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Third Order Differential Subordination for Analytic Functions Involving Convolution Operator

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Abstract:

In the present paper, by making use of the new generalized operator, some results of third order differential subordination and differential superordination consequence for analytic functions are obtained. Also, some sandwich-type theorems are presented.

Keywords: Analytic functions, Differential subordination, Hadamard product, Univalent functions.

Introduction:

Let U be the open unit disk $U = \{w: w \in \mathbb{C}, |w| < 1\}$, and let $\Upsilon(U)$ be the class of analytic functions in U. For $n \in N = \{1,2,3,4,\dots\}$ and $b \in \mathbb{C}$ define the subclass $\Upsilon(U)$ by

$$\Upsilon[b,n] = \{f \in \mathcal{A} : f(w) = b + b_n w^n + b_{n+1} w^{n+1} + \dots\dots\dots\},$$

with Υ_1=[1,1] and $\Upsilon_0 = [0,1]$.

Let $\mathcal{A}$ denote the class of all analytic functions in U to satisfy the condition $f(0) = f'(0) - 1 = 0$ and write in the form

$$f(w) = w + \sum_{n=2}^{\infty} b_n w^n \,(w \in U)\ . \quad (1)$$

For a function $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ are defined by

$$g(w) = w + \sum_{n=2}^{\infty} c_n w^n \,(w \in U)\ . \quad (2)$$

The convolution (or Hadmard product) of $f(w)$ and g(w) is defined by

$$(f * g)(w) = w + \sum_{n=2}^{\infty} b_n c_n w^n = (g * f)(w)(w \in U)\,.$$

For two functions f and g are analytic in U. A function f is subordinate to g (or g superordinate to f), written as :

$$f \prec g \text{ in } U \text{ or } g(w) \prec f(w)\,, \quad (w \in U),$$

if there exists a Schwarz function $\hbar(w)$ which (by definition) is analytic in U satisfies the following conditions (see [1-9]),

$\hbar(0) = 0$ and $|\hbar(w)| < 1$ for all $(w \in U)$,

such that $f(w) = g(\hbar(w))$ $(w \in U)$.

Indeed it is known that, $f(w) \prec g(w)$, $(w \in U) \Longrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

In a special case, if g is a univalent function in open unit disk U, then the reverse implication also holds (see [10-11]),

$f(w) \prec g(w)\,,(w \in U) \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition1.(see[12]). Let $\mu(w)$ is the univalent function in U, let $\psi : \mathbb{C}^4 \times U \to \mathbb{C}$. If $\lambda(w)$ be an analytic function in open unit disk U that satisfies the next third order differential subordination:

$$\psi(\lambda(w), w\lambda'(w), w^2\lambda''(w), w^3\lambda'''(w); w) \prec \mu(w) \quad . \quad (3)$$

From(3), $\lambda(w)$ is namely a solution of the differential subordination. Moreover, q(w) is a univalent function which is namely a dominant of the solution of the differential subordination (3), or more simply, a dominant if $\lambda(w) \prec$ q(w) for all $\lambda(w)$ satisfying (3). A dominant $\tilde{q}(w)$ satisfies $\tilde{q}(w) \prec$ q(w) for all dominates q(w) of (3) is called the best dominant. Note that the best dominant is unique up to a rotation of U.

Definition2.(see [12]). Let $\mu(w)$ is a univalent function in the open unit disk, let $\psi: \mathbb{C}^4 \times U \to \mathbb{C}$. If the function $\lambda(w)$ is analytic in U that satisfies the next third order differential subordination :

$$\mu(w) \prec \psi(\lambda(w), w\lambda'(w), w^2\lambda''(w), w^3\lambda'''(w); w). \quad (4)$$

Then $\lambda(w)$, is namely a solution of the differential superordnation given by (4). Moreover, an analytic function q(w) is namely a subordinate of the solutions of the differential superordination provided by(4), or more simply a subordinate if q(w) subordination $\lambda(w)$ for all $\lambda(w)$ it should be satisfy (4). A univalent subordinate $\tilde{q}(w)$ that satisfies $q(w) \prec \tilde{q}(w)$ for subordination q(w) of (4) is called the best subordinate of the differential superordination given by (4). Note that the best subordinate is unique up to a rotation off.

The process of admissible functions (also known as the differential subordinations method) was first introduced by Miller and Mocanu in 1978 (see [13]), and the theory started to improve in 1981 (see [14]). For more details, (see [15]).

Definition 3. (see [16]). For $f \in \mathcal{A}$,the generalized derivative operator $\mathcal{T}_{s,\eta}^m : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$\mathcal{T}_{s,\eta}^m f(w) = w + \sum_{n=2}^{\infty}[1+\eta(n-1)]^m C(s,n)\, b_n w^n,\ w \in U \quad (5)$$

where $m, s \in N_0 = \{0,1,2,\ldots\}, \eta > 0$ and $C(s,n) = \binom{n+\eta-1}{s} = \frac{(s+1)_{n-1}}{(1)_{n-1}}$.

It would be easily to see that $\mathcal{T}_{0,\eta}^0 f(w) = f(w)$and $\mathcal{T}_{0,\eta}^1 f(w) = wf'(w)$

For $m, s \in N_0 = N \cup \{0\} = \{0,1,2,\ldots\}, \eta > 0$, it is easy from (5), that

$$\mathcal{T}_{s,\eta}^{m+1} f(w) = (1-\eta)\mathcal{T}_{s,\eta}^m f(w) + \eta w(\mathcal{T}_{s,\eta}^m f(w))' \quad (6)$$

and

$$w(\mathcal{T}_{s,\eta}^m f(w))' = (1+s)\mathcal{T}_{s+1,\eta}^m f(w) - s\mathcal{T}_{s,\eta}^m f(w) \quad (7)$$

Definition 4. (see [17]). For $f \in \mathcal{A}$, the Srivastava – Attiya operator is defined by

$$\mathcal{N}_{k,c} = w + \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^k b_n w^n,\ w \in U, \quad (8)$$

where $\in \mathbb{C}$ and $c \in \mathbb{C}/\mathbb{Z}_0^-$.

From (8) it is easy that

$$w(\mathcal{N}_{k+1,c}f(w))' = (1+c)\mathcal{N}_{k,c}f(w) - c\mathcal{N}_{k+1,c}f(w). \quad (9)$$

Definition 5.For $f \in \mathcal{A}$, the operator $\mathcal{NT}_{s,\eta}^m : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by convolution of the Srivastava – Attiya operator $\mathcal{N}_{k,c}$ and the generalized operator $\mathcal{T}_{s,\eta}^m$

$$\mathcal{NT}_{s,\eta,c}^{m,k} f(w) = (\mathcal{N}_{k,c} * \mathcal{T}_{s,\eta}^m) f(w), (w \in U)$$

and

$$\mathcal{NT}_{s,\eta,c}^{m,k} f(w) = w + \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^k [1+\eta(n-1)]^m C(s,n)\, b_n^2 w^n,\ w \in U. \quad (10)$$

From (10), the following identity relations can be obtained:

$$w(\mathcal{NT}_{s,\eta,c}^{m,k} f(w))' = (1+s)\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w) - s\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \quad (11)$$

also $$\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w) = (1-\eta)\mathcal{NT}_{s,\eta,c}^{m,k} f(w) + \eta w(\mathcal{NT}_{s,\eta,c}^{m,k} f(w))' \quad (12)$$

and

$$w(\mathcal{NT}_{s,\eta,c}^{m,k+1} f(w))' = (1+c)\mathcal{NT}_{s,\eta,c}^{m,k} f(w) - c\mathcal{NT}_{s,\eta,c}^{m,k+1} f(w). \quad (13)$$

Note that, the following are special cases of operator $\mathcal{NT}_{s,\eta}^m$

1- When $\mathcal{T}_{0,0}^m = 1$ include the Srivastava-Attiya operator $\mathcal{N}_{k,c}$ (see [17]).

2- When $\mathcal{N}_{0,c}$ include the generalizedderivative operator$\mathcal{T}_{s,\eta}^m$ (see [18]).

3- when $m = 0, \eta = 1$, $\mathcal{T}_{s,\eta}^m$ reduces to $\mathcal{T}_{s,1}^0$ which is introduced by Ruscheweyh derivative operator (see [19]).

4- When $= 0, \eta = 1$, $\mathcal{T}_{s,\eta}^m$ reduces to $\mathcal{T}_{0,1}^m$ which is introduced by Salagean derivative operator(see [20]).

5- When s = 0 , $\mathcal{T}_{s,\eta}^m$ reduces to$\mathcal{T}_{0,\eta}^m$ which is introduced by generalized Salagean derivative operator (or Al-Oboudi derivative) (see [21]).

6- When m = 0 , $\mathcal{T}_{s,\eta}^m$ reduces to$\mathcal{T}_{s,\eta}^0$ which is introduced bygeneralized Ruscheweyh derivative operator (or Al-Shaqsi – Darus derivative operator) (see [22]).

7- When $\eta = 0$,$\mathcal{T}_{s,\eta}^m$ reduces to $\mathcal{T}_{s,0}^m$ which is introduced by Srivastava- Attiya derivative operator(see [17]).

8- When k=1 , c = 0 , $\mathcal{N}_{k,c}$ reduces to $\mathcal{N}_{1,0}$ which is introduced by Alexander integral operator (see [23]).

9- When k = 1 , $c = \lambda$,$\mathcal{N}_{k,c}$ reduces to $\mathcal{N}_{1,\lambda}$ which is introduced by Bernardi integral operator (see [24]).

10- When $k = \sigma, c = 1$, $\mathcal{N}_{\sigma,1}$ reduces to $\mathcal{N}_{1,0}$ which is introduced by Jung-Kim-Srivastava integral operator (see [25]).

Definition 6. (see [26]) is symbolized by **Q,** a collection of all function q that is univalent and analytic on closed unit disk except $E(q)$ and denote $\overline{U}/$ E(q)where $\overline{U}$ is the closed unit disk

$\overline{U} = U \cup \{w \in \partial U\} = \{w \in \mathbb{C} : |z| \leq 1\}$

andE(q) = $\left\{ \zeta : \zeta \in \partial U : \lim_{w \to \zeta} q(w) = \infty \right\}$.

(14)

Such that $\min|q'(\zeta)| = \alpha > o$ for $\zeta \in \partial U/E(q)$.**Q**(b) denote the subclass of **Q** for which $q(0) = b$ with $\mathbf{Q}_{(0)} = \mathbf{Q_0}$ and $\mathbf{Q}_{(1)} = \mathbf{Q}_1$ **.**

Definition 7. (see [6]). Let Ωbe a set in complex plane $\mathbb{C}$. Also, let $q \in \boldsymbol{Q}$and $n \in N/\{1\}$, N be the set of positive integers. The class of admissible function $\psi_n[\Omega, q]$ consists of those functions $\Psi: \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the next admissibility conditions :

$\Psi(v, u, t, r, w) \notin \Omega,$

whenever

$v = q(\zeta)$, $u = \kappa\zeta\, q'(\zeta)$,

$$\operatorname{Re}\left(1+\frac{t}{u}\right) \geq \kappa \operatorname{Re}\left(1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \quad \text{and}$$

$$\operatorname{Re}\left(\frac{r}{u}\right) \geq \kappa^2 \operatorname{Re}\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $w \in U$, $\zeta \in \partial U/E(q)$ and $\kappa \geq n$

Definition 8. (see [22]). Let $\boldsymbol{\Omega}$be a set in complex plane $\mathbb{C}$, let $q \in \Upsilon[\boldsymbol{b}, \boldsymbol{n}]$ be in the subclass and $q'(w) \neq 0$. The class $\psi_n'[\Omega, q]$ of admissible function $\psi_n[\Omega, q]$ consists of function $\Psi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfies the next admissibility conditions :

$\Psi(v, u, t, r : \zeta) \in \Omega,$

whenever

$v = q(w)$, $u = \dfrac{wq'(w)}{j}$

$$\operatorname{Re}\left(1+\frac{t}{u}\right) \geq \frac{1}{j}\operatorname{Re}\left(1+\frac{\zeta q''(w)}{q'(w)}\right) \text{and}$$

$$\operatorname{Re}\left(\frac{r}{u}\right) \leq \frac{1}{j^2}\operatorname{Re}\left(\frac{w^2 q'''(w)}{q'(w)}\right),$$

where $w \in U$, $\zeta \in \partial U$ and $j \geq n \geq 2$.

Lemma 1.(see [6]). Let $p \in \Upsilon[\boldsymbol{b}, \boldsymbol{n}]$ with $n \geq 2$ and $q \in \boldsymbol{Q}(b)$ satisfy the next condition: $\operatorname{Re}\left(\dfrac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0$ and $\left|\dfrac{wq'(w)}{q'(\zeta)}\right| \leq m$ where $w \in U$, $\zeta \in \partial U/E(q)$ and $\geq n$. If $\Omega$is a set in complex plane $\mathbb{C}$, $\Psi \in \Psi_n[\Omega, q]$ and $\psi(\lambda(w), w\lambda'(w), w^2\lambda''(w), w^3\lambda'''(w); w) \subset \Omega$,

then $\lambda(w) \prec q(w)$.

Lemma 2. (see [6]). Let $\lambda \in \Upsilon[\boldsymbol{b}, \boldsymbol{n}]$ be in this subclass with $\psi \in \psi_n'[\Omega., q]$. If $\psi(\lambda(w), w\lambda'(w), w^2\lambda''(w), w^3\lambda'''(w); w)$, be univalent function in open unit disk U and $\lambda \in \boldsymbol{Q}(b)$ satisfying the next condition

$$\operatorname{Re}\left(\frac{wq''(w)}{q'(w)}\right) \geq 0 \text{ and } \left|\frac{w\lambda'(w)}{q'(w)}\right| \leq j ,$$

where$w \in U$, $\zeta \in \partial U$ and $j \geq n \geq 2$,

then,

$$\Omega \subset \{\psi(\lambda(w), w\lambda'(w), w^2\lambda''(w), w^3\lambda'''(w); w) : w \in U\},$$

it means that$q(w) \prec \lambda(w)$ $(w \in U)$.

Third-Order Differential Subordination Results Involving the Operator: $\mathcal{NT}_{s,\eta,c}^{m,k}\, f(w)$

In this section, some results of differential subordination are obtained. Also studying the class of admissible functions involving the generalized derivative operator and Srivastava – Attiya operator defined by (10)

Definition 9. Let Ω be a set in complex plane $\mathbb{C}$and $q \in Q_0 \cap \Upsilon[0,1]$.The class $\emptyset_n[\Omega, q]$ of admissible function consists of those function $\emptyset : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy next admissibility conditions:

$\emptyset(a, b, d, e; w) \notin \Omega,$

whenever

$a = q(\zeta)$, $b = \dfrac{\kappa\zeta q'(w) + sq(w)}{(s+1)}$,

$$\operatorname{Re}\left(\frac{(s+1)^2 d - s^2 a}{(b(s+1) - ba)} - 2s\right) \geq \kappa \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \text{ and}$$

$$\operatorname{Re}\left(\frac{(s+1)^2[(s+1)e - 3(s+1)d] + (2s^3 a + 3s^2 a)}{(s(b-a)+b)} + 3s^2 + 6s + 2\right) \geq \kappa^2 \operatorname{Re}\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right)$$

,

where $w \in U$, $\zeta \in \partial U/E(q)$ and $\kappa \geq 2$

Theorem 1. Let $\emptyset \in \Phi_n[\Omega, q]$be in this class. If the function f belongs to $\mathcal{A}$ and q belongs to $\mathbf{Q_0}$satisfying the following condition :

$$\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \text{ and } \left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k}\, f(w)}{q'(\zeta)}\right| \leq \kappa$$

(15)

and $\left\{\emptyset\left(\mathcal{NT}_{s,\eta,c}^{m,k}\, f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k}\, f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k}\, f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k}\, f(w); w\right) : w \in U\right\} \subset \Omega$, (16)

then $\mathcal{NT}_{s,\eta,c}^{m,k}\, f(w) \prec q(w).$

proof: From the relation between (10) and (11), gives

$\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w)=\frac{w(\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w))'+s\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)}{(s+1)}$.(17)

Let g(w) be analytic in open unit disk defined by

g(w)=$\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)$ (18)

Then

$\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w)=\frac{wg'(w)+sg(w)}{(s+1)}$. (19)

Based on that

$\mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w)=$

$\frac{w^2g''(w)+(2s+1)wg'(w)+s^2g(w)}{(s+1)^2}$ (20)

and

$\mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w)=$

$\frac{w^3g'''(w)+3(s+1)w^2g''(w)+(3s^2+3s+1)wg'(w)+s^3g(w)}{(s+1)^3}$

. (21)

Now, the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ defined by

a(v, u, t, r) = v , $b(v,u,t,r)=\frac{u+sv}{s+1}$, $d(v,u,t,r)=\frac{t+(2s+1)u+s^2v}{(s+1)^2}$ (22)

and

$e(v,u,t,r)=\frac{r+3(s+1)t+(3s^2+3s+1)u+s^3v}{(s+1)^3}$ (23)

Let

$\Psi(v,u,t,r)=\phi(a,b,d,e)=$

$\phi\left(\begin{matrix} v,\frac{u+sv}{s+1},\frac{t+(2s+1)u+s^2v}{(s+1)^2}, \\ \frac{r+3(s+1)t+(3s^2+3s+1)u+s^3v}{(s+1)^3};w\end{matrix}\right)$ (24)

The proof shall make use of Lemma1. By using (18) to (21), and from equation (24), get

$$\psi(\lambda(w),w\lambda'(w),w^2\lambda''(w),w^3\lambda'''(w);w)=\emptyset\left(\begin{matrix}\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w),\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w), \\ \mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w),\mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w);w\end{matrix}\right)\quad . \qquad (25)$$

Therefore,(16) gives

$\psi(g(w),wg'(w),w^2g''(w),w^3g'''(w);w)\in\Omega$

Such that $1+\frac{t}{u}=\frac{(s+1)^2d-s^2a}{(b(s+1)-ba)}-2s$ and

$\frac{r}{u}=\frac{(s+1)^2[(s+1)e-3(s+1)d]+(2s^3a+3s^2a)}{(s(b-a)+b)}$.

Now, since the admissibility condition for $\emptyset\in\Phi_n[\Omega,q]\equiv\Psi\in\Psi_2[\Omega,q]$ given in Definition7 with n = 2.

Thus, using (15) and Lemma1, implies that $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)\prec q(w)$.

This complete proof Theorem 1.

The next consequence is expansion of Theorem 1 for the case where the attitude of q(w) on ∂U is unknown.

Corollary1. Let q(w) be an univalent function in open unit disk U with $q(0)=0$ and let $\Omega\subset\mathbb{C}$. Let $\emptyset\in\Phi_n[\Omega,q_\rho]$ for some $\rho\in(0,1)$, where $q_\rho(w)=q(\rho w)$. If the function $f(w)$ belongs to $\mathcal{A}$ and q_ρ satisfies :

$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)\geq 0$ and $\left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w)}{q'(\zeta)}\right|\leq\kappa$ where $w\in U$, $\zeta\ \in\partial U/E(q_\rho)$ and $\kappa\geq 2$ and

$\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w),\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w),\mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w),\mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w);w)\in$ Ω .

Then $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)\prec q(w)$.

Proof: Since q_ρ is an univalent function in U therefore, $E(q_\rho)=\emptyset$ and $q_\rho\in Q_b$.

The class $\emptyset\in\Phi_n[\Omega,q_\rho]$ is an admissible class and from Theorem 1 yields

$\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)\prec q_\rho(w)$. $w\in U$

The consequences certain by Corollary1 is just a summary from the next subordination quality

$q_\rho(w)\prec q(w)$ $,w\in U$. Since $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)\prec q_\rho(w)$, get $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)\prec q(w)$ $,w\in U$.

This is the end of the proof Corollary1.

In case of $\Omega\neq\mathbb{C}$ this means it is a connected domain, then $\Omega=\mu(U)$ for some conformal mapping in open unit disk U onto Ω. In this case, the class $\Phi_n[\mu(U),q]$, it can be formed as $\Phi_n[\mu,q]$. This leads to the following immediate consequence of Theorem1.

Theorem 2. Let $\emptyset\in\Phi_n[\mu,q]$. If the function f belongs to $\mathcal{A}$ and $q\in\boldsymbol{Q_0}$ satisfies the following condition :

$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)\geq 0$, $\left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w)}{q'(\zeta)}\right|\leq\kappa$, (26)

and

$\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w),\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w),$

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$\mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w); w) \prec \mu(w)$,
(27)

then $\mathcal{NT}_{s,\eta,c}^{m,k}\ fw \prec q(w) \qquad w \in U$.

Corollary2. Let q(w) be an univalent function in open unit disk U with $q(0) = 0$. Also, let $\Omega \subset \mathbb{C}$ be a subset of the complex plane and $\emptyset \in \Phi_n[\mu, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(w) = q(\rho w)$. If the function f(w) belong to $\mathcal{A}$ and q_ρ satisfies :

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \text{ and } \left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w)}{q'(\zeta)}\right| \leq \kappa,$$

where $w \in U$, $\zeta \in \partial U / E(q_\rho)$ and $\kappa \geq 2$ and

$$\emptyset\left(\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w); w\right) \prec \mu(w),$$

then $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w) \prec q(w) \qquad w \in U.$.

Proof :

Case 1. To prove this Corollary by using Theorem 1, and have $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w) \prec q_\rho(w)$ and since $q_\rho(w) \prec q(w)$ Implies that $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w) \prec q(w)$.

Case 2. Let $g_\rho(w) = \mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)_\rho(w) = \mathcal{NT}_{s,\eta,c}^{m,k}\ f(\rho w) = g(\rho w)$. Then

$$\phi(\mathrm{g}_\rho(w), w\mathrm{g}'_\rho(w), w^2\mathrm{g}''_\rho(w), w^3\mathrm{g}'''_\rho(w); \rho w) = \phi(g(\rho w), wg'(\rho w), w^2 g''(\rho w), w^3 g'''(\rho w); \rho w) \in \mu_\rho(U)$$

By applying Theorem1, deduce:

$$\emptyset\left(\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w); \lambda(w)\right)$$

where $\lambda(w) = \rho w$ is any mapping from U into itself, $g_\rho(w) \prec q_\rho(w)$ for $\rho \in (\rho_0, 1)$. By $\rho \to 1^-$, get $g(w) \prec q(w)$.

Hence $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w) \prec q(w)$.

Theorem 3. Let μ(w) be a univalent function in the open unit disk. Also let $\emptyset : \mathbb{C}^4 \times U \to \mathbb{C}$ and Ψ be given by (24). Suppose that

$\psi(q(w), wq'(w), w^2 q''(w), w^3 q'''(w); w) = \mu(w)$,
(28)

the differential equation has a solution $q(w) \in Q_0$ which satisfies (15). If $f(w)$ belongs to $\mathcal{A}$ satisfies condition (27) and if $\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w); w)$ is analytic in open unit disk U, then $\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w) \prec q(w) \qquad w \in U$, q(w) is therefore, the best dominant.

Proof: From Theorem 1, it views q as a dominant of (27) because q satisfies (28), and also a solution of (27). Thus, q shall be dominated by all dominants. Hence q is the best dominate. This completes the proof of Theorem3.

From Definition 9, see that the special case when q(w) = Mw (M > 0), the class $\Phi_n[\Omega, M]$, is stated as follows.

Definition10. Let Ω be a set in complex plane $\mathbb{C}$ and M > 0. The class $\Phi_n[\Omega, M]$ of the admissible function consists of the function $\emptyset : \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\emptyset\left(Me^{i\theta}, \frac{(\kappa + s)Me^{i\theta}}{s+1}, \frac{L + [(2s+1)\kappa + s^2]Me^{i\theta}}{(s+1)^2}, \frac{\mathrm{N} + (3s+3)L + [(3s^2+3s+1)\kappa + s^3]Me^{i\theta}}{(s+1)^3}; w\right) \notin \Omega$$

, (29)

where

$w \in U$, $Re(Le^{-i\theta}) \geq (\kappa - 1)\kappa M$ and $Re(Ne^{-i\theta}) \geq 0$, ($\forall\ \theta \in \mathbb{R}, \kappa \geq 2$).

Corollary3. Let $\emptyset \in \Phi_n[\Omega, M]$ be in this class. If the function f belongs to $\mathcal{A}$, then it satisfies the following conditions:

$\left|\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w)\right| \leq \kappa M \qquad (w \in U\ , \kappa \geq 2; M > 0)$ and

$\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w); w) \in \Omega$, then

$$\left|\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)\right| < M .$$

The special case of the above Corollary 3 when $\Omega = q(U) = \{y: |y| < M\}$,the class $\Phi_n[\Omega, M]$ is simply symbolized by its $\Phi_n[M]$. Corollary4 can be rewritten as follows.

Corollary4. Let $\emptyset \in \Phi_n[M]$ be in this class. If $f \in \mathcal{A}$ is a function, it would satisfy the following conditions:

$\left|\mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w)\right| \leq \kappa M \qquad (w \in U\ , \kappa \geq 2; M > 0)$ and

$\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k}\ f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k}\ f(w); w) \in \Omega$, then $\left|\mathcal{NT}_{s,\eta,c}^{m,k}\ f(w)\right| < M$.

Corollary 5. A non-zeros s belongs to the complex plane $\mathbb{C}$, let $\kappa \geq 2$ and M > 0. If $f \in \mathcal{A}$ it would satisfy the conditions :

$\left|\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)\right| \leq \kappa M$ and

$\left|\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w) - \mathcal{NT}_{s,\eta,c}^{m,k} f(w)\right| < \frac{M}{|s+1|}$, then

$\left|\mathcal{NT}_{s,\eta,c}^{m,k} f(w)\right| < M$.

Proof: Let $\emptyset(a, b, d, e; w) = \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w) - \mathcal{NT}_{s,\eta,c}^{m,k} f(w) = b - a$ be equal and $\Omega = \mu(U)$, where

$\mu(z) = \frac{Mw}{|s+1|}$, (M> 0) .

Use Corollary3 that can be shown as $\emptyset \in \Phi_n[\Omega, M]$. This means that the admissibility condition (29), is satisfied. This follows easily, because

$$\left|\emptyset\left(\begin{array}{c} Me^{i\theta}, \frac{(\kappa+s)Me^{i\theta}}{s+1}, \frac{L+[(2s+1)\kappa+s^2]Me^{i\theta}}{(s+1)^2}, \\ \frac{N+(3s+3)L+[(3s^2+3s+1)\kappa+s^3]Me^{i\theta}}{(s+1)^3}; z \end{array}\right)\right| =$$

$$\left|\frac{(\kappa-1)}{s+1} Me^{i\theta}\right| \geq \frac{M}{|s+2|},$$

where $w \in U$ $\theta \in \mathbb{R}, \kappa \geq 2)$. The required results now follow from Corollary 3.

Definition11. Let $q \in Q_1 \cap \Upsilon_1[1,1]$ and Ω be a set in complex plane $\mathbb{C}$.The class $\emptyset_{n,1}[\Omega, q]$ of admissible function consists of those function $\emptyset : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfies the next admissibility conditions:

$\emptyset(a, b, d, e; w) \notin \Omega$,

whenever

$$a = q(\zeta), b = \frac{\kappa\zeta q'(w) + (s+1)q(w)}{(s+1)},$$

$$\mathrm{Re}\left(\frac{(s+1)^2)d - (s+1)^2 a}{((s+1)b - (s+1)a)} - 2(s+1)\right) \geq \kappa \, \mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)$$

and

$$\mathrm{Re}\left(\frac{(s+1)^3 e - (3s+6)[(s+1)^2 d - (s+1)^2 a] - (s+1)^3 a}{(s+1)b - (s+1)a} + 3s^2 + 12s + 11\right) \geq \kappa^2 \, \mathrm{Re}\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right)$$

,

where $w \in U$, $\zeta \in \partial U / E(q)$ and $\kappa \geq 2$

Theorem4. Let $\emptyset \in \Phi_{n,1}[\Omega, q]$. If a function f belongs to $\mathcal{A}$ and q belongs to **Q_1**satisfying the following condition :

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \text{ and } \left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{q'(\zeta)}\right| \leq \kappa ,$$

(30)

and

$$\left\{\begin{array}{c} \emptyset(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \\ \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w) : w \in U \end{array}\right\} \subset \Omega ,$$

(31)

then $\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w} \prec q(w)$.

proof: From the relation between (10) and (11), gives

$$\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w) = \frac{w(\mathcal{NT}_{s,\eta,c}^{m,k} f(w))' + (s+1)\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{(s+1)}.$$

(32)

Let g(w) be analytic in open unit disk defined by :

$$g(w) = \frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}. \quad (33)$$

Then$\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w} = \frac{wg'(w) + (s+1)g(w)}{(s+1)}$. (34)

Based on that

$$\frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w} = \frac{w^2 g''(w) + (2s+3)wg'(w) + (s+1)^2 g(w)}{(s+1)^2}$$

(35)

and

$$\frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w} = \frac{w^3 g'''(w) + 3(s+2)w^2 g''(w) + (3s^2 + 9s + 7)wg'(w) + (s+1)^3 g(w)}{(s+1)^3}$$

(36)

Now, the transformation from$\mathbb{C}^4$ to $\mathbb{C}$ defined by a(v, u, t, r) = v , $b(v, u, t, r) = \frac{u+(s+1)v}{s+1}$, $d(v, u, t, r) = \frac{t+(2s+3)u+(s+1)^2 v}{(s+1)^2}$,

(37)

and $e(v, u, t, r) = \frac{r+3(s+2)t+(3s^2+9s+7)u+(s+1)^3 v}{(s+1)^3}$

. (38)

Let $\Psi(v, u, t, r) = \phi(a, b, d, e)$

$$= \phi\left(\begin{array}{c} v, \frac{u+(s+1)v}{s+1}, \frac{t+(2s+3)u+(s+1)^2 v}{(s+1)^2}, \\ \frac{r+3(s+2)t+(3s^2+9s+7)u+(s+1)^3 v}{(s+1)^3}; w \end{array}\right) . \quad (39)$$

The proof will make use of Lemma1 .Using (33) to (36) , from equation (39), get

$$\psi(\lambda(w), w\lambda'(w), w^2\lambda''(w), w^3\lambda'''(w); w) = \emptyset\left(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w\right). \quad (40)$$

Therefore,(31) gets

$$\psi(g(w), wg'(w), w^2g''(w), w^3g'''(w); w) \in \Omega, \quad (41)$$

such that

$1+\frac{t}{u} = \frac{(s+1)^2)d-(s+1)^2 a}{((s+1)b-(s+1)a)} - 2(s+1)$ and

$\frac{r}{u} = \frac{(s+1)^3 e - (3s+6)[(s+1)^2 d - (s+1)^2 a] - (s+1)^3 a}{(s+1)b-(s+1)a} + 3s^2 + 12s + 11$

.

And since the admissibility condition for $\emptyset \in \Phi_{n,1}[\Omega, q] \equiv \Psi \in \Psi_2[\Omega, q]$, given in Definition11 with n = 2 .

Thus, using (30) and Lemma1, it implies that $\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w} \prec q(w)$.

In case $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = \mu(U)$ for some conformal mapping in open unit disk onto Ω. In this case, the class $\Phi_{n,1}[\mu(U), q]$ is rewritten as $\Phi_{n,1}[\mu, q]$. Thus can show below the result of Theorem 4.

Theorem 5. Let $\emptyset \in \Phi_{n,1}[\mu, q]$ be in this class. If the function $f \in \mathcal{A}$ and q belong to $\mathbf{Q_1}$satisfying the following condition :

$$\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \text{ and } \left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{q'(\zeta)}\right| \leq \kappa \quad (42)$$

and

$$\left\{\emptyset\left(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w\right): w \in U\right\} \prec \mu(w), \quad (43)$$

then $\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w} \prec q(w) \quad , w \in U$.

In a particular case where q(w) = Mw , where (M > 0), and in saw of Definition11 , the class $\Phi_{n,1}[\Omega, q]$of admissibility functions and symbol by $\Phi_{n,1}[\Omega, M]$ is :

Definition 12. Let Ω be a set in complex plane $\mathbb{C}$ and M > 0. The class $\Phi_{n,1}[\Omega, M]$ of the admissible function consists of the function $\emptyset : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ such that

$$\emptyset\left(Me^{i\theta}, \frac{(\kappa+1+s)Me^{i\theta}}{s+1}, \frac{L+[(2s+3)\kappa+(s+1)^2]Me^{i\theta}}{(s+1)^2}, \frac{N+(3s+6)L+[(3s^2+9s+7)\kappa+(s+1)^3]Me^{i\theta}}{(s+1)^3}; z\right) \notin \Omega \quad (44)$$

where

$w \in U$, $Re(Le^{-i\theta}) \geq (\kappa-1)\kappa Mm$ and $Re(Ne^{-i\theta}) \geq 0$, $(\forall\ \theta \in \mathbb{R}, \kappa \geq 2)$

Corollary 6. If $f \in \mathcal{A}$ and let $\emptyset \in \Phi_{n,1}[\Omega, M]$ satisfies the following conditions:

$\left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}\right| \leq \kappa M \qquad (w \in U, \kappa \geq 2; M > 0)$ and

$$\emptyset\left(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w\right) \in \Omega,$$

then $\left|\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}\right| < M$.

The particular case of above Corollary 6 when $\Omega = q(U) = \{y: |y| < M\}$, the class $\Phi_{n,1}[\Omega, M]$ is simply denoted by $\Phi_{n,1}[M]$. Corollary6 can be rewritten as shape.

Corollary 7: Let $\emptyset \in \Phi_{n,1}[M]$ be in this class. If the function f belong to $\mathcal{A}$ satisfying the following conditions:

$\left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}\right| \leq \kappa M \quad (w \in U, \kappa \geq 2; M > 0)$

and

$$\left|\emptyset\left(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w\right)\right| < M,$$

then $\frac{\left|\mathcal{NT}_{s,\eta,c}^{m,k} f(w)\right|}{w} < M$.

Some Properties of the Third-Order Differential Subordination Results Involving the Operator: $\mathcal{NT}_{s,\eta,c}^{m,k} f(w)$

Note that, making use of the recurrence relation (12), certain differential subordination consequences associated with the operator $\mathcal{NT}_{s,\eta,c}^{m,k} f(w)$ are obtained. Because the proofs of the consequences contained in this part are similar to those from the prior part, they will be omitted.

Definition13. Let $q \in Q_0 \cap \Upsilon[0,1]$and Ω be a set in complex plane $\mathbb{C}$.The class $\emptyset_n[\Omega, q]$ of admissible function consists of the function $\emptyset : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ that satisfies the following conditions:

$\emptyset(a, b, d, e; w) \notin \Omega$,

whenever

$a = q(\zeta)$, $\quad b = \dfrac{\kappa\zeta q'(w) + \left(\frac{1-\eta}{\eta}\right) q(w)}{\left(\frac{1}{\eta}\right)}$, and

$$\mathrm{Re}\left(\frac{\left(\frac{1}{\eta}\right)^2 d - \left(\frac{1-\eta}{\eta}\right)^2 a}{(b\left(\frac{1}{\eta}\right) - ba)} - 2\left(\frac{1-\eta}{\eta}\right) \right) \geq \kappa \,\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)$$

$$\mathrm{Re}\left(\frac{\left(\frac{1}{\eta}\right)^2 \left[\left(\frac{1}{\eta}\right) e - 3\left(\frac{1}{\eta}\right) d\right] + (2\left(\frac{1-\eta}{\eta}\right)^3 a + 3\left(\frac{1-\eta}{\eta}\right)^2 a)}{\left(\left(\frac{1-\eta}{\eta}\right)(b-a) + b\right)} + 3\left(\frac{1-\eta}{\eta}\right)^2 + 6\left(\frac{1-\eta}{\eta}\right) + 2 \right) \geq \kappa^2 \,\mathrm{Re}\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $w \in U$, $\zeta \in \partial U / E(q)$ and $\kappa \geq 2$.

Theorem 6. If the function $\in \mathcal{A}$. Let $\emptyset \in \Phi_n[\Omega, q]$ and q that belongs to $\mathbf{Q_0}$ satisfy the next condition :

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \text{ and } \left|\frac{\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w))}{q'(\zeta)}\right| \leq \kappa \;, \quad (45)$$

and

$$\left\{ \begin{matrix} \emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+1,k} f(w), \\ \mathcal{NT}_{s,\eta,c}^{m+2,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+3,k} f(w); w) : w \in U \end{matrix} \right\} \subset \Omega, \quad (46)$$

then $\mathcal{NT}_{s,\eta,c}^{m,k} f(w) \prec q(w)$.

In a particular case $\Omega \neq \mathbb{C}$. This means it is a simply connected domain, then $\Omega = \mu(U)$ for some conformal mapping in open unit disk U onto Ω. In this status, the class $\Phi_n[\mu(U), q]$ is written as $\Phi_n[\mu, q]$. The next consequence is immediate of Theorem 6.

Theorem7. Let $\emptyset \in \Phi_n[\mu, q]$. If $f \in \mathcal{A}$ is a function and $q \in \boldsymbol{Q_0}$ satisfy the next condition :

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0 \text{ and } \left|\frac{\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w))}{q'(\zeta)}\right| \leq \kappa \quad (47)$$

and

$$\emptyset\left(\begin{matrix} \mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+1,k} f(w), \\ \mathcal{NT}_{s,\eta,c}^{m+2,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+3,k} f(w); w \end{matrix} \right) \prec \mu(w), \quad (48)$$

then $\mathcal{NT}_{s,\eta,c}^{m,k} f(w) \prec q(w)$.

In a particular case when q(w) = Mw (M > 0), the class $\Phi_n[\Omega, M]$, is stated as follows.

Definition14. Let M > 0 and Ω be a set in the complex plan ℂ. The class $\Phi_n[\Omega, M]$ of the admissible function consists of those function $\emptyset : \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\emptyset\left(Me^{i\theta}, \frac{\left(\kappa + \left(\frac{1-\eta}{\eta}\right)\right) Me^{i\theta}}{\frac{1}{\eta}}, \frac{L + \left[\left(2\left(\frac{1-\eta}{\eta}\right) + 1\right)\kappa + \left(\frac{1-\eta}{\eta}\right)^2\right] Me^{i\theta}}{\left(\frac{1}{\eta}\right)^2}, \frac{N + \left(3\left(\frac{1-\eta}{\eta}\right) + 3\right) L + \left[\left(3\left(\frac{1-\eta}{\eta}\right)^2 + 3\left(\frac{1-\eta}{\eta}\right) + 1\right)\kappa + \left(\frac{1-\eta}{\eta}\right)^3\right] Me^{i\theta}}{\left(\frac{1}{\eta}\right)^3}; z \right) \notin \Omega \quad (49)$$

where $w \in U$, $Re(Le^{-i\theta}) \geq (\kappa - 1)\kappa M$ and $Re(Ne^{-i\theta}) \geq 0$, $(\forall\ \theta \in \mathbb{R}, \kappa \geq 2)$.

Corollary 8. Let $\emptyset \in \Phi_n[\Omega, M]$ be in this class. If the function $f \in \mathcal{A}$ that satisfies the following conditions:

$\left|\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w)\right| \leq \kappa M \quad (w \in U, \kappa \geq 2; M > 0)$ and

$\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+1,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+2,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+3,k} f(w); w) \in \Omega$, then

$$\left|\mathcal{NT}_{s,\eta,c}^{m,k} f(w)\right| < M .$$

The particular case of the above Corollary 8 when $\Omega = q(U) = \{y : |y| < M\}$,the class

$\Phi_n[\Omega, M]$simply symbolizes it$\Phi_n[M]$. Corollary 9 can be written as the following form.

Corollary 9. Let $\emptyset \in \Phi_n[M]$. If $\in \mathcal{A}$, satisfies the following conditions:

$\left|\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w))\right| \le \kappa M \qquad (w \in U \ , \kappa \ge 2; M > 0)$ and

$\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+1,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+2,k} f(w), \mathcal{NT}_{s,\eta,c}^{m+3,k} f(w); w) \in \Omega$ then

$\left|\mathcal{NT}_{s,\eta,c}^{m,k} f(w)\right| < M$.

Corollary 10. A non-zeros m that belong to the complex plane $\mathbb{C}$ let $\kappa \ge 2$ and M > 0. If $f \in \mathcal{A}$ satisfies conditions :

$\left|\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w)\right| \le \kappa M$ and $\left|\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w) - \mathcal{NT}_{s,\eta,c}^{m,k} f(w)\right| < \frac{M}{\left|\frac{1}{\eta}\right|}$, then

$\left|\mathcal{NT}_{s,\eta,c}^{m,k} f(w)\right| < M$.

Definition15. Let $q \in Q_1 \cap \Upsilon_1[1,1]and\Omega$ be a set in complex plane $\mathbb{C}$.The class $\emptyset_{n,1}[\Omega, q]$ of admissible function consists of those function $\emptyset : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfies the next conditions:

$\emptyset(a, b, d, e; w) \notin \Omega$,

whenever

$$a = q(\zeta), \ b = \frac{\kappa\zeta q'(w) + \left(\frac{1-\eta}{\eta}+1\right)q(w)}{\left(\frac{1}{\eta}\right)}, \ \mathrm{Re}\left(\frac{\left(\frac{1}{\eta}\right)^2 d - \left(\frac{1-\eta}{\eta}+1\right)^2 a}{\left(\left(\frac{1}{\eta}\right)b - \left(\frac{1-\eta}{\eta}+1\right)a\right)} - 2(s+1)\right) \ge \kappa\,\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \text{ and}$$

$$\mathrm{Re}\left(\frac{\left(\frac{1}{\eta}\right)^3 e - \left(3\left(\frac{1-\eta}{\eta}\right)+6\right)\left[\left(\frac{1}{\eta}\right)^2 d - \left(\frac{1-\eta}{\eta}+1\right)^2 a\right] - \left(\frac{1-\eta}{\eta}+1\right)^3 a}{\left(\frac{1}{\eta}\right)b - \left(\frac{1-\eta}{\eta}+1\right)a} + 3\left(\frac{1-\eta}{\eta}\right)^2 + 12\left(\frac{1-\eta}{\eta}\right) + 11\right) \ge \kappa^2\,\mathrm{Re}\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $w \in U$, $\zeta \in \partial U / E(q)$ and $\kappa \ge 2$.

Theorem 8. Let $\emptyset \in \Phi_{n,1}[\Omega, q]$ be in this class. If $f \in \mathcal{A}$ and q belong to $\mathbf{Q_1}$ satisfying the following condition :

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \ge 0 \text{ and } \left|\frac{\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w)}{q'(\zeta)}\right| \le \kappa \qquad (50)$$

and

$$\left\{\begin{array}{c}\emptyset\left(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s,\eta,c}^{m+1,k} f(w)}{w},\right.\\ \left.\frac{\mathcal{NT}_{s,\eta,c}^{m+2,k} f(w)}{w}, \frac{\mathcal{NT}_{s,\eta,c}^{m+3,k} f(w)}{w}; w\right) : w \in U\end{array}\right\} \subset \Omega \ , \qquad (51)$$

then $\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w} \prec q(w)$.

Main Results of the Third-Order Differential Superordination Involving the Operator: $\mathcal{NT}_{s,\eta,c}^{m,k} f(w)$

In this part, the third order differential superordination for the operator $\mathcal{NT}_{s,\eta,c}^{m,k} f(z)$ defined in (10) is obtained and proving many Theorems, for the purpose, consider the next class of admissible functions.

Definition16. Let $q \in \Upsilon[0,1]$ with $q'(w) \neq 0$ and Ω be a set in complex plane $\mathbb{C}$. The class $\phi_n'[\Omega, q]$ of admissible function consists of those function $\emptyset : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfies the next admissibility conditions:

$\emptyset(a, b, d, e; \zeta) \in \Omega$,

whenever

$$a = q(\zeta), b = \frac{wq'(w) + msq(w)}{m(s+1)},$$

$$\mathrm{Re}\left(\frac{(s+1)^2[(s+1)e - 3(s+1)d] + (2s^3 a + 3s^2 a)}{(s(b-a)+b)} + 3s^2 + 6s + 2\right) \le \frac{1}{j^2}\mathrm{Re}\left(\frac{w^2 q'''(w)}{q'(w)}\right),$$

$$\mathrm{Re}\left(\frac{(s+1)^2 d - s^2 a}{(b(s+1) - ba)} - 2s\right) \le \frac{1}{j}\mathrm{Re}\left(\frac{wq''(w)}{q'(w)}\right) \text{ and}$$

where $w \in U$, $\zeta \in \partial U$ and $j \ge 2$.

Theorem10. Let $\phi \in \Phi_n'[\Omega, q]$. If f belongs to $\mathcal{A}$,with $\mathcal{NT}_{s,\eta,c}^{m,k} f(w) \in \mathbf{Q_0}$,and $q \in \Upsilon[0,1]$ *with* $q'(w) \neq 0$satisfying the following condition :

$$\mathrm{Re}\left(\frac{wq''(w)}{q'(w)}\right) \ge 0 \text{ and } \left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{q'(w)}\right| \le j \qquad (52)$$

and $\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w)$ is an univalent in open unit disk U, then

$$\Omega \subset \left\{\begin{array}{c}\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w),\\ \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w) : w \in U\end{array}\right\} . \qquad (53)$$

It means that $q(w) \prec \mathcal{NT}_{s,\eta,c}^{m,k} f(w)$.
Proof. Let g(w) be a function defined by (18), and Ψ defined by (24). Since $\phi \in \Phi'_n[\Omega, q]$, from (25) and (53) yield,

$$\Omega \subset \left\{ \begin{matrix} \emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w), \\ \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w): w \in U \end{matrix} \right\}$$

. From (22) and (23) deduce that the admissibility condition for $\Psi \in \Psi_2[\Omega, q]$ given in Definition8 with n = 2. Subsequently $\psi \in \psi'_2[\Omega, q]$ and, by using (53) and Lemma2, get
$q(w) \prec g(w)$this equivalently,

$$q(w) \prec \mathcal{NT}_{s,\eta,c}^{m,k} f(w) \quad . \qquad w \in U$$

In case of $\Omega \neq \mathbb{C}$ this means it is a connected domain, then $\Omega = \mu(U)$ for some conformal mapping μ(w) of open unit disk U on to Ω. In this case, the class $\phi \in \Phi'_n[\mu(U), q]$ is formed as $\phi \in \Phi'_n[\mu, q]$. The next is instant consequence of Theorem10.

Theorem11. Let $\phi \in \Phi'_n[\mu, q]$ and μ be analytic in unit disk U. If the function $f \in \mathcal{A}$ and $\mathcal{NT}_{s,\eta,c}^{m,k} f(w) \in \boldsymbol{Q_0}$, and if $q \in \Upsilon[0,1] with$ $q'(w) \neq 0$ satisfying the condition(52) and the function $\emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w),$ $\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w)$is univalent in unit disk U ,then

$$\mathrm{M}(w) \prec \emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w). \qquad (54)$$

It means that $q(w) \prec \mathcal{NT}_{s,\eta,c}^{m,k} f(w) \qquad w \in U$.
The next Theorem proves the presence of the best subordination of (54) for appropriate $\emptyset$.

Theorem 12 . Let $\emptyset : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ and μ be univalent function in unit disk U, let Ψ be given by (24). Suppose that :

$$\psi(q(w), wq'(w), w^2q''(w), w^3q'''(w); w) = \mu(w) ,$$
(55)

the differential equation has a solution q(w) belong to $\mathbf{Q_0}$ and if $q \in [0,1]$ with $q'(w)$not equal zero satisfying the condition (52) and $\emptyset\begin{pmatrix} \mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w), \\ \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w \end{pmatrix}$ is univalent in unit disk U, then

$$\mu(w) \prec \emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w) .$$

It means that$q(w) \prec \mathcal{NT}_{s,\eta,c}^{m,k} f(w)(w \in U)$ and q(w) is the best dominant.
Proof: To prove this Theorem using Theorem10 and Theorem 11, draw q is a subordination (54). Since that q satisfies(55), it is also a solution of (54) and, subsequently, q will be subordinate by all subordinate. Hence q is the best subordination.

Definition17. Let Ω be a set in complex plane $\mathbb{C}$ and $q \in \Upsilon_1[1,1]$ with $q'(w) \neq 0$. The class $\phi'_{n,1}[\Omega, q]$ of admissible function consists of those function $\emptyset : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfies the next admissibility conditions:
$\emptyset (a , b , d, e; \zeta) \in \Omega$,
whenever

$$a = q(\zeta), \qquad b = \frac{wq'(w) + m(s+1)q(w)}{m(s+1)},$$

$$\mathrm{Re}\left(\frac{(s+1)^2)d - (s+1)^2 a}{((s+1)b - (s+1)a)} - 2(s+1) \right) \leq \frac{1}{j} \mathrm{Re}\left(\frac{wq''(w)}{q'(w)} \right)$$

and

$$\mathrm{Re}\left(\frac{(s+1)^3 e - (3s+6)[(s+1)^2 d - (s+1)^2 a] - (s+1)^3 a}{(s+1)b - (s+1)a} + 3s^2 + 12s + 11 \right) \leq \frac{1}{j^2} \ \mathrm{Re}\left(\frac{w^2 q'''(w)}{q'(w)} \right),$$

where $w \in U$, $\zeta \in \partial U$ and $j \geq 2$.

Theorem13:Let $\varphi \in \Phi'_{n,1}[\Omega, q]$. If $f \in \mathcal{A}$ with $\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}$ belongs to $\mathbf{Q_1}$and q belongs to $\Upsilon_1[1,1]$with $q'(w)$ not equal to zero satisfying the condition:

$$\mathrm{Re}\left(\frac{wq''(w)}{q'(w)} \right) \geq 0 \text{ and } \left| \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{q'(w)} \right| \leq j$$
(56)

and the function $\emptyset(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w)$ is univalent in open unit disk U, then

$$\Omega \subset \left\{ \begin{matrix} \emptyset(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \\ \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w): w \in U \end{matrix} \right\} .$$
(57)

It means that $q(w) \prec \frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}$.
Proof: Let g(w) be a function defined by (33) and Ψ defined by (39). Since $\phi \in \Phi'_n[\Omega, q]$, from (40) and (57) yield,

$$\Omega \subset \left\{ \begin{matrix} \emptyset(\mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w), \\ \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w) : w \in U \end{matrix} \right\}$$

. From (37) and (38), deduce that the admissibility condition for Ψ belongs to $\Psi_2[\Omega, q]$ given in Definition8 with n=2. Subsequently $\psi \in \psi_2'[\Omega, q]$ and, by using (56) and Lemma 2, it implies that

$q(w) \prec \frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}$, $w \in U$. This is the complete proof of Theorem1

In case of $\Omega \neq \mathbb{C}$ this means that it is a simply connected domain, then $\Omega = \mu(U)$ for some conformal mapping $\mu(w)$ of open unit disk U onto Ω. In this case, the class $\phi \in \Phi_{n,1}'[\mu(U), q]$ is formed as $\phi \in \Phi_{n,1}'[\mu, q]$. The next is an instant consequence of Theorem13.

Theorem 14. Let $\varphi \in \Phi_{n,1}'[\mu, q]$, and let μ be analytic function in unit disk U. If f belong to $\mathcal{A}$ and q belong to $\Upsilon_1[1,1]$with $q'(w)$ not equal to zero satisfying the condition:

$$\mathrm{Re}\left(\frac{wq''(w)}{q'(w)}\right) \geq 0 \text{ and } \left|\frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{q'(w)}\right| \leq j \quad (58)$$

and the function $\emptyset\left(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w\right)$ is univalent in unit disk U, then

$$\Omega \subset \left\{ \begin{matrix} \emptyset\left(\frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+1,\eta,c}^{m,k} f(w)}{w}, \right. \\ \left. \frac{\mathcal{NT}_{s+2,\eta,c}^{m,k} f(w)}{w}, \frac{\mathcal{NT}_{s+3,\eta,c}^{m,k} f(w)}{w}; w\right) : w \in U \end{matrix} \right\} . \quad (59)$$

It means that $q(w) \prec \frac{\mathcal{NT}_{s,\eta,c}^{m,k} f(w)}{w}$.

A Set of Sandwich – Type Results

Collect Theorem2 and Theorem11. The next sandwich-type Theorem is obtained.

Theorem 15: Let μ_1 and q_1 be analytic functions in U, μ_2 is an univalent function in open unit disk U, $q_2 \in \boldsymbol{Q_0}$with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_n[\mu_2, q_2] \cap \Phi_n'[\mu_1, q_1]$. If the function $f \in \mathcal{A}$ with $\mathcal{NT}_{s,\eta,c}^{m,k} f(w) \in \boldsymbol{Q_0} \cap \Upsilon[0,1]$and the function

$$\emptyset\left(\begin{matrix} \mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w), \\ \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w \end{matrix} \right)$$

is univalent in open unit disk , and if the conditions (15) and (52) are satisfied

$$\mu_1(w) \prec \emptyset\left(\begin{matrix} \mathcal{NT}_{s,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+1,\eta,c}^{m,k} f(w), \\ \mathcal{NT}_{s+2,\eta,c}^{m,k} f(w), \mathcal{NT}_{s+3,\eta,c}^{m,k} f(w); w \end{matrix} \right) \prec \mu_2(w),$$

it means that $q_1(w) \prec \mathcal{NT}_{s,\eta,c}^{m,k} f(w) \prec q_2(w)$. (60)

Conclusion:

In this work, in the case of applying the new operator to the geometric functions, it was concluded that they remain preserving their geometric features. If are put some restrictions on the functions that belong to these subclasses characteristics, can get new results inside the unit disk.

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All authors contributed equally to the writing of this paper.

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التابعية التفاضلية من الرتبة الثالثة للدوال التحليلية التي تتضمن مؤثر الالتفاف

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الخلاصة:

في هذا البحث، من خلال استخدام المؤثر المعمم الجديد تم الحصول بعض نتائج التابعية التفاضلية من الدرجة الثالثة ونتائج التابعية التفاضلية العليا للدوال التحليلية .كذلك تم تقديم بعض نظريات من نوع الساندوج .

الكلمات المفتاحية :الدوال التحليلية، التابعية التفاضلية ، ضرب هادمرد، الدوال احادية التكافؤ .