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Faber Polynomial for Holomorphic Functions Involving Differential Operator

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ABSTRACT

The purpose of this paper, is to present differential operator for the univalent functions employ a Faber Polynomial . In addition, we will introduce some inclusion properties of the operator that were obtained employ the principle of subordination between holomorphic functions.

1.Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, (z \in \mathcal{U}) \tag{1.1}$$

which are holomorphic and normalized in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Definition:1.1. Let $g(z)$ be holomorphic and univalent in $\mathcal{U}$ and f also holomorphic in $\mathcal{U}$, then f is subordinate to g if there be a Schwartz functions w(z) it is also holomorphic in $\mathcal{U}$ with w(0)= 0 and $|\mathrm{w}(z)| < 1, \forall z \in \mathcal{U}$ such that $f(z) = g(\mathrm{w}(z))$.

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This denoted by $f \prec g$. Furthmore, assume g is univalent in $\mathcal{U}$. Then the next part holds[4,5,8,10]:

$$f \prec g \Leftrightarrow f(0) = g(0) \; and \; f(\mathcal{U}) \subset g(\mathcal{U}).$$

For the function *f* of the form (1.1) ,we will present result the following*: f* is starlike and convex, respectively, with respect to the origin, if and only if

$$\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \left|z\right| < 1,$$

and

$$\mathcal{R}e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \left|z\right| < 1,$$

f is starlike and convex, respectively, of order τ,if only and if

$$\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > \tau, 0 \leq \tau < 1, \left|z\right| < 1,$$

and

$$\mathcal{R}e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \tau, 0 \leq \tau < 1, \left|z\right| < 1.$$

Remark1.1 The function *f* is convex, if and only if $zf'(z)$ is starlike. This is according to the Alexander's Theorem[3].

Definition:1.2. Let *f*∈ $\mathcal{A}$ and g is starlike of order τ , i.e. $g \in S^*(\tau)$, then $f \in K(\varepsilon, \tau)$, if and only if, $\mathcal{R}e\left\{\frac{zf'(z)}{g(z)}\right\} > \varepsilon, z \in \mathcal{U}$, we named this function closed-to-convex function of order ε, type τ . For $f \in \mathcal{A}$, the next subclasses of starlike, convex and closed-to-convex functions $S^*(\zeta, \varpi)$, $C(\zeta, \varpi)$ and $K(\zeta, \rho; \varpi, \varphi)$ of order ζ,with many authors have studied[7,8,9,10] ,and respectively defined by:

$$S^*(\eta, \vartheta) = \left\{f \in \mathcal{A}: \frac{1}{1-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec \vartheta(z), z \in \mathcal{U}\right\}$$

$$C(\eta, \vartheta) = \left\{f \in \mathcal{A}: \frac{1}{1-\eta}\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right) \prec \vartheta(z), \; z \in \mathcal{U}\right\}$$

$$K(\eta, \xi; \vartheta, \varphi) = \left\{f \in \mathcal{A}: \frac{1}{1-\xi}\left(\frac{Zf'(z)}{g(z)} - \xi\right) \prec \varphi(z), \; z \in \mathcal{U}, g(z) \in S^*(\eta, \vartheta)\right\}$$

From (1.1),we have

$$f(z)^\alpha = z^\alpha\left[1 + \alpha(a_2 z + a_3 z^2 + a_4 z^3 + \cdots) + \frac{\alpha(\alpha-1)}{2!}(a_2 z + a_3 z^2 + a_4 z^3 + \cdots)^2 + \cdots\right]$$

$$= z^{\alpha} + \alpha a_2 z^{\alpha+1} + \left(\alpha a_3 + \frac{\alpha(\alpha-1)}{2!} a_2^2\right) z^{\alpha+2} + \left(\alpha a_4 + \frac{\alpha(\alpha-1)}{2!} 2a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\right) z^{\alpha+3}$$

$$+ \left(\alpha a_5 + \frac{\alpha(\alpha-1)}{2!}(2a_2 a_4 + a_3^2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} 3a_2^2 a_3 + 6 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}\right) z^{\alpha+4} \quad ,$$

$$= z^{\alpha} + \alpha a_2 z^{\alpha+1} + \alpha a_3 z^{\alpha+2} + \alpha a_4 z^{\alpha+3} + \cdots$$

definition we give $(f(z))^{\alpha}$by

$$f^{\alpha}(z) = z^{\alpha} + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}. \qquad (1.2)$$

Thus as $\alpha \to 1$, to have :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

For $f \in \mathcal{A}$, the generalized differential operator $\mathcal{T}_{\gamma,\mu}^{m,\delta}: \mathcal{A}_{\alpha} \to \mathcal{A}_{\alpha}$ is defined

$$\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)^{\alpha} = \mathcal{T}\left(\mathcal{T}_{\gamma,\mu}^{m-1,\delta} f(z)^{\alpha}\right)\left(\frac{1-(\gamma-\delta)}{\mu+\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m-1,\delta} f(z)^{\alpha}\right) + \frac{\gamma-\delta}{\mu+\delta} z\left(\mathcal{T}_{\gamma,\mu}^{m-1,\delta} f(z)^{\alpha}\right)' \qquad (1.3)$$

$$= \left(\frac{1+(1-\alpha)(\delta-\gamma)}{\mu+\delta}\right)^m z^{\alpha} + \sum_{k=2}^{\infty}\left(\frac{1+(1-\alpha)(\alpha+k-1)}{\mu+\delta}\right)^m a_k(\alpha) z^{\alpha+k-1},$$

and

where $(\mu > 0, \delta, \gamma \geq 0; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \alpha > 0, z \in \mathcal{U})$. For more details see[6].

From equation (1.3),we get

$$\mathcal{T}_{\gamma,\mu}^{m+1,\delta} f(z) = \left(\frac{1-(\gamma-\delta)}{\mu+\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)\right) + \frac{\gamma-\delta}{\mu+\delta}\left(\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)\right)'$$

$$\frac{\gamma-\delta}{\mu+\delta} z\left(\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)\right)' = \mathcal{T}_{\gamma,\mu}^{m+1,\delta} f(z) - \left(\frac{1-(\gamma-\delta)}{\mu+\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)\right). \qquad (1.4)$$

We indicate by $\mathcal{H}$, the class of all function ϑ which are holomorphic and univalent in $\mathcal{U}$ each of them $\vartheta(\mathcal{U})$ is convex such that $\vartheta(0) = 1$ and $\mathcal{Re}(\vartheta(z)) > 0, z \in \mathcal{U}$.

In this paper, we present differential operator from in relation to (1.1) and (1.2).We must use the principle of subordination between holomorphic functions to present the subclasses of starlike, convex and close-to-convex functions $S^*(\eta,\vartheta)$, $C(\eta,\vartheta)$ and $K(\eta,\xi;\varphi,\vartheta)$ of order η respectively, for the function $\vartheta, \varphi \in \mathcal{H}$ which are defined by:

$$S_{\alpha,\gamma,\mu}^{m,\delta}(\eta,\vartheta) = \left\{f(z) \in \mathcal{A}: \mathcal{T}_{\gamma,\mu}^{m,\delta} f(z) \in S^*(\eta,\vartheta)\right\}$$

$$C_{\alpha,\gamma,\mu}^{m,\delta}(\eta,\vartheta) = \left\{f(z) \in \mathcal{A}: \mathcal{T}_{\gamma,\mu}^{m,\delta} f(z) \in C(\eta,\vartheta)\right\},$$

$$\mathrm{K}_{\alpha,\gamma,\mu}^{m,\delta}(\eta,\xi;\varphi,\vartheta) = \left\{f(z) \in \mathcal{A}: \mathcal{T}_{\gamma,\mu}^{m,\delta} f(z) \in \mathrm{K}(\eta,\xi;\varphi,\vartheta)\right\}.$$

Next ,we will give introductory results that will be used in the show that of the main results of the paper.

Lemma:1.1.[1,2,10] Let ϖ be convex, univalent in $\mathcal{U}$ with $\varpi(0) = 1$ and $\mathcal{R}e\{\pi\varpi(z) + \eta\} \geq 0, \pi, \eta \in \mathbb{C}$. If $\mathcal{P}$ is holomorphic in $\mathcal{U}$ with $\mathcal{P}(0) = 1$, then

$$\mathcal{P}(z) + \frac{z\mathcal{P}'(z)}{\pi\mathcal{P}(z)+\eta} \prec \varpi(z),\ z \in \mathcal{U}, \text{means } \mathcal{P}(z) \prec \varpi(z), z \in \mathcal{U}\ .$$

Lemma:1.2.[8,10] Let ϖ be convex, univalent in $\mathcal{U}$ and w be holomorphic in $\mathcal{U}$ with $\mathcal{R}e(w(z)) \geq 0$. If $\mathcal{P}$ is holomorphic in $\mathcal{U}$ with $\mathcal{P}(0) = \varpi(0)$, then

$$\mathcal{P}(z) + w(z)z\mathcal{P}'(z) \prec \varpi(z), z \in \mathcal{U}, \text{ means } \mathcal{P}(z) \prec \varpi(z), z \in \mathcal{U}.$$

We will introduce some inclusion properties of the operator $\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)^{\alpha}$,by the *principle of subordination.*

2.Main Results

Theorem:2.1 Let $f \in \mathcal{A}$and let $\vartheta \in \mathcal{H}$ with $\mathcal{R}e\left((1-\eta)\vartheta(z) + \eta + \frac{1-(\gamma-\delta)}{\mu+\delta}\right) > 0$. Then,

$$S_{\alpha,\gamma,\mu}^{m+1,\delta}(\eta,\vartheta) \subset S_{\alpha,\gamma,\mu}^{m,\delta}(\eta,\vartheta)$$

Proof: *Let* $f(z) \in S_{\alpha,\gamma,\mu}^{m+1,\delta}(z)$ *and let*

$$\mathcal{P}(z) = \frac{1}{1-\eta}\left(\frac{z\left(\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)\right)'}{\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)} - \eta\right). \qquad (2.1)$$

Applying (1.5) in (2.1), we get:

$$\mathcal{P}(z) = \frac{1}{1-\eta}\left(\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta} f(z) - \left(\frac{1-(\gamma-\delta)}{\mu+\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)\right)}{\frac{\lambda-\delta}{\mu+\delta}\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)} - \eta\right)$$

$$\left(\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta} f(z) - \left(\frac{1-(\gamma-\delta)}{\mu+\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)\right)}{\frac{\lambda-\delta}{\mu+\delta}\mathcal{T}_{\gamma,\mu}^{m,\delta} f(z)} - \eta\right) = (1-\eta)\mathcal{P}(z)$$

$$\left(\frac{\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z) - \left(\frac{1-(\gamma-\delta)}{\mu+\delta}\right)\left(\mathcal{J}_{\lambda,\mu}^{m,\delta} f(z)\right)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)}\right) = (1-\eta)\mathcal{P}(z)+\eta$$

$$\left(\frac{\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)} - \frac{\left(\frac{1-(\gamma-\delta)}{\mu+\delta}\right)\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)}\right) = (1-\eta)\mathcal{P}(z)+\eta$$

$$\frac{\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)} = (1-\eta)\mathcal{P}(z)+\eta+\left(\frac{1-(\gamma-\delta)}{\gamma-\delta}\right). \tag{2.2}$$

Differentiating (2.2) logarithmically with respect to z

$$Log\left(\frac{\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)}\right) = Log\left((1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right)$$

$$Log\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right) - Log\left(\frac{\gamma-\delta}{\mu+\delta}\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right) = Log\left((1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right)$$

$$\frac{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)} - \frac{\left(\frac{\gamma-\delta}{\mu+\delta}\right)\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)'}{\left(\frac{\gamma-\delta}{\mu+\delta}\right)\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)} = \frac{(1-\eta)\mathcal{P}'(z)}{(1-\eta)\mathcal{P}(z)+\frac{1-(\gamma-\delta)}{\gamma-\delta}}$$

$$\frac{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)} = \frac{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)} + \frac{(1-\eta)\mathcal{P}'(z)}{(1-\eta)\mathcal{P}(z)+\frac{1-(\gamma-\delta)}{\gamma-\delta}}. \tag{2.3}$$

On the other hand:

$$\frac{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)} = \frac{(1-\eta)\mathcal{P}(z)+\eta}{z}, \tag{2.4}$$

using (2.3) and (2.4)

$$\frac{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)} = \frac{(1-\eta)\mathcal{P}(z)+\eta}{z} + \frac{(1-\eta)\mathcal{P}'(z)}{(1-\eta)\mathcal{P}(z)+\frac{1-(\gamma-\delta)}{\gamma-\delta}}$$

$$\frac{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)} = \frac{(1-\eta)\mathcal{P}(z)}{z} + \frac{\eta}{z} + \frac{(1-\eta)\mathcal{P}'(z)}{(1-\eta)\mathcal{P}(z)+\frac{1-(\gamma-\delta)}{\gamma-\delta}}$$

$$\frac{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta}f(z)\right)}-\frac{\eta}{z}=\frac{(1-\eta)\mathcal{P}(z)}{z}+\frac{(1-\eta)\mathcal{P}'(z)}{(1-\eta)\mathcal{P}(z)+\frac{1-(\gamma-\delta)}{\gamma-\delta}}$$

$$\frac{z\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta}f(z)\right)}-\eta=\frac{(1-\eta)\left[\mathcal{P}(z)\left((1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right)\right]+z\mathcal{P}'(z)}{\left[(1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right]}$$

$$\frac{1}{1-\eta}\left(\frac{z\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta}f(z)\right)}-\eta\right)=\frac{\mathcal{P}(z)\left((1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right)}{\left((1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right)}+\frac{z\mathcal{P}'(z)}{\left((1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right)}$$

$$\frac{1}{1-\eta}\left(\frac{z\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta}f(z)\right)}-\eta\right)=\mathcal{P}(z)+\frac{z\mathcal{P}'(z)}{\left((1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right)}. \qquad (2.5)$$

Applying Lemma (1.1) to (2.5),we get

$$\mathcal{R}e\{\pi\vartheta(z)+\tau\}\geq 0,$$

$$\pi\mathcal{R}e\,(\vartheta(z))+\tau\geq 0$$

Since ϑ convex, order τ

$$\mathcal{R}e\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\pi\left|\frac{zf''(z)}{f'(z)}\right|+\tau, z\in\mathcal{U}, 0\leq\tau<1, \pi\geq 0, \pi+\tau\geq 0$$

$$\mathcal{R}e\left\{(1-\eta)\vartheta(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right\}>0, \delta,\gamma\geq 0$$

$$(1-\eta)\mathcal{R}e(\vartheta(z))+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}>0, \vartheta(0)=1, \mathcal{R}e\big(\vartheta(z)\big)>0, z\in\mathcal{U},$$

$$(1-\eta)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}>0,$$

then

$$\mathcal{P}(z)+\frac{z\mathcal{P}'(z)}{(1-\eta)\mathcal{P}(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}}\prec\vartheta(z), z\in\mathcal{U}, \text{means } \mathcal{P}(z)\prec\vartheta(z), z\in\mathcal{U},$$

i.e. $f(z)\in \mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)$.

Thus,

$$S_{\alpha,\gamma,\mu}^{m+1,\delta}f(z)\subset S_{\alpha,\gamma,\mu}^{m,\delta}f(z), \qquad \square$$

Theorem:2.2. Let $f\in A$ and let $\vartheta\in\mathcal{H}$ with $\mathcal{R}e\left\{(1-\eta)\vartheta(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right\}>0$. Then,

$$C^{m+1,\delta}_{\alpha,\gamma,\mu} f(z) \subset C^{m,\delta}_{\alpha,\gamma,\mu} f(z).$$

Proof: From Remark 1.1**,** we have**:** f ∈ $\mathcal{A}$ is starlike and convex, respectively, order τ, if and only if

$$\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > \tau, o \le \tau < 1, |z| < 1,$$

and

$$\mathcal{R}e\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \tau, o \le \tau < 1, |z| < 1.$$

A function f ∈ $\mathcal{A}$ is uniformly starlike and uniformly convex, respectively, of order τ, if and only if [5]

$$\mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > \pi\left|\frac{zf'(z)}{f(z)} - 1\right| + \tau, z \in \mathcal{U}$$

and

$$\mathcal{R}e\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \pi\left|\frac{zf''(z)}{f'(z)}\right| + \tau, z \in \mathcal{U},$$

where $\pi \ge 0,\ 0 \le \tau < 1\ and\ \pi + \tau \ge 0.$

The relation between the classes of starlike and convex functions, obviously ,led us to the next relation .

$$f \in C^{m+1,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta) \Leftrightarrow zf' \in S^{m+1,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta).$$

From Theorem 2.1,we get

$$f \in C^{m+1,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta) \Leftrightarrow zf' \in S^{m+1,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta) \subset S^{m,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta)$$

$$\Longrightarrow zf' \in S^{m,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta)$$

$$\Longrightarrow zf' \in C^{m,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta).$$

Thus,

$$C^{m+1,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta) \subset C^{m,\delta}_{\alpha,\gamma,\mu}(\eta,\vartheta).$$

□

The function $\vartheta(z) = \frac{1-\mathcal{A}z}{1-\mathcal{B}z}$ is holomorphic and satisfies $\vartheta(0) = 1$.Thus,we get the next corollaries:

Corollary:2.1. Let $f \in \mathcal{A}\ and\ \vartheta = \frac{1+\mathcal{A}z}{1-\mathcal{B}z}, -1 \le \mathcal{B} \le \mathcal{A} \le 1$ in Theorem 2.1.Then:

$$S^{m+1,\delta}_{\alpha,\gamma,\mu}(\eta,\mathcal{A},\mathcal{B}) \subset S^{m,\delta}_{\alpha,\gamma,\mu}(\eta,\mathcal{A},\mathcal{B}).$$

Corollary:2.2. Let $f \in \mathcal{A}$ and $\vartheta = \frac{1+\mathcal{A}z}{1-\mathcal{B}z}, -1 \le \mathcal{B} \le \mathcal{A} \le 1$ in Theorem 2.1.Then:

$$\mathrm{K}^{m+1,\delta}_{\alpha,\gamma,\mu}(\eta,\mathcal{A},\mathcal{B}) \subset \mathrm{K}^{m,\delta}_{\alpha,\gamma,\mu}(\eta,\mathcal{A},\mathcal{B}).$$

Theorem:2.3 Let $f \in \mathcal{A}$ and let $\vartheta, \varphi \in \mathcal{H}$ with $\mathcal{R}\mathrm{e}\left\{(1-\eta)\vartheta(z)+\eta+\frac{1-(\gamma-\delta)}{\gamma-\delta}\right\}>0$. Then:

$$\mathrm{K}_{\alpha,\gamma,\mu}^{m+1,\delta}(\eta,\xi;\ \vartheta,\varphi)\ \subset\ \mathrm{K}_{\alpha,\gamma,\mu}^{m,\delta}(\eta,\xi;\ \vartheta,\varphi)\,.$$

Proof : Let $f \in\ K_{\alpha,\gamma,\mu}^{m+1,\delta}\ (\eta,\xi;\ \vartheta,\varphi)$, and let $g \in\ S_{\alpha,\gamma,\mu}^{m+1,\delta}(\eta,\vartheta)$ such that:

$$\mathcal{R}e\left\{\frac{z\left(\mathcal{T}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ g(z)}\right\}>\xi, z\in U.$$

The next should, we get

$$\frac{1}{1-\xi}\left(\frac{z\left(\mathcal{T}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ g(z)}-\xi\right)\prec\varphi, z\in\mathcal{U}.$$

Let

$$\mathcal{P}(z)=\frac{1}{1-\xi}\left(\frac{z\left(\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ f(z)\right)'}{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ g(z)}-\xi\right) \tag{2.6}$$

$$\left(\frac{z\left(\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ f(z)\right)'}{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ g(z)}-\xi\right)=\ (1-\xi)\mathcal{P}(z)$$

$$\frac{z\left(\mathcal{T}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ g(z)}=\ (1-\xi)\mathcal{P}(z)+\xi.$$

From (1.6),we get

$$\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}f(z)-\left(\frac{1-(\gamma-\delta)}{\mu-\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}\ f(z)\right)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{T}_{\gamma,\mu}^{m,\delta}\ g(z)}=(1-\xi)\mathcal{P}(z)+\xi$$

$$\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ f(z)-\left(\frac{1-(\gamma-\delta)}{\mu-\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}f(z)\right)}{\frac{\gamma-\delta}{\mu+\delta}}=[(1-\xi)\mathcal{P}(z)+\xi]\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}\ g(z)\right)$$

$$\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}\ f(z)}{\frac{\gamma-\delta}{\mu+\delta}}-\frac{\left(\frac{1-(\gamma-\delta)}{\mu-\delta}\right)}{\frac{\gamma-\delta}{\mu+\delta}}\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}\ f(z)\right)=[(1-\xi)\mathcal{P}(z)+\xi]\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right).$$

Mean that

$$\frac{\left(\mathcal{T}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\frac{\gamma-\delta}{\mu+\delta}} = \left(\frac{1-(\gamma-\delta)}{\mu-\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}f(z)\right)' + [(1-\xi)\mathcal{P}(z)+\xi]\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right)'$$

$$+\left[(1-\xi)\mathcal{P}'(z)\right]\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right).$$

$$\frac{z(\mu+\delta)}{\gamma-\delta}\left(\mathcal{T}_{\gamma,\mu}^{m+1,\delta}f(z)\right)' = \left(\frac{1-(\gamma-\delta)}{\gamma-\delta}\right)z\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}f(z)\right)' + [(1-\xi)\mathcal{P}(z)+\xi]z\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right)'$$

$$+\left[(1-\xi)z\mathcal{P}'(z)\right]\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right). \qquad (2.7)$$

Also, by Theorem 2.1, $g \in S_{\gamma,\mu}^{m+1,\delta}(\eta,\vartheta) \to g \in S_{\gamma,\mu}^{m,\delta}(\eta,\vartheta)$.

Thus, let

$$q(z) = \frac{1}{1-\eta}\left(\frac{z\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right)'}{\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)} - \eta\right). \qquad (2.8)$$

Using (1.6) in (2.8),we get

$$\left(\frac{z\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right)'}{\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)}\right) = (1-\eta)q(z)+\eta$$

$$\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}g(z) - \left(\frac{1-(\gamma-\delta)}{\mu-\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)} = (1-\eta)q(z)+\eta$$

$$\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}g(z)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)} - \frac{\left(\frac{1-(\gamma-\delta)}{\mu-\delta}\right)\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)} = (1-\eta)q(z)+\eta$$

$$\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}g(z)}{\frac{\gamma-\delta}{\mu+\delta}\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)} = (1-\eta)q(z)+\eta+\frac{\left(1-(\gamma-\delta)\right)}{\gamma-\delta}. \qquad (2.9)$$

And further, from (2.7) and (2.9), we get

$$\frac{\frac{\mu+\delta}{\gamma-\delta}\left(\mathcal{T}_{\gamma,\mu}^{m+1,\delta}f(z)\right)'}{\frac{\mu+\delta}{\gamma-\delta}\frac{\mathcal{T}_{\gamma,\mu}^{m+1,\delta}g(z)}{\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)}} = \frac{\left(\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right)z\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}f(z)\right)' + [(1-\xi)\mathcal{P}(z)+\xi]z\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right)'}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$+\frac{[(1-\xi)z\mathcal{P}'(z)]\left(\mathcal{T}_{\gamma,\mu}^{m,\delta}g(z)\right)}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$\frac{\frac{\mu+\delta}{\gamma-\delta}\, z\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)'}{\frac{\mu+\delta}{\gamma-\delta}\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} g(z)\right)} = \frac{\left(\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right) z\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)' + [(1-\xi)\mathcal{P}(z)+\xi] z\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$+\frac{[(1-\xi)z\mathcal{P}'(z)]\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)}{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$=\frac{\left(\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right) z\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]} + \frac{[(1-\xi)\mathcal{P}(z)+\xi] z\left(\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$+\frac{[(1-\xi)z\mathcal{P}'(z)]\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)}{\left(\mathcal{J}_{\gamma,\mu}^{m,\delta} g(z)\right)\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$=\frac{\frac{(1-(\gamma-\delta))}{\gamma-\delta}[(1-\xi)\mathcal{P}(z)+\xi]}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]} + \frac{[(1-\xi)\mathcal{P}(z)+\xi][(1-\eta)q(z)+\eta]}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]} + \frac{[(1-\xi)z\mathcal{P}'(z)]}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$=\frac{\left(\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right)[(1-\xi)\mathcal{P}(z)+\xi] + [(1-\xi)\mathcal{P}(z)+\xi][(1-\eta)q(z)+\eta]}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]} + \frac{[(1-\xi)z\mathcal{P}'(z)]}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$\frac{z\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} g(z)\right)} = [(1-\xi)\mathcal{P}(z)+\xi] + \frac{[(1-\xi)z\mathcal{P}'(z)]}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}. \tag{2.10}$$

Algebraic manipulation in (2.10) gives:

$$\frac{z\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} g(z)\right)} - \xi = [(1-\xi)\mathcal{P}(z)] + \frac{[(1-\xi)z\mathcal{P}'(z)]}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

$$\frac{1}{1-\xi}\left(\frac{z\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} f(z)\right)'}{\left(\mathcal{J}_{\gamma,\mu}^{m+1,\delta} g(z)\right)} - \xi\right) = \mathcal{P}(z) + \frac{[z\mathcal{P}'(z)]}{\left[(1-\eta)q(z)+\eta+\frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}.$$

So, we get

$$w(z) = \frac{1}{\left[(1-\eta)q(z) + \eta + \frac{(1-(\gamma-\delta))}{\gamma-\delta}\right]}$$

Applying Lemma 1.2, we get

$$\mathcal{P}(z) \prec \varphi(z),\ mean\ that\ f \in \mathrm{K}_{\gamma,\mu}^{m+1,\delta}(\eta,\xi;\vartheta,\varphi) \qquad \square$$

3. Conclusion: In this article we concluded that in this case of applying the differential operator for univalent function using a Faber polynomial it remains preserving it's geometric properties and there are other conditions added to obtain results inside the unit disk.

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