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ORIGINAL RESEARCH

Symmetric-periodic solutions for some types of generalized neutral equations

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Abstract The existence of symmetric-periodic outcomes for a class of fractional differential equations has been increasingly studied. Such study has used various methods such as fixed point theory, critical point theory, and approximation theory. In this work, we study the m-pseudo almost automorphic (m-PΛΛ) outcomes for a category of fractional neutral differential equations. To satisfy this aim, we introduce composition results under suitable conditions and employ them to establish some extant outcomes using interpolation theory mixed with fixed point technique. Examples are illustrated.

Keywords Fractional calculus · Fractional differential equations · Periodic symmetric solution

Mathematics Subject Classification 34A08

Introduction

The symmetry in the field of differential equations is a transformation that preserves its domestic of results invariant. Symmetry study can be utilized to resolve some classes of ordinary, partial, fractional differential equations, though defining the symmetries can be computationally concentrated like other mathematical methods. The best method for symmetry is by finding the periodic solution of the differential equation.

In 1962, Bochner [1] introduced the concept of almost automorphy, which is an important generalization of almost periodicity. The concept of almost periodic functions was introduced by Bohr [2]. It was named as PAA functions because they originally presented themselves, in their work in differential geometry, as scalars and tensors on manifolds with (discrete) groups of automorphisms [3]. Later, PΛΛ function has become one of the most attractive topics in the qualitative theory of evolution equations, and there have been several interesting, natural and powerful generalizations of the classical PΛΛ functions [4–9]. Recently, Digana et al., studied the concept for different classes of ODF and PDE (see [10–14]). Xiao et al. [15] introduced the concept of PΛΛ functions for a natural and a significant extension of PΛΛ functions. Moreover, they proved that the space of PΛΛ functions is complete; so they solve a key fundamental problem on this issue and pave the road to further study the applications of PΛΛ functions. They investigated the existence of PΛΛ to

$$u'(t) = \Lambda(t)u(t) + f(t)$$

and

$$u'(t) = \Lambda(t)u(t) + f(t, u(t))$$

in a Banach space. Chang and Luos [16] presented a composition theorem for m-PΛΛ function, which was proved under appropriate conditions. They applied this theorem to investigate whether the m-PΛΛ solutions exist in the neutral differential equation as follows:

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$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t)+f(t,u(t))]=\Lambda u(t)+g(t,u(t)), \quad t\in\mathbb{R}.$$

Periodic motion is a very important and special phenomena not only in natural science, but also in social science, such as climate, food supplement, insecticide population and sustainable development. Periodic solutions are desired property in differential equations, constituting one of the most important research directions in the theory of differential equations. The existence of periodic solutions is often a desired property in dynamical systems, constituting one of the most important research directions in the theory of dynamical systems, with applications ranging from celestial mechanics to biology and finance. Fractional differential equations (FDEs) are the most important generalizations of the field of ODE [17–21]. Recent investigations in physics, engineering, biological sciences and other fields have demonstrated that the dynamics of many systems are described more accurately using FDEs, and that FDE with delay are often more realistic to describe natural phenomena than those without delay. Periodic solution fractional differential equations have been studied by many researchers. They studied periodic solutions of the equation (see [17, 18])

$$D^{\alpha}u(t)+BD^{\beta}u(t)\Lambda u(t)=f(t), \quad 0\le t\le 2\pi,$$

where A and B are closed linear operators defined on a complex Banach space X with domains $D(A)$ and $D(B)$, respectively, $0\le\beta<\alpha\le 2$.

The aim of this paper is to study the existence of periodic solutions for the following FDE:

$$D^{\mu}\Big(\upsilon(t)+\varphi(t,\upsilon(t))\Big)=\Lambda\upsilon(t)+\vartheta(t,\upsilon(t)), \quad t\in\Re \tag{1}$$

$\forall\mu\in(0,1]$, where $\Lambda: dom(\Lambda)\subset\chi\to\chi$ is considered the operator of a hyperbolic analytic semigroup $T(t)_{t\ge 0}$, and $\varphi:\Re\times\chi\to\chi_{\delta}(<\lambda<\delta<),\vartheta\ \ \Re\times\chi\to\chi$ are appropriate continuous functions; χ_{δ} refers to the appropriate interpolation space and D^{μ} is the Riemann–Liouville fractional differential operator (R–L operator).

This paper is classified as follows. In "Setting", we present some basic definitions, lemmas, and setting results which will be used in this study. In "Findings", we introduce some existence results of almost-periodic and mild solutions of the fractional neutral differential equation. Examples are illustrated in the sequel.

Setting

The researchers allocated this section to investigate some results required in the sequel. In this paper, the notations $(\chi,\|\bullet\|)$ and $(\Upsilon,\|\bullet\|_{\Upsilon})$ denote the two Banach spaces, whereas $BC(\Re,\chi)$ refers to the Banach space of all bounded continuous functions from $\Re$ to χ, qualified with the supremum norm $\|\varphi\|_{\infty}=sup_{t\in\Re}\|\varphi(t)\|$. Let χ_{λ} be a space mediated between $dom(\Lambda)$ and $\chi.B(\Re,\chi_{\lambda})$ for $\lambda\in(0,1)$ refers to Banach space of all bounded continuous functions $\sigma:\Re\to\chi_{\lambda}$ when supported with the $\lambda-sup$ norm:

$$\|\sigma\|_{\lambda,\infty}:=sup_{t\in\Re}\|\sigma(t)\|_{\lambda}$$

for $\sigma\in BC(\Re,\chi_{\lambda})$. Throughout this paper, $\wp$ denotes the Lebesgue field of $\Re$ and $\aleph$ the set of all positive measures m on $\wp$ satisfying $m(\Re)=+\infty$ and $m([a,b])<+\infty$, for all $a,b\in\Re(\mathfrak{a}<\mathfrak{b})$.

Definition 2.1 [3] A continuous function $\varphi:\Re\to\chi$ is referred to as automorphic in the case that every sequence of real numbers $(\varsigma_{\eta})_{\eta\in N}$ has a subsequence $(\varsigma'_{\eta})_{\eta\in N}\subset(\varsigma_{\eta})_{\eta\in N}$, such that

$$\lim_{\eta,n\to\infty}\|\varphi(t+\varsigma_{\eta}-\varsigma_{n})-\varphi(t)\|=0.$$

Define

$$\rho\Lambda\Lambda_0(\Re,\chi)=\left\{\Phi\in\mathfrak{BC}(\Re,\chi)\ \lim_{\mathfrak{T}\to\infty}\frac{}{\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}\|\varphi(\tau)\|\mathrm{d}\tau=\right\}.$$

Likely, $\rho\Lambda\Lambda_0(\Re\times\chi,\chi)$ is defined as the gathering of combined continuous functions $\varphi:\Re\times\chi\to\chi$ which belong to $BC(\Re\times\chi,\chi)$ and satisfy

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|\varphi(\tau,x)\|\mathrm{d}\tau=0$$

uniformly in a compact subset of χ.

Definition 2.2 [15] A continuous function $\varphi:\Re\to\chi$ (respectively, $\Re\times\chi\to\chi$) represents pseudo automorphic when decomposed as $\varphi=\vartheta+\Phi$, where $\vartheta\in\Lambda\Lambda(\Re,\chi)$ (respectively, $\Lambda\Lambda(\Re\times\chi,\chi)$) and $\Phi\in\rho\Lambda\Lambda_0(\Re,\chi)$(respectively, $\rho\Lambda\Lambda_0(\Re\times\chi,\chi)$)). Denote by $\rho\Lambda\Lambda(\Re,\chi)$ (respectively, $\rho\Lambda\Lambda(\Re\times\chi,\chi)$) the set of all such functions.

Definition 2.3 [19] Let $m\in\aleph$. A bounded continuous function $\varphi:\Re\to\chi$ is referred to as m-ergodic if φ is ergodic with respect to m (measure) i.e.

$$\lim_{r\to\infty}\frac{1}{m([-r,r])}\int_{[-r,r]}\|\varphi(t)\|\mathrm{d}m(t)=0.$$

The space of all such functions is denoted as $\zeta(\Re,\chi,\mathfrak{m})$ and $(\zeta(\Re,\chi,\mathfrak{m}),\|\bullet\|_{\infty})$ is a Banach space (see [19], Proposition 2.13]). The word ergodic (work) is employed to explain the dynamical system which has the same behavior averaged during the item time as averaged over the phase space.

Definition 2.4 [19] Let $m\in\aleph$. A continuous function $\varphi:\Re\to\chi$ is stated to be m-P$\Lambda\Lambda$ if φ comes in the form

$\varphi = \vartheta + \Phi$, where $\vartheta \in \Lambda\Lambda(\Re, \chi)$ and $\Phi \in \zeta(\Re, \chi, \mathfrak{m})$. So, all such functions have a space denoted by $\rho\Lambda\Lambda(\Re, \chi, \mathfrak{m})$. Most clearly, we have $\Lambda\Lambda(\Re, \chi) \subset \rho\Lambda\Lambda(\Re, \chi, \mathfrak{m}) \subset BC(\Re, \chi)$.

Lemma 2.5 [20, Theorem 2.2.6] *If* $\varphi : \Re \times \chi \mapsto \chi$ is P$\Lambda\Lambda$, *and assume that* $\varphi(t, \bullet)$ *is uniformly continuous on each bounded subset* $\kappa \subset \chi$ *uniformly for* $t \in \Re$, *that is for any* $\zeta > 0$, *there exists* $S > 0$ *such that* $x, y \in \kappa$ *and* $\|x - y\| < S$ *imply that* $\|\varphi(t, x) - \varphi(t, y)\| < \zeta$ *for all* $t \in \Re$. Let $\Phi : \Re \mapsto \chi$ *be* $P\ \Lambda\Lambda$. *Then the function* $F : \Re \mapsto \chi$ *defined by* $F(t) = \varphi(t, \Phi(t))$ *is* P$\Lambda\Lambda$.

Lemma 2.6 [19, Theorem 4.1] *Let* $m \in \aleph$ *and* $\varphi \in \rho\Lambda\Lambda(\Re, \chi, \mathfrak{m})$ *be such that* $\varphi = \vartheta + \Phi$, *where* $\vartheta \in \Lambda\Lambda(\Re, \chi)$ *and* $\Phi \in \zeta(\Re, \chi, \mathfrak{m})$. *If* $\rho\Lambda\Lambda(\Re, \chi, \mathfrak{m})$ *is translation invariant, then*

$$\{\vartheta(t) : t \in \Re\} \subset \overline{\{\varphi(t)\ \ t \in \Re\}}$$

(the closure of the range of φ*).*

Lemma 2.7 [19, Theorem 2.14] *Let* $m \in \aleph$ *and* I *be the bounded interval (eventually* $I = \emptyset$*). Suppose that* $\varphi \in BC(\Re, \chi)$. *The assertions indicated as following are equivalent*.

(i) $\varphi \in \zeta(\Re, \chi, \mathfrak{m})$;

(ii) $\lim_{r \to +\infty} \frac{1}{m([-r,r] \setminus I)} \int_{[-r,r] \setminus I} \|\varphi(t)\| \mathrm{d}m(t) = 0$;

(iii) For any $\zeta > 0, \lim_{r \to +\infty} \frac{m(\{t \in [-r,r] \setminus I : \|\varphi(t)\| > \zeta\})}{m([-r,r] \setminus I)} = 0$.

In the sequel, we need some notions and properties of intermediate spaces and hyperbolic semi groups. Let χ and Z be Banach spaces, with norms $\| \bullet \|_{\chi}, \| \bullet \|_Z$, respectively, and assume that Z is continuously embedded in χ, that is, $Z \hookrightarrow \chi$.

Definition 2.8 The Riemann–Liouville fractional integral is defined as follows:

$$I^{\mu} u(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - \varsigma)^{\mu - 1} u(\varsigma) \mathrm{d}\varsigma,$$

where Γ denotes the gamma function (see [22, 23]).

Definition 2.9 The Riemann–Liouville fractional derivative is defined as follows:

$$D^{\mu} u(t) = \frac{1}{\Gamma(1 - \mu)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t - \varsigma)^{-\mu} u(\varsigma) \mathrm{d}\varsigma, \quad 0 < t < \infty.$$

Definition 2.10 [8, *Definition* 2.5] A semi group $(T(t))_{t \geq 0}$ on χ is stated to be hyperbolic if there is a projection ρ and constants $\aleph, S > 0$ such that each $T(t)$ commutes with $\rho, Ker\rho$ is invariant with respect to $T(t), T(t) : ImQ \to ImQ$ is invertible and for every $x \in \chi$

$$\|T(t)\rho x\| \leq M\varrho^{-st}\|x\|, \quad for\, t \geq 0; \tag{2}$$

$$\|T(t)Qx\| \leq M\varrho^{-st}\|x\|, \quad for\, t \leq 0, \tag{3}$$

where $Q := I - \rho$ and, for $t < 0, T(t) = T(-t)^{-1}$.

Definition 2.11 [14] A linear operator $\Lambda : dom(\Lambda) \subset \chi \to \chi$ (not necessarily densely defined) is referred to as sectorial if the following hold: there exist constants $\mho \in \Re, \theta \in (\frac{\pi}{2}, \pi)$ and $M > 0$ such that

$$p(\Lambda) \subset \varsigma_{\theta,\mho} := \{\alpha \in \ell : \alpha \neq \mho, |arg(\alpha - \mho)| < \theta\},$$

$$\|\Re(\alpha, \Lambda)\| \leq \frac{\mathfrak{M}}{|\alpha - \mho|'}, \quad \alpha \in \varsigma_{\theta,\mho}.$$

Definition 2.12 [8, *Definition* 2.7] Let $0 \leq \lambda \leq 1$. A Banach space Υ such that $Z \hookrightarrow \Upsilon \hookrightarrow \chi$ refers to the class J_{λ} between χ and Z if there is a constant $c > 0$ such that

$$\|x\|_{\Upsilon} \leq c\|x\|^{1-\lambda}\|x\|_Z^{\lambda}(x \in Z).$$

In this case, we write $\Upsilon \in J_{\lambda}((X), Z)$.

Definition 2.13 [8, *Definition* 2.8] Let $\Lambda : dom(\Lambda) \subset \chi \to \chi$ be a sectorial operator. A Banach space $(\chi_{\lambda}, \| \bullet \|_{\lambda}), \lambda \in (0, 1)$ is said to be an intermediate space between χ and $dom(\Lambda)$ if $\chi_{\lambda} \in J_{\lambda}$.

For the problem (1), we list the following assumptions:

(H1) If $0 \leq \lambda < \delta < 1$, then we let k_1 be the bound of the embedding $\chi_{\lambda} \hookrightarrow \chi$, that is

$$\|\upsilon\| \leq k_1\|\upsilon\|_{\lambda} \quad \text{for}\, \upsilon \in \chi_{\lambda}.$$

(H2) Let $0 \leq \lambda < \delta < 1$ and the function $\varphi : \Re \times \chi \to \chi_{\delta}$ belongs to $\rho\Lambda\Lambda(\Re, \chi_{\delta}, \mathfrak{m})$, while $\vartheta : \Re \times \chi \to \chi$ belongs to $\rho\Lambda\Lambda(\Re, \chi, \mathfrak{m})$. Moreover, the functions φ, ϑ are uniformly Lipschitz in rotation to the second following argument: there exist $K > 0$ such that

$$\|\varphi(t, \upsilon) - \varphi(t, \upsilon)\|_{\delta} \leq K\|\upsilon - v\|$$

and

$$\|\vartheta(t, \upsilon) - \vartheta(t, \upsilon)\| \leq K\|\upsilon - v\|$$

for all $\upsilon, v \in \chi$ and $t \in \Re$.

Findings

In the present section, a composition theorem is proved for m-P$\Lambda\Lambda$ functions under appropriate conditions. Then, we apply this composition theorem to obtain some results regarding Eq. (1).

Auxiliary outcomes

Theorem 3.1 *Let $m \in \aleph$ and $\varphi = \vartheta + \hbar \in \rho\Lambda\Lambda(\Re \times \chi, \chi, \mathfrak{m})$. Suppose that*

(H3) *$\varphi(t,\chi)$ is uniformly continuous on any bounded subset $\kappa \subset \chi$ uniformly in $t \in \Re$;*

(H4) *$\vartheta(t,\chi)$ is uniformly continuous on any bounded subset $\kappa \subset \chi$ uniformly in $t \in \Re$.*

If $\Phi \in \rho\Lambda\Lambda(\Re, \chi, \mathfrak{m})$ then $F(\bullet) := \varphi(\bullet, \Phi(\bullet)) \in \rho\Lambda\Lambda(\Re \times \chi, \mathfrak{m})$.

Proof Let $\varphi = \vartheta + \hbar$ with $\vartheta \in \Lambda\Lambda(\Re \times \chi, \chi), \mathfrak{h} \in \zeta(\Re \times \chi, \chi, \mathfrak{m})$ and $\Phi = \upsilon + \nu$, with $\upsilon \in \Lambda\Lambda(\Re, \chi)$ and $\nu \in \zeta(\Re, \chi, \mathfrak{m})$. Define a function f as follows:

$$\begin{aligned} f(t) &:= \vartheta(t, \upsilon(t)) + \varphi(t, \Phi(t)) - \vartheta(t, \upsilon(t)) \\ &= \vartheta(t, \upsilon(t)) + \varphi(t, \Phi(t)) - \varphi(t, \upsilon(t)) + \hbar(t, \upsilon(t)). \end{aligned}$$

Let us restate

$$G(t) = \vartheta(t, \upsilon(t)), \phi(t) = \varphi(t, \Phi(t)) - \varphi(t, \upsilon(t)), H(t) = \hbar(t, \upsilon(t)).$$

Therefore, we obtain $f(t) = G(t) + \phi(t) + H(t)$. By Lemma 2.5, we conclude that $G(t) \in \Lambda\Lambda(\Re, \chi)$ and obviously $\phi(t) \in BC(\Re, \chi)$. We proceed to show that $\phi(t) \in \zeta(\Re, \chi, \mathfrak{m})$. It suffices to prove that

$$\mathrm{Lim}_{r\to\infty} \frac{1}{m([-r,r])} \int_{[-r,r]} \|\phi(t)\| \mathrm{d}m(t) = 0.$$

By Lemma 2.6 , $\upsilon(\Re) \subset \overline{\Phi(\Re)}$ which is a bounded set. From the hypothesis $(H3)$ with $\kappa = \overline{\Phi(\Re)}$ yields that for each $\zeta > 0$, there is an existence of a constant $S > 0$ such that for all $t \in \Re$,

$$\|\Phi - \upsilon\| \le S \Rightarrow \|\varphi(t, \Phi(t)) - \varphi(t, \upsilon(t))\| \le \zeta.$$

From the following set: $\Lambda_{r,\zeta} = \{t \in [-r,r] : \|\varphi(t)\| > \zeta\}$, we get

$$\begin{aligned} \Lambda_{r,\zeta}(\phi) &= \Lambda_{r,\zeta}(\varphi(t, \Phi(t)) - \varphi(t, u(t))) \\ &\subset \Lambda_{r,S}(\Phi(t) - \upsilon(t)) = \Lambda_{r,S}(\nu). \end{aligned}$$

Therefore, the following inequality carries:

$$\begin{aligned} &\frac{m(\{t \in [-r,r] : \|\varphi(t, \Phi(t)) - \varphi(t, \upsilon(t))\| > \zeta\})}{m([-r,r])} \\ &\le \frac{m(\{t \in [-r,r] : \|\Phi(t) - \upsilon(t)\| > S\})}{m([-r,r])}. \end{aligned}$$

Since $\Phi(t) = \upsilon(t) + \nu(t)$ and $\nu \in \zeta(\Re, \chi, \mathfrak{m})$, Lemma 2.7 states that for the above-mentioned S, we have

$$\lim_{r\to\infty} \frac{m(\{t \in [-r,r] : \|\varphi(t - \upsilon(t)\| > \zeta\})}{m([-r,r])} = 0,$$

and then we get

$$\lim_{r\to\infty} \frac{m(\{t \in [-r,r] : \|\varphi(t, \Phi(t)) - \varphi(t, \upsilon(t))\| > \zeta\})}{m([-r,r])} = 0. \quad (4)$$

Again, in view of Lemma 2.7 and Eq. (8), we attain $\phi(t) \in \zeta(\Re, \chi, \mathfrak{m})$. Finally, we have to prove $H(t) = \hbar(t, \upsilon(t)) \in \zeta(\Re, \chi, \mathfrak{m})$. Since υ is continuous on $\Re$ as P$\Lambda\Lambda$ function, the set $\upsilon([-r,r])$ can be taken as compact. Therefore, the function $\vartheta \in \Lambda\Lambda(\Re \times \chi, \chi)$, and ϑ is uniformly continuous on $[-r,r] \times \upsilon([-r,r])$. Then, (H3) implies that $\hbar(t,\chi)$ is uniformly continuous with $X \in \upsilon([-r,r])$ uniformly in $t \in ([-r,r])$. Thus for any $\zeta > 0$, a constant $S > 0$ exists such that for $X_1, X_2 \in \upsilon([-r,r])$ with $\|X_1 - X_2\| < S$, we have

$$\|\hbar(t, X_1) - \hbar(t, X_2)\| < \frac{\zeta}{2}, \quad \forall t \in [-r,r]. \quad (5)$$

By the compactness of the set $\upsilon([-r,r])$, we conclude that there is an existence of finite balls Θ_K with $\delta_K \in \upsilon([-r,r]), K = 1, \ldots, n$ and radius S indicated above, such that $\upsilon([-r,r]) \subset \cup_n^K \Theta_K$.

Then the sets $U_K := \{t \in [-r,r] : \upsilon(t) \in \Theta_K\}, K = 1, \ldots, n$ are open in $[-r,r] = \cup_{K=1}^n U_K$. Define V_K by

$$V_1 = U_1, V_K = U_K - \cup_{i=1}^{K-1} U_i, \quad 2 \le K \le n.$$

Then it is clear that $V_i \cap V_j = \emptyset$, if $i \ne j, 1 \le i,j \le n$. So, we obtain

$$\begin{aligned} \wedge &:= \{t \in [-r,r] : \|H(t)\| \ge \zeta\} = \{t \in [-r,r] : \|\hbar(t, \upsilon(T))\| \ge \zeta\} \\ &\subset \cup_{K=1}^n \{t \in V_K : \|\hbar(t, \upsilon(t)) - \hbar(t, \delta_K)\| + \|\hbar(t, \delta_K)\| \ge \zeta\} \\ &\subset \cup_{K=1}^n \left(\left\{ t \in V_K : \|\hbar(t, \upsilon(t)) - \hbar(t, \delta_K)\| \ge \frac{\zeta}{2} \right\} \right. \\ &\left. \cup \left\{ t \in V_K : \|\hbar(t, \delta_K)\| \ge \frac{\zeta}{2} \right\} \right). \end{aligned}$$

By (9), we obtain

$$\left\{ t \in V_K : \|\hbar(t, \upsilon(t)) - \hbar(t, \delta_K)\| \ge \frac{\zeta}{2} \right\} = \emptyset, \quad K = 1, \ldots, n.$$

Thus , if we set $\Lambda_{r,\frac{\zeta}{2}}(\hbar_K) := \Lambda_{r,\frac{\zeta}{2}}(\hbar(t, \delta_K))$, then $\Lambda_{r,\zeta}(H) \subset U_{K=1}^n \Lambda_{r,\frac{\zeta}{2}}(\hbar_K)$ and

$$\frac{1}{m([-r,r])} \int_{[-r,r]} \|H(t)\| \mathrm{d}m(t) \le \sum_{K=1}^n \frac{1}{m([r,-r])} \int_{[r,-r]} \|H(t)\| \mathrm{d}m(t).$$

Since $\hbar \in \zeta(\Re \times \chi, \chi, \mathfrak{m})$, we have

$$\lim_{r\to\infty} \frac{1}{m([r,-r])} \int_{[r,-r]} \|\hbar_K(t)\| \mathrm{d}m(t) = 0, \quad K = 1, \ldots, n.$$

It indicates that $\lim_{r\to\infty} \frac{1}{m([r,-r])} \int_{[r,-r]} \|H(t)\| \mathrm{d}m(t) = 0$. According to Lemma 2.7, we impose

$H(t) = \hbar(t, \upsilon(t)) \in \zeta(\Re, \chi, \mathfrak{m})$.

This ends the proof. □

Throughout the remaining parts of this paper, it is proposed that there is existence of two real numbers λ, δ such that $0 < \lambda < \delta < 1$ and $2\delta > \lambda + 1$. Moreover, we define the following fractional operators $\gamma_1^\mu, \gamma_2^\mu, \gamma_3^\mu$, and γ_4^μ by

$$(\gamma_1^\mu(\upsilon)(t)) := \int_{-\infty}^t \frac{\Lambda T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \varphi(\varsigma, \upsilon(\varsigma)) \mathrm{d}\varsigma,$$

$$(\gamma_2^\mu(\upsilon)(t)) := \int_t^\infty \frac{\Lambda T^\mu(t-\varsigma)Q}{\Gamma(\mu)} \varphi(\varsigma, \upsilon(\varsigma)) \mathrm{d}\varsigma,$$

$$(\gamma_3^\mu(\upsilon)(t)) := \int_{-\infty}^t \frac{\Lambda T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \vartheta(\varsigma, \upsilon(\varsigma)) \mathrm{d}\varsigma,$$

$$(\gamma_4^\mu(\upsilon)(t)) := \int_t^\infty \frac{\Lambda T^\mu(t-\varsigma)Q}{\Gamma(\mu)} \vartheta(\varsigma, \upsilon(\varsigma)) \mathrm{d}\varsigma,$$

where $\mu \in (0, 1]$ and $T(t)_{t \geq 0}$ is the analytic semigroup. It is clear that, if $T^\mu = (t - \varsigma)^{\mu-1}$, then we obtain the RL-integral operator.

Lemma 3.2 *Let $m \in \aleph$, $\upsilon \in \rho\Lambda\Lambda(\Re, \chi_\lambda, \mathfrak{m})$ and (H1)–(H2) hold. Then,*

$$\gamma_3^\mu, \gamma_4^\mu : \rho\Lambda\Lambda(\Re, \chi_\lambda, \mathfrak{m}) \longrightarrow \rho\Lambda\Lambda(\Re, \chi_\lambda, \mathfrak{m}).$$

Proof Let $\upsilon \in \rho\Lambda\Lambda(\Re, \chi_\lambda, \mathfrak{m})$. Putting $\hbar(t) = \vartheta(t, \upsilon(t))$ and by Theorem 3.1, it indicates that $\hbar \in \rho\Lambda\Lambda(\Re, \chi, \mathfrak{m})$ for each $\upsilon \in \rho\Lambda\Lambda(\Re, \chi_\lambda, \mathfrak{m})$. Setting $\hbar = \Phi + \xi$, where $\Phi \in \Lambda\Lambda(\Re, \chi)$ and $\xi \in \zeta(\Re, \chi, \mathfrak{m})$. Therefore, $\gamma_3^\mu \upsilon$ can be read as

$$(\gamma_3^\mu(\upsilon)(t)) := \int_{-\infty}^t \frac{T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \Phi(\varsigma) \mathrm{d}\varsigma + \int_{-\infty}^t \frac{T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \xi(\varsigma) \mathrm{d}\varsigma.$$

Let

$$\phi(t) = \int_{-\infty}^t \frac{T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \Phi(\varsigma) \mathrm{d}\varsigma$$

and

$$\psi(t) = \int_{-\infty}^t \frac{T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \xi(\varsigma) \mathrm{d}\varsigma,$$

for each $t \in \Re$. It can be realized that $\phi \in \Lambda\Lambda(\Re, \chi_\lambda)$. Consider a sequence $(\varsigma'_\eta)_{\eta \in N}$, then there is a subsequence $(\varsigma_\eta)_{\eta \in N}$ such that

$$\lim_{\eta, n \to \infty} \|\Phi(t + \varsigma_\eta - \varsigma_n) - \Phi(t)\| = 0. \tag{6}$$

Moreover, we have

$$\begin{aligned}\phi(t + \varsigma_\eta - \varsigma_n) - \phi(t) &= \int_{-\infty}^{t+\varsigma_\eta-\varsigma_n} \frac{T^\mu(t + \varsigma_\eta - \varsigma_n - \varsigma)\rho}{\Gamma(\mu)} \Phi(\varsigma) \mathrm{d}\varsigma \\ &\quad - \int_{-\infty}^t \frac{T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \Phi(\varsigma) \mathrm{d}\varsigma \\ &= \int_{-\infty}^0 \frac{T^\mu(-\varsigma)\rho}{\Gamma(\mu)} [\Phi(\varsigma + t + \varsigma_\eta - \varsigma_n) \\ &\quad - \Phi(\varsigma + t)] \mathrm{d}\varsigma.\end{aligned}$$

Then, we get

$$\begin{aligned}&\|\phi(t + \varsigma_\eta - \varsigma_n) - \phi(t)\|_\lambda \\ &\quad \leq \int_{-\infty}^0 \left\| \frac{T^\mu(-\varsigma)\rho}{\Gamma(\mu)} [\Phi(\varsigma + t + \varsigma_\eta - \varsigma_n) - \Phi(\varsigma + t)] \right\|_\lambda \mathrm{d}\varsigma.\end{aligned}$$

Hence, by (4) and the fact that $\|T^\mu\|_\lambda \leq \|T\|_\lambda$, we conclude

$$\begin{aligned}&\|\phi(t + \varsigma_\eta - \varsigma_n) - \phi(t)\|_\lambda \\ &\quad \leq \int_{-\infty}^0 \frac{M(\lambda)\varsigma^{-\lambda}\varrho^{-\epsilon\varsigma}}{\Gamma(\mu)} \|\Phi(\varsigma + t + \varsigma_\eta - \varsigma_n) - \Phi(\varsigma + t)\| \mathrm{d}\varsigma.\end{aligned}$$

The outcome obtains from (10) and the Lebesgue dominated convergence theorem. Lastly, we aim to show that $\psi(t) \in \zeta(\Re, \chi_\lambda, \mathfrak{m})$. A computation yields

$$\begin{aligned}&\frac{1}{m([-r,r])} \int_{[-r,r]} \|\psi(t)\|_\lambda \mathrm{d}m(t) \\ &\quad = \frac{1}{m([-r,r])} \int_{[-r,r]} \left\| \int_{-\infty}^t \frac{T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \xi(\varsigma) \mathrm{d}\varsigma \right\|_\lambda \mathrm{d}m(t) \\ &\quad \leq \frac{1}{m([-r,r])} \int_{[-r,r]} \int_{-\infty}^t \left\| \frac{T^\mu(t-\varsigma)\rho}{\Gamma(\mu)} \xi(\varsigma) \right\| \mathrm{d}\varsigma \mathrm{d}m(t) \\ &\quad \leq \frac{1}{m([-r,r])} \int_{[-r,r]} \int_{-\infty}^t \frac{M(\lambda)(t-\varsigma)^{-\mu\lambda}\varrho^{-\epsilon(t-\varsigma)}}{\Gamma(\mu)} \|\xi(\varsigma)\| \mathrm{d}\varsigma \mathrm{d}m(t) \\ &\quad \leq \frac{M(\lambda)}{\Gamma(\mu)} \int_0^\infty \varsigma^{-\mu\lambda}\varrho^{-\epsilon\varsigma} \left(\frac{1}{m([-r,r])} \int_{[-r,r]} \|\xi(t-\varsigma)\| \mathrm{d}m(t) \right) \mathrm{d}\varsigma.\end{aligned}$$

In fact, the space $\zeta(\Re, \chi, \mathfrak{m})$ is invariant (preserved by some function); it shows that $t \mapsto \xi(t - \varsigma)$ belongs to $\zeta(\Re, \chi, \mathfrak{m})$ for each $\varsigma \in \Re$ and hence

$$\lim_{r \to \infty} \frac{1}{m([-r,r])} \int_{[-r,r]} \|\xi(t-\varsigma)\| \mathrm{d}m(t) = 0.$$

Consequently, by utilizing the Lebesgue dominated convergence theorem, we have

$$\lim_{r \to \infty} \frac{M(\lambda)}{\Gamma(\mu)} \int_0^\infty \varsigma^{-\lambda}\varrho^{-\epsilon\varsigma} \left(\frac{1}{m([-r,r])} \int_{[-r,r]} \|\xi(t-\varsigma)\| \mathrm{d}m(t) \right) \mathrm{d}\varsigma = 0,$$

similarly, by applying (5) to $\gamma_4^\mu \upsilon$. This completes the proof. □

Lemma 3.3 *Let $m \in \aleph$, and $\upsilon \in \rho\Lambda\Lambda(\Re, \chi, \mathfrak{m})$. If (H1)–(H2) are satisfied, then*

$$\gamma_1^\mu, \gamma_2^\mu : \rho\Lambda\Lambda(\Re, \chi_\delta, \mathfrak{m}) \longrightarrow \rho\Lambda\Lambda(\Re, \chi_\delta, \mathfrak{m}).$$

Proof Let $\upsilon \in \rho\Lambda\Lambda(\Re, \chi, \mathfrak{m})$ and $\hbar(t) = \varphi(t, \chi, \upsilon(t))$. Then in view of Theorem 3.1, it implies that $\hbar \in \rho\Lambda\Lambda(\Re, \chi_\delta, \mathfrak{m})$ whenever $\upsilon \in \rho\Lambda\Lambda(\Re, \chi_\lambda, \mathfrak{m})$. In particular,

$$\|\hbar\|_{\infty,\delta} = \sup_{t\in\Re}\|\varphi(t, \upsilon(t))\|_\delta < \infty.$$

Now, we write $\hbar = \Phi + \Psi$, where $\Phi \in \Lambda\Lambda(\Re, \chi_\delta), \Psi \in \zeta(\Re, \chi_\delta, \mathfrak{m})$, that is, $\gamma_1^\mu \hbar = E\Phi + E\Psi$ where

$$E\Phi(t) := \int_{-\infty}^{t} \frac{\Lambda T^\mu(t-\varsigma)\rho}{\Gamma(\mu)}\Phi(\varsigma)d\varsigma,$$

$$E\Psi(t) := \int_{-\infty}^{t} \frac{\Lambda T^\mu(t-\varsigma)\rho}{\Gamma(\mu)}\Psi(\varsigma)d\varsigma.$$

First, we need to show that $E\Phi(t) \in \Lambda\Lambda(\Re, \chi_\lambda)$. Consider a sequence $(\varsigma'_\eta)_{\eta\in N}$ in $t \in \Re$, since $\Phi(t) \in \Lambda\Lambda(\Re, \chi_\delta)$, a subsequence $(S_\eta)_{\eta\in N}$ exists such that

$$\lim_{\eta,n\to\infty} \|\Phi(t, +\varsigma_\eta - \varsigma_n) - \Phi(t)\|_\delta = 0. \tag{7}$$

In addition, since

$$\begin{aligned} &E\Phi(t+\varsigma_\eta-\varsigma_n) - E\Phi(t) \\ &= \int_{-\infty}^{t+\varsigma_\eta-\varsigma_n} \frac{\Lambda T^\mu(t+\varsigma_\eta-\varsigma_n-\varsigma)\rho}{\Gamma(\mu)}\Phi(\varsigma)d\varsigma \\ &\quad - \int_{-\infty}^{t} \frac{\Lambda T^\mu(t-\varsigma)\rho}{\Gamma(\mu)}\Phi(\varsigma)d\varsigma \\ &= \int_{-\infty}^{0} \frac{AT^\mu(-\varsigma)\rho[\Phi(\varsigma+t+\varsigma_\eta-\varsigma_n) - \Phi(\varsigma+t)]}{\Gamma(\mu)}d\varsigma. \end{aligned}$$

Then, a computation implies

$$\begin{aligned} &\|E\Phi(t+\varsigma_\eta-\varsigma_n) - E\Phi(t)\|_\lambda \\ &\le \int_{-\infty}^{0} \left\| \frac{\Lambda T^\mu(-\varsigma)\rho[\Phi(\varsigma+t+\varsigma_\eta-\varsigma_n) - \Phi(\varsigma+t)]}{\Gamma(\mu)} \right\|_\lambda d\varsigma. \end{aligned}$$

Hence, by (6) and the fact that $\|\Lambda(T^\mu)\| \le \|\Lambda(T)\|$, we receive

$$\begin{aligned} &\|E\Phi(t+\varsigma_\eta-\varsigma_n) - E\Phi(t)\|_\lambda \\ &\le \int_{-\infty}^{0} \frac{c\varsigma^{\delta-\lambda-1}\varrho^{-\epsilon\varsigma}}{\Gamma(\mu)}\|\Phi(\varsigma+t+\varsigma_\eta+\varsigma_n) - \Phi(\varsigma+t)\|_\delta d\varsigma. \end{aligned}$$

The result comes from Eq. (11) and the Lebesgue's dominated theorem. Finally, we reveal that $E\Psi(t) \in \zeta(\Re, \chi_\lambda, \mathfrak{m})$. We have

$$\begin{aligned} &\frac{1}{m([-r,r])}\int_{[-r,r]} \|E\Psi(t)\|_\lambda dm(t) \\ &= \frac{1}{m([-r,r])}\int_{[-r,r]} \left\| \int_{-\infty}^{t} \frac{AT^\mu(t-\varsigma)\rho}{\Gamma(\mu)}\Psi(\varsigma)d\varsigma \right\|_\lambda dm(t) \\ &\le \frac{1}{m([-r,r])}\int_{[-r,r]} \int_{-\infty}^{t} \left\| \frac{AT^\mu(t-\varsigma)\rho}{\Gamma(\mu)}\Psi(\varsigma) \right\|_\lambda d\varsigma dm(t) \\ &\le \frac{1}{m([-r,r])}\int_{[-r,r]} \int_{-\infty}^{t} \frac{c(t-\varsigma)^{\delta-\lambda-1}\varrho^{-\epsilon(t-\varsigma)}}{\Gamma(\mu)}\|\Psi(\varsigma)\|_\delta d\varsigma dm(t) \\ &\le \frac{c}{\Gamma(\mu)}\int_0^\infty \varsigma^{\delta-\lambda-1}\varrho^{-\epsilon\varsigma}\left(\frac{1}{m([-r,r])}\int_{[-r,r]} \|\Psi(t-\varsigma)\|_\delta dm(t)\right)d\varsigma. \end{aligned}$$

Therefore, we obtain

$$\lim_{r\to\infty} \frac{1}{([-r,r])}\int_{[-r,r]} \|\Psi(t-\varsigma)\|_\delta dm(t) = 0$$

as

$$\varsigma \longrightarrow \Psi(t-\varsigma) \in \zeta(\Re, \chi_\delta, \mathfrak{m})$$

for every $\varsigma \in \Re$. The proof is completed by applying the Lebesgue's dominated convergence theorem and similarly for $\gamma_2^\mu\upsilon$ using (7). □

$m-$P$\Lambda\Lambda$M outcomes

The rest of this section is conducted to find the existence of m-P$\Lambda\Lambda$ mild solutions (m-P$\Lambda\Lambda$M) of Eq. (1). Recently, Ibrahim et al. studied the mild solution of a class of FDE, by utilizing the fractional resolvent concept (see [24, 25]).

Definition 3.4 Let $\lambda \in (0, 1)$. A bounded continuous function $\upsilon : \Re \to \chi_\lambda$ is stated to be a mild solution to (1) indicate that the function $\varsigma \longrightarrow \frac{\Lambda T^\mu(t-\varsigma\rho}{\Gamma(\mu)}\varphi(\varsigma, \upsilon(\varsigma))$ is integrable on $(-\infty, t), \varsigma \to AT^\mu(t-\varsigma)Q\varphi(\varsigma, \upsilon(\varsigma))$ is integrable on (t, ∞) and

$$\begin{aligned} \upsilon(t) &= -\varphi(t, \upsilon(t)) - \int_{-\infty}^{t} \frac{\Lambda T^\mu(t,\varsigma)\rho}{\Gamma(\mu)}\varphi(\varsigma, \upsilon(\varsigma))d\varsigma \\ &\quad + \int_t^\infty \frac{\Lambda T^\mu(t,\varsigma)Q}{\Gamma(\mu)}\varphi(\varsigma, \upsilon(\varsigma))d\varsigma + \int_{-\infty}^{t} \frac{T^\mu(t,\varsigma)\rho}{\Gamma(\mu)}\vartheta(\varsigma, \upsilon(\varsigma))d\varsigma \\ &\quad - \int_t^\infty \frac{\Lambda T^\mu(t,\varsigma)Q}{\Gamma(\mu)}\vartheta(\varsigma, \upsilon(\varsigma))d\varsigma \end{aligned}$$

for each $t \in \Re$.

Theorem 3.5 *Let $m \in \aleph$. Under the assumptions (H1) and (H2), Eq. (1) admits a unique m-P$\Lambda\Lambda$M solution for some constants $K > 0$.*

Proof Consider the fractional integral operator $\wedge : \rho\Lambda\Lambda(\Re, \chi_\lambda, \mathfrak{m}) \longrightarrow \rho\Lambda\Lambda(\Re, \chi_\lambda, \mathfrak{m})$ such that

$$\wedge\upsilon(t) := -\varphi(t,\upsilon(t)) - \int_{-\infty}^{t}\frac{\Lambda T^{\mu}(t,\varsigma)\rho}{\Gamma(\mu)}\varphi(\varsigma,\upsilon(\varsigma))\mathrm{d}\varsigma + \int_{t}^{\infty}\frac{\Lambda T^{\mu}(t,\varsigma)Q}{\Gamma(\mu)}\varphi(\varsigma,\upsilon(\varsigma))\mathrm{d}\varsigma + \int_{-\infty}^{t}\frac{T^{\mu}(t,\varsigma)\rho}{\Gamma(\mu)}\vartheta(\varsigma,\upsilon(\varsigma))\mathrm{d}\varsigma - \int_{t}^{\infty}\frac{AT^{\mu}(t,\varsigma)Q}{\Gamma(\mu)}\vartheta(\varsigma,\upsilon(\varsigma))\mathrm{d}\varsigma.$$

It has been formerly shown that for every $\upsilon \in \rho\Lambda\Lambda(\mathfrak{R},\chi_\lambda,\mathfrak{m})$, $\varphi(\bullet,\upsilon(\bullet)) \in \rho\Lambda\Lambda(\mathfrak{R},\chi_\lambda,\mathfrak{m})$ (see Theorem 3.1). In view of Lemmas 3.2 and 3.3, it documents that $\wedge : \rho\Lambda\Lambda(\mathfrak{R},\chi_\lambda,\mathfrak{m}) \longrightarrow \rho\Lambda\Lambda(\mathfrak{R},\chi_\lambda,\mathfrak{m})$. Our aim is to show that $\wedge$ has a unique fixed point. For this purpose, we employ Lemmas 2.14 and 2.15. Let $v,\varpi \in \rho\Lambda\Lambda(\mathfrak{R},\chi_\lambda,\mathfrak{m})$, then for γ_1^μ, we conclude

$$\begin{aligned}\|\gamma_1^\mu(v)(t)-\gamma_1^\mu(\varpi)(t)\|_\lambda &\le \int_{-\infty}^{t}\left\|\frac{\Lambda T^{\mu}(t-\varsigma)\rho}{\Gamma(\mu)}[\varphi(\varsigma,v(\varsigma))-\varphi(\varsigma,\varpi(\varsigma))]\right\|_\lambda \mathrm{d}\varsigma\\ &\le \int_{-\infty}^{t}\left\|\frac{\Lambda T(t-\varsigma)\rho}{\Gamma(\mu)}[\varphi(\varsigma,v(\varsigma))-\varphi(\varsigma,\varpi(\varsigma))]\right\|_\lambda \mathrm{d}\varsigma\\ &\le \int_{-\infty}^{t}\frac{c(t-\varsigma)^{\delta-\lambda-1}\varrho^{-\epsilon(t-\varsigma)}}{\Gamma(\mu)}\|\varphi(\varsigma,v(\varsigma))\\ &\quad -\varphi(\varsigma,\varpi(\varsigma))\|_\delta \mathrm{d}\varsigma \le \frac{\kappa_1 K}{\Gamma(\mu)}\|v-\varpi\|_{\lambda,\infty}.\end{aligned}$$

Now, for γ_2^μ, we obtain

$$\begin{aligned}\|\gamma_2^\mu(v)(t)-\gamma_2^\mu(\varpi)(t)\|_\lambda &\le \int_{t}^{\infty}\left\|\frac{\Lambda T^{\mu}(t-\varsigma)Q}{\Gamma(\mu)}[\varphi(\varsigma,v(\varsigma))-\varphi(\varsigma,\varpi(\varsigma))]\right\|_\lambda \mathrm{d}\varsigma\\ &\le \int_{t}^{\infty}\left\|\frac{\Lambda T(t-\varsigma)Q}{\Gamma(\mu)}[\varphi(\varsigma,v(\varsigma))-\varphi(\varsigma,\varpi(\varsigma))]\right\|_\lambda \mathrm{d}\varsigma\\ &\le \int_{t}^{\infty}\frac{c\varrho^{s(t-\varsigma)}}{\Gamma(\mu)}\|\varphi(\varsigma,v(\varsigma))-\varphi(\varsigma,\varpi(\varsigma))\|_\delta \mathrm{d}\varsigma\\ &\le \frac{\kappa_2 K}{\Gamma(\mu)}\|v-\varpi\|_{\lambda,\infty}.\end{aligned}$$

Now, for γ_3^μ and γ_4^μ, the following approximations can be given:

$$\begin{aligned}\|\gamma_3^\mu(v)(t)-\gamma_3^\mu(\varpi)(t)\|_\lambda &\le \int_{-\infty}^{t}\left\|\frac{T^{\mu}(t-\varsigma)\rho}{\Gamma(\mu)}[\vartheta(\varsigma,v(\varsigma))-\vartheta(\varsigma,\varpi(\varsigma))]\right\|_\lambda \mathrm{d}\varsigma\\ &\le \int_{-\infty}^{t}\left\|\frac{T(t-\varsigma)\rho}{\Gamma(\mu)}[\vartheta(\varsigma,v(\varsigma))-\vartheta(\varsigma,\varpi(\varsigma))]\right\|_\lambda \mathrm{d}\varsigma\\ &\le \int_{-\infty}^{t}\frac{M(\lambda)(t-\varsigma)^{-\lambda}\varrho^{-\epsilon(t-\varsigma)}}{\Gamma(\mu)}\|\vartheta(\varsigma,v(\varsigma))\\ &\quad -\vartheta(\varsigma,\varpi(\varsigma))\|\mathrm{d}\varsigma \le \frac{\kappa_3 K}{\Gamma(\mu)}\|v-\varpi\|_{\lambda,\infty}.\end{aligned}$$

Also, we attain

$$\begin{aligned}\|\gamma_4^\mu(v)(t)-\gamma_4^\mu(\varpi)(t)\|_\lambda &\le \int_{t}^{\infty}\left\|\frac{T^{\mu}(t-\varsigma)Q}{\Gamma(\mu)}[\vartheta(\varsigma,v(\varsigma))-\vartheta(\varsigma,\varpi(\varsigma))]\right\|_\lambda \mathrm{d}\varsigma\\ &\le \int_{t}^{\infty}\left\|\frac{T(t-\varsigma)Q}{\Gamma(\mu)}[\vartheta(\varsigma,v(\varsigma))-\vartheta(\varsigma,\varpi(\varsigma))]\right\|_\lambda \mathrm{d}\varsigma\\ &\le \int_{t}^{\infty}\frac{C(\lambda)\varrho^{s(t-\varsigma)}}{\Gamma(\mu)}\|\vartheta(\varsigma,v(\varsigma))-\vartheta(\varsigma,\varpi(\varsigma))\|\mathrm{d}\varsigma\\ &\le \frac{\kappa_4 K}{\Gamma(\mu)}\|v-\varpi\|_{\lambda,\infty}.\end{aligned}$$

Joining the above inequalities yields $\|\wedge v - \wedge\varpi\|_{\lambda,\infty} \le K\Xi$, where

$$\Xi := \frac{\kappa_1+\kappa_2+\kappa_3+\kappa_4}{\Gamma(\mu)}.$$

Therefore, if $K < \Xi^{-1}$, then in view of the Banach fixed point theorem, Eq. (1) has a unique solution, which clearly is the only m-P$\Lambda\Lambda$M solution. This completes the proof. □

Example 3.6 Consider the equation

$$D^\mu\Big((1+t)\upsilon(t)\Big) = \Big((1+t)\upsilon(t)\Big), \quad t\in[0,1],\ \mu\in(0,1]. \tag{8}$$

Let $\mu = 0.15$. It is clear that $\varphi(t,\upsilon(t)) = \vartheta(t,\upsilon(t)) = t\upsilon(t)$. Thus, they are Lipschitz with $K = 1$, $t\in[0,1]$. Moreover, $\kappa_1=\kappa_2=\kappa_3=\kappa_4 = 2.7/2$, this implies that $\frac{\kappa_1+\kappa_2+\kappa_3+\kappa_4}{\Gamma(0.15)} = \frac{4\times1.11}{6.22} = 0.874 \to \Xi^{-1} = 1.14 > K = 1$, $k_1 = 1$ such that $\lambda = 1/2$ and $\|\upsilon\| \le k_1\|\upsilon\|_\lambda$. In addition, the functions φ and ϑ are bounded and uniformly continuous for all $t\in[0,1]$. Hence, all the conditions of Theorem 3.5 are achieved; therefore, Eq. (12) has a unique periodic solution. Note that if $\mu = 0.5$, we obtain $\frac{\kappa_1+\kappa_2+\kappa_3+\kappa_4}{\Gamma(0.5)} = \frac{4\times1.11}{1.77} = 3.0 \to \Xi^{-1} = 0.3 < K = 1$, then Theorem 3.5 is field. From the above computation, Eq. (12) has a unique periodic solution when $0<\mu<0.4$.

Example 3.7 Consider the equation

$$D^\mu\Big((1/4+\sin t)\upsilon(t)\Big) = \Big((1/2+\cos t)\upsilon(t)\Big), \quad t\in[0,2\pi],\ \mu\in(0,1]. \tag{9}$$

Obviously, $\varphi(t,\upsilon(t)) = \sin t\,\upsilon(t)$, $\vartheta(t,\upsilon(t)) = \cos t\,\upsilon(t)$. Thus, they are Lipschitz with $K = \max_{t\in[0,2\pi]}\{\sin(t),\cos(t)\} = 1$, $\kappa_1=\kappa_2=\kappa_3=\kappa_4 = 2.7/4$; this implies that $\frac{\kappa_1+\kappa_2+\kappa_3+\kappa_4}{\Gamma(0.15)} = 0.4 \to \Xi^{-1} = 2.5 > K = 1$. Moreover, $k_1 = 1/2$ such that $\lambda = 1/4$ and $\|\upsilon\| \le k_1\|\upsilon\|_\lambda$. In addition, the functions φ and ϑ are bounded and uniformly continuous for all $t\in[0,2\pi]$. Hence, all the conditions of Theorem 3.5 are achieved; therefore, Eq. (13) admits a unique periodic solution. If $\mu = 0.5$, then we have $\Xi^{-1} = 0.66 < K = 1$ and hence Theorem 3.5 is true when $0<\mu<0.4$.

Conclusions

In the current study, we suggested symmetry of a class of fractional differential equations (the fractional calculus depends on the RL fractional operators) by utilizing its periodic solutions. We studied a special class of FDEs. This class is a generalization of the neutral equation. We proved a composition theorem for m-P$\Lambda\Lambda$ functions under appropriate conditions. Our technique is based on interpolation theory and Banach's fixed point theorem. Therefore, the solution, in this case, is unique. Moreover, we investigated the mild solution, for such a class by illustrating a new fractional resolvent concept. This functional is constructed to keep the periodicity of the solution and consequently its symmetry.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no competing interests

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