_{k=1}^n b_k + \frac{T^{3/2}}{\pi^2} \sum_{k=1}^n a_k \right) < r, \quad r > 0.$$

Proof: Let Λ, χ and Υ be given as in Theorem 3.1. We define a differential operator Θ as follows:

$$(\Theta u)(\tau) := \kappa \left[r u'(\tau) + \left(\sum_{k=1}^n \vartheta'_k(\tau, 0) \right) u(\tau) - u(\tau) \right],$$

$$\kappa > 0.$$

Our aim is to apply Lemma 2.5. Therefore, we must show that the operator Λ is convex in Υ. By utilizing

the assumptions (H_1)–(H_7) and $\sum_{k=1}^{n} \vartheta_k(\tau,0) = \varrho(\tau)$, a computation implies that

$$\|\Lambda u - \Theta u\|$$

$$= \left\| \kappa \left[\varphi(u'(\tau)) + \sum_{k=1}^{n} \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau))) - u(\tau) - \varrho(\tau) \right] \right.$$

$$\left. - \kappa \left[ru'(\tau) + \left(\sum_{k=1}^{n} \vartheta'_k(\tau, 0) \right) u(\tau) - u(\tau) \right] \right\|$$

$$\leq \kappa \left\| \left(\frac{\varphi(u'(\tau))}{u'(\tau)} - r \right) + \sum_{k=1}^{n} \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau))) \right.$$

$$\left. - \left(\sum_{k=1}^{n} \vartheta'_k(\tau, 0) \right) u(\tau) - \varrho(\tau) \right\|$$

$$= \kappa \left\| \left(\frac{\varphi(u'(\tau))}{u'(\tau)} - r \right) + \sum_{k=1}^{n} \vartheta_k(\tau, u(\tau - \varepsilon_k(\tau))) \right.$$

$$- \sum_{k=1}^{n} \vartheta_k(\tau, 0)) + \sum_{k=1}^{n} \vartheta_k(\tau, 0))$$

$$\left. - \left(\sum_{k=1}^{n} \vartheta'_k(\tau, 0) \right) u(\tau) - \varrho(\tau) \right\| \leq \wp \|u\|, \quad \wp := \kappa |c|_J > 0.$$

But κ is arbitrary, this yields

$$\lim_{\|u\| \to 0} \frac{\|\Lambda u - \Lambda 0 - \Theta u\|}{\|u\|} = 0, \quad \Lambda 0 = 0. \quad (26)$$

Thus, we get $(\Theta u) = \Lambda'(u)$. Since $(\Theta u) \geq (\Theta 0)$, we have

$$\frac{\|\Lambda u\|}{\|u\|} \geq \|\Lambda'(0)\|,$$

which implies that Λ is a continuously differentiable convex function. Hence, Λ has at least one periodic fixed point, which corresponds to the solution of Equation (1). In the same manner of Theorem 3.1, and

$$\int_0^T \varphi(u'(\tau)) u'(\tau) \, d\tau = \int_0^T \frac{\varphi(u'(\tau))}{u'(\tau)} (u'(\tau))^2 \, d\tau$$

$$\leq r \int_0^T |u'(\tau)|^2 \, d\tau,$$

we have

$$r\|u'\|_{L^2}^2 \leq \left(\sum_{k=1}^{n} J_k \sqrt{2S_k} + \frac{T^2}{\pi^2} \sum_{k=1}^{n} b_k + \frac{T^{3/2}}{\pi^2} \sum_{k=1}^{n} a_k \right)$$

$$\times \|u'\|_{L^2}^2 + 2\|\varrho\|_{L^2}] \|u'\|_{L^2}$$

$$= \Xi \|u'\|_{L^2}^2 + 2\|\varrho\|_{L^2}] \|u'\|_{L^2}, \quad (27)$$

which is equivalent to

$$\|u'\|_{L^2}^2 \leq \frac{2\|\varrho\|_{L^2}] \|u'\|_{L^2}}{r - \Xi}.$$

for sufficient $r > \Xi$; thus, we get

$$\|u\|_\infty \leq M_2, \quad M_2 > 0. \quad (28)$$

This leads that the periodic solution of Equation (1) is bounded. This completes the proof. ∎

4. An example

Consider $\vartheta_k(u) = \frac{1}{20} u$, $k = 2$, $\varphi(u'(\tau)) = u'(\tau)$ and $\varrho(\tau) = 1$; in the following fractional system:

$$D^{2\mu} u(\tau) + u'(\tau) + \vartheta_1(\tau, u(\tau - \varepsilon_1(\tau)))$$

$$+ \vartheta_2(\tau, u(\tau - \varepsilon_2(\tau))) = \varrho(\tau), \quad (29)$$

such that $T = 1$, $\epsilon = \frac{1}{2} < r = 1$, $S_k = |\frac{1}{2} - 1| = \frac{1}{2} < T = 1$, $\varepsilon_k = \frac{1}{2}$, $J_k = \frac{1}{20}$ and $a_k = b_k = 0.1$, $k = 1, 2$. This implies

$$\left(\sum_{k=1}^{n} J_k \sqrt{2S_k} + \frac{T^2}{\pi^2} \sum_{k=1}^{n} b_k + \frac{T^{3/2}}{\pi^2} \sum_{k=1}^{n} a_k \right)$$

$$= 0.14 < 0.5 = \epsilon$$

and if $\mu = 0.1$ we get $\alpha = 1 - \Gamma(2\mu + 1) = 1 - \Gamma(1.2) = 1 - 0.918 = 0.082$ then in view of Theorem 3.1, the system (29) has at least one periodic solution in $[0, 1]$. Note that if $\mu \geq 1/2$, Theorem 3.1 is failing because $\alpha > 1$.

Now, we let $\vartheta_k(u) = \frac{1}{20} u + \frac{1}{2}$, $k = 1, 2$, $\varrho(\tau) = 1$. It is clear that $\sum_{k=1}^{2} \vartheta_k(0) = 1 = \varrho(\tau)$. Hence, under the above data, we conclude that

$$\left(\sum_{k=1}^{n} J_k \sqrt{2S_k} + \frac{T^2}{\pi^2} \sum_{k=1}^{n} b_k + \frac{T^{3/2}}{\pi^2} \sum_{k=1}^{n} a_k \right)$$

$$= 0.14 < 1 = r,$$

which yields, by Theorem 3.2, the system (29) has at least one periodic solution.

5. Conclusions

By utilizing various types of fixed-point theorems, we showed that a system of FDEs-type Rayleigh equation of n-arguments has at least one periodic solution. We imposed two cases of this system, depending on the n-arguments (homogenous and non-homogenous). The conditions are little different in both cases. The main function of this system is $\varphi(u')$, which has upper and lower bounds (see (H_1) and (H_7)). The delay is computed for the periodic solution by the argument functions ϑ_k. In the future works, we shall discuss the periodicity of solutions of fractional differential systems, including the generalized term $D^{3\mu}$ by utilizing the continuation theorem.

Authors' contributions

The authors jointly worked on deriving the results and approved the final manuscript.

Acknowledgments

Disclosure statement

No potential conflict of interest was reported by the authors.

ORCiD

R.W. Ibrahim http://orcid.org/0000-0001-9341-025X

References

Burton, T.A. (1985). *Stability and periodic solutions of ordinary and functional-differential equations, Vol. 178*. Orlando, FL: Academic Press.

Chaddha, A., & Pandey, D. N. (2016). Approximations of solutions for an impulsive fractional differential equation with a deviated argument. *International Journal of Applied and Computational Mathematics*, *2*(2), 269–289.

Feng, M. (2013). Periodic solutions and nontrivial periodic solutions for a class of Rayleigh-type equation with two deviating arguments. *Journal of Function Spaces*, *2013*, 7 p.

Huang, C., He, Y., Huang, L., & Tan, W. (2007). New results on the periodic solutions for a kind of Rayleigh equation with two deviating arguments. *Mathematical and Computer Modelling*, *46*(5), 604–611.

Ibrahim, R. W. (2012). Existence of deviating fractional differential equation. *Cubo (Temuco)*, *14*(3), 129–142.

Ibrahim, R. W., Ahmad, M. Z., & Mohammed, M. J. (2016). Periodicity and positivity of a class of fractional differential equations. *Springer Plus*, *5*(1), 1–10.

Ibrahim, R. W., & Jahangiri, J. M. (2015). Existence of fractional differential chains and factorizations based on transformations. *Mathematical Methods in the Applied Sciences*, *38*(12), 2630–2635.

Jumarie, G. (2013). *Fractional differential calculus for non-differentiable functions. Mechanics, Geometry, 1010 Stochastics, Information theory*. Saarbrucken: Lambert Academic Publishing.

Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations, Vol. 204*. Amsterdam: Elsevier Science Limited.

Lizama, C., & Poblete, V. (2011). Periodic solutions of fractional differential equations with delay. *Journal of Evolution Equations*, *11*(1), 57–70.

Luo, Y., Wang, Z., Liang, J., Wei, G., & Alsaadi, F. E. (2016). H_{∞} control for 2-D Fuzzy systems with interval time-varying delays and missing measurements. *IEEE Transactions on Cybernetics*, doi:10.1109/TCYB.2016.2514846

Luo, Y., Wang, Z., Wei, G., & Alsaadi, F (2016). Robust H-infinity filtering for a class of two-dimensional uncertain fuzzy systems with randomly occurring mixed delays. *IEEE Transactions on Fuzzy Systems*, doi:10.1109/TFUZZ.2016.2556001

Peng, L., Liu, B., Zhou, Q., & Huang, L. (2006). Periodic solutions for a kind of Rayleigh equation with two deviating arguments. *Journal of the Franklin Institute*, *343*(7), 676–687.

Podlubny, I. (1999). *Fractional differential equations*. San Diego, CA: Academic Press.

Sirma, A., Tunç, C., & Özlem, S. (2010). Existence and uniqueness of periodic solutions for a kind of Rayleigh equation with finitely many deviating arguments. *Nonlinear Analysis: Theory, Methods & Applications* , *73*(2), 358–366.

Tarasov, V. (2010). *Fractional dynamics: Applications of fractional calculus to dynamics of particles, fields and media*. New York: Springer.

Wang, Y., & Zhang, L. (2009). Existence of asymptotically stable periodic solutions of a Rayleigh type equation. *Nonlinear Analysis: Theory, Methods & Applications* *71*(5), 1728–1735.