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Univalence Criteria for Holomorphic Functions Involving Srivastava-Attiya Operator

Univalence Criteria for Holomorphic Functions Involving Srivastava-Attiya Operator

Huda F. Hussian, Abdul Rahman S. Juma

Department of Mathematics, University of Anbar, Ramadi, Iraq

E-mail: dr_juma@hotmail.com

Abstract. The purpose of present paper is to introduce and investigate the univalence criteria of holomorphic functions by employ a basically general form of Srivastava-Attiya operator. In specific, we derive several sufficient conditions of univalence for the generalized Srivastava-Attiya operator .Furthermore, number of famous univalent conditions would follow across specializing the parameters involved.Relevant connections with other related previous works are also indicating.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k , \quad (1)$$

which are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.
Let S be the subclass of $\mathcal{A}$, which consists of functions of the form (1) that are univalent and normalized by the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1 \text{ in U.}$$

In geometric function theory, the Univalence of complex functions considered as substantial property. However, it is complicated, and in many situations impossible to show immediately that a certain complex function is univalent. because of that many authors found different kinds of sufficient conditions of univalence. On of the most substantial of these conditions of univalence in the domains U and the exterior of the closed unit disk is the well-known criterion of Becker [2].Becker us the generalized Loewner differential equation and theory of Loewner chains cleverly. Extension of these criterias were given by Deniz and Orhan [4], Ali et al. [1] and Nehari [6].
For $f \in \mathcal{A}$, the generalized Srivastava-Attiya operator $\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} : \mathcal{A} \to \mathcal{A}$ is defined by

$$\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} f(z) = z + \sum_{k=2}^{\infty} \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right) \left(\frac{a+1}{a+k}\right)^s a_k z^k \quad (2)$$

$(\beta_i \in \mathbb{C}(i = 1, \dots, p);\ \alpha_i \in \mathbb{C} \setminus Z_0^-(i = 1, \dots, q); z \in U; p \le q + 1;$ $\min\{\mathcal{R}(a), \mathcal{R}(s)\} > 0; \beta > 0$ when $\mathcal{R}(b) > 0$ and $S \in \mathbb{C}; a \in \mathbb{C} \setminus Z_0^-$ when $b = 0)$. For more details see [9,10]

In this paper, we derive sufficient conditions of univalence for the generalized Srivastava-Attiya operator $\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} f(z)$.

Furthermore, a number of known univalent conditions would follow across specializing the parameters involved. We will use the following lemmas to prove our results.

Lemma1.1 [2] Let $f \in \mathcal{A}$. If for all $z \in U$

$$(1-|z|^2)\left|\frac{z\,f'(z)}{f''(z)}\right| \le 1, \tag{3}$$

then f is univalent in U.

Lemma1.2 [7] Let $f \in \mathcal{A}$. If for all $z \in U$

$$\left|\frac{z^2\,f'(z)}{f^2(z)} - 1\right| \le 1, \tag{4}$$

then f is univalent in U.

Lemma1.3 [11] Let be real number $\eta > \frac{1}{2}$ and $f \in \mathcal{A}$. If for all $z \in U$

$$(1-|z|^{2\eta})\left|\frac{z\,f''(z)}{f'(z)} + 1 - \eta\right| \le \eta, \tag{5}$$

then f is univalent in U.

Lemma 1.4 [5] If $f \in S$. If for all $z \in U$,

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k, \tag{6}$$

then $\sum_{k=1}^{\infty}(k-1)|b_k| \le 1.$

Lemma 1.5 [8] Let $v \in \mathbb{C}, \mathcal{R}e(v) \ge 0$ and $f \in \mathcal{A}$. If for all $z \in U$

$$\frac{1-|z|^{2\mathcal{R}e(v)}}{\mathcal{R}e(v)}\left|\frac{z\,f''(z)}{f'(z)}\right| \le 1, \tag{7}$$

then a function

$$T_v(z) = \left(v\int_0^z y^{v-1}\,f'(y)dy\right)^{\frac{1}{v}}$$

is univalent in U.

2. Main Results

In this section, we determine the sufficient conditions to get univalence for holomorphic functions by using the Srivastava – Attiya operator.

Theorem 2.1 Let $f \in \mathcal{A}$. If for all $z \in U$

$$\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^{s}[k(2k-1)]\,|a_k| \le 1,$$

then $\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)$ is univalent in U. (8)

Proof . Let $f \in \mathcal{A}$. Then for all $z \in U$, we have

$$(1-|z|^2)\left|\frac{z\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)''}{\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)'}\right| \le (1-|-z|^2)\frac{|z|\left|\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)''\right|}{\left|\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)'\right|}$$

$$\leq (1+|z|^2)\frac{|z|\left|\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s a_k\, z^k\right]''\right|}{\left|\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s a_k\, z^k\right]'\right|}$$

$$= (1+|z|^2)\frac{|z|\left|\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)\, z^{k-2}\right|}{|1|-\left|\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}.\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right).\left(\frac{a+1}{a+k}\right)^s k\, a_k\, z^{k-1}\right|}$$

$$\leq (1+|z|^2)\frac{|z|\left[\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)\, |a_k|\, |z^{k-2}|\right]}{\left[1-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k\, |a_k|\, |z^{k-1}|\right]}$$

$$\leq \frac{2\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k|}{1-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k\,|a_k|}$$

Applying Lemma 1.1,we get

$$\frac{2\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k|}{1-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k\,|a_k|} \leq 1$$

then

$$2\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k| \leq 1-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k\,|a_k|,$$

therefor,

$$\left[2\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k| + \sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k|\right] \leq 1,$$

and we have

$$\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s |a_k|\,[k(2k-1)] \leq 1$$

Theorem 2.2 Let $f \in \mathcal{A}$. If for all $z \in U$

$$\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s |a_k| \leq \frac{1}{\sqrt{7}}, \quad (9)$$

then $\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} f(z)$ is univalent in U.

Proof. Let $f \in \mathcal{A}$. We must show that

$$\left|\frac{z^2\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)'}{2\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)^2}\right|\leq 1\ ,$$

thus

$$\left|\frac{z^2\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)'}{2\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)^2}\right|=\frac{\left|z^2\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)'\right|}{2\left|\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)^2\right|}$$

$$\leq\frac{|z|^2\left|\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s a_k z^k\right]'\right|}{2\left|\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s a_k z^k\right]^2\right|}$$

$$\leq\frac{|z|^2\left|1+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k\,a_k z^{k-1}\right|}{2\left|\left[z^2+2z\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s a_k z^k\right]+\left(\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s a_k z^k\right)^2\right|}$$

$$\leq\frac{|z|^2\left[1+\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\cdot\left(\frac{a+1}{a+k}\right)^s k|a_k|\,|z^{k-1}|\right]}{2\left[|z|^2-2|-z|\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\,|z^k|\right]-\left|-\sum_{k=2}^{\infty}\left(\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s\right|\,|a_k|\,|z^k|\right)^2}$$

$$\leq\frac{1+\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\cdot\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\cdot\left(\frac{a+1}{a+k}\right)^s k|a_k|}{2\left[1-2\sum_{k=2}^{\infty}\left(\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s\right)|a_k|\right]-\sum_{k=2}^{\infty}\left(\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right)^2}$$

$$\leq\frac{1+\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k|\right]^2}{2\left[1-2\left(\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right)^2-2\left(\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right)^2\right]}.$$

Applying Lemma 1.2,we get

$$\frac{1+\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right]^2}{2\left[1-2\left(\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right)^2-2\left(\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{a+1}{a+k}\right)^s\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)|a_k|\right)^2\right]}\leq 1$$

$$1+\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right]^2\leq 2-$$
$$4\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right]^2-2\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right]^2,$$

then

$$1+\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right]^2$$
$$+4\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right]^2$$

$$2+\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right]^2\le 1,$$

therefor,

$$7\left[\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s|a_k|\right]^2\le 1,$$

and we have

$$\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s|a_k\le\frac{1}{\sqrt{7}}$$

Theorem 2.3 Let $f\in\mathcal{A}$. If for all $z\in U$

$$\sum_{k=1}^{\infty}k\left[2(k-1)+(2\eta-1)\left[\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}.\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right).\left(\frac{a+1}{a+k}\right)^s|a_k|\right]\right]\le 2\eta-1,\ \eta>\frac{1}{2}, \quad (10)$$

then $\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)$ is univalent in U.

Proof. Let $f\in\mathcal{A}$. If for all $z\in U$, we have

$$(1-|z|^{2\eta}).\left|\frac{z\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)''}{\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)'}+1-\eta\right|$$

$$\le(1-|z|^{2\eta}).\frac{|z|\left|\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b}f(z)\right)''\right|}{\left|\left(\psi^{s,a,\beta}_{(\beta_p),((\propto_q)),b}f(z)\right)'\right|}+|1-\eta|$$

$$=(1+|z|^2).\frac{|z|\left|\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s a_k z^k\right]''\right|}{\left|\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s a_k z^k\right]'\right|}+|1-\eta|$$

$$\le(1+|z|^2).\frac{|z|\left[\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta)}{B(a+1,b,s,\beta)}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k||z^{k-2}|\right]}{|1|-\left|-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k\right||a_k|\,|z^{k-1}|}+|1-\eta|$$

$$\le\frac{2\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k|}{1-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k|}+|1-\eta|.$$

Applying Lemma 1.3,we get

$$\frac{2\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k|}{1-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k|}+|1-\eta|\le\eta,$$

then

$$\frac{2\sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k|}{1-\sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k|} \le 2\eta - 1,$$

therefor,

$$2\sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k| + 2\eta\sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k| - \sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k| \le 2\eta - 1,$$

and we have

$$\sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k| \, [2(k-1)+(2\eta-1)] \le 2\eta - 1.$$

As applications of Theorems 2.1, 2.2, and 2.3, we have the following Theorem.

Theorem 2.4 Let $f \in \mathcal{A}$. If for all $z \in U$. One of inequality (9-11) holds then $\sum_{k=1}^{\infty}(k-1)\,|b_k| \le 1$, ...(11)

where $\frac{z}{\psi_{(\beta_p),(\propto_q),b}^{s,a,\beta} f(z)} = 1 + \sum_{k=1}^{\infty}(k-1)\, b_k\, z^k$

Proof. Let $f \in \mathcal{A}$. Then in view of theorems 2.1 , 2.2 , 2.3

$\psi_{(\beta_p),(\propto_q),b}^{s,a,\beta} f(z)$ is univalent in U.

Using Theorem 2.1,

$$(1-|z|^2)\left|\frac{z\left(\psi_{(\beta_p),(\propto_q),b}^{s,a,\beta} f(z)\right)''}{\left(\psi_{(\beta_p),(\propto_q),b}^{s,a,\beta} f(z)\right)'}\right| = (1-|z|^2)\left|\frac{z\left(\psi_{(\beta_p)(\propto_q),b}^{s,a,\beta+2} f(z)\right)}{\psi_{,(\beta_p)(\propto_q),b}^{s,a,\beta+1} f(z)}\right|$$

$$= (1-|z|^2)\left|\frac{Z\ \psi_{(\beta_p),(\propto_q),b}^{s,a,\beta+1} f(z)}{\psi_{(\beta_p)(\propto_q),b,}^{s,a,\beta} f(z)}\right|$$

$$= (1-|z|^2)\left|\frac{z}{\psi_{(\beta_p),(\propto_q),b}^{s,a,\beta} f(z)}\ \psi_{(\beta_p)(\propto_q),b}^{s,a,\beta+1} f(z)\right|$$

$$= (1-|z|^2)\left|[1+\sum_{k=1}^{\infty} b_k z^k]\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s a_k z^k\right]'\right|$$

$$\le (1-|-z|^2).\left[1+\sum_{k=1}^{\infty}|b_k|\,|z^k|\right]\left[|1|-|-\sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k\,|a_k|\,|z^{k-1}|\right]$$

$$\le (1+|z|^2)\left[1+\sum_{k=1}^{\infty}|b_k|\,|z^k|\right]\left[1-\sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k\,|a_k|\,|z^{k-1}|\right]$$

$$\le 2\left[1-\sum_{k=2}^{\infty} \mid \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \mid \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k|\right][1+\sum_{k=1}^{\infty}|b_k|]$$

$$\leq 2\left[1+\sum_{k=1}^{\infty}|b_k| - \sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k|\right] +$$
$$\left(\sum_{k=1}^{\infty}|b_k|\right)\left(-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k|\right).$$

Applying Lemma 1.4, we get

$$2\left[1+\sum_{k=1}^{\infty}|b_k| - 2\sum_{k=1}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s |a_k| - 2\left(\sum_{k=1}^{\infty}|b_k|\right)\left(\sum_{k=1}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s |a_k|\right)\right] \leq 1,$$

therefor,

$$2\left[1+\sum_{k=1}^{\infty}|b_k| - 2\sum_{k=1}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s |a_k| - 2\left(\sum_{k=1}^{\infty}|b_k|\right)\left(\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s |a_k|\right)\right] \leq \frac{(k-1)|b_k|}{(k-1)|b_k|},$$

and we have

$$\sum_{k=1}^{\infty}(k-1)|b_k|^2 \leq 1$$

Theorem 2.5 Let $f \in \mathcal{A}$. If for all $z \in U$

$$\sum_{k=1}^{\infty} k[2(k-1)+\mathcal{R}e(v)]\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s |a_k| \leq \mathcal{R}e(v), \mathcal{R}e(v) > 0 \qquad (12)$$

then

$$G_v(z) = \left(v\int_0^z y^{v-1}\left[\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} f(z)\right]' dy\right)^{\frac{1}{v}} \text{ is univalent in } U$$

Proof. Let $f \in \mathcal{A}$. Then for all $z \in U$

$$\frac{1-|z|^{2\mathcal{R}e(v)}}{\mathcal{R}e(v)}.\left|\frac{z\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} f(z)\right)''}{\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} f(z)\right)'}\right| \leq \frac{1-|-z|^{2\mathcal{R}e(v)}}{\mathcal{R}e(v)}.\frac{|z|\left|\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} f(z)\right)''\right|}{\left|\left(\psi^{s,a,\beta}_{(\beta_p),(\propto_q),b} f(z)\right)'\right|}$$

$$\leq \frac{1+|Z|^{2\mathcal{R}e(v)}}{\mathcal{R}e(v)}.\frac{|z|\left|\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}.\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right).\left(\frac{a+1}{a+k}\right)^s a_k z^k\right]''\right|}{\left|\left[z+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}.\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right).\left(\frac{a+1}{a+k}\right)^s a_k z^k\right]'\right|}$$

$$= \frac{1+|z|^{2\mathcal{R}e(v)}}{\mathcal{R}e(v)}\ \frac{|z|\left|1+\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)a_k z^{k-2}\right|}{|1|-\left|-\sum_{k=2}^{\infty}\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k\, a_k z^{k-1}\right|}$$

$$\leq \frac{1+|z|^{2\mathcal{R}e(v)}}{\mathcal{R}e(v)}.\frac{|z|\left[\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k(k-1)|a_k|\left|z^{k-2}\right|\right]}{1-\sum_{k=2}^{\infty}\left|\frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}}\right|\left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^s k|a_k||z^{k-1}|}$$

$$\leq \frac{2\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k(k-1)|a_k|}{\mathcal{R}e(v)\left[1-\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k|a_k|\right]}$$

Applying Lemma 1.5, we get

$$\frac{2\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k(k-1)|a_k|}{\mathcal{R}e(v)\left[1-\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k|a_k|\right]} \leq 1$$

then

$$2\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k(k-1)|a_k| \leq \mathcal{R}e(v)\left[1-\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k|a_k|\right],$$

therefor,

$$2\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k(k-1)|a_k| + \mathcal{R}e(v)\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k|a_k| \leq \mathcal{R}e(v),$$

and we have

$$\sum_{k=2}^{\infty} \left| \frac{\prod_{i=1}^{p}(1+\beta_i)_{k-1}}{\prod_{i=1}^{q}(1+\propto_i)_{k-1}} \right| \left(\frac{B(a+k,b,s,\beta}{B(a+1,b,s,\beta}\right)\left(\frac{a+1}{a+k}\right)^{s} k[2(k-1)+\mathcal{R}e(v)]\,|a_k| \leq \mathcal{R}e(v).$$

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