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ABSTRACT

Numerical solution of the modified equal width equation is getting by quartic trigonometric B-spline scheme. The approach based on finite difference scheme with the help of Crank-Nicolson formulation. The finite difference scheme is used for time integration and quartic trigonometric B-spline function for space integration. Performance and accuracy of the scheme is validated through testing two problems by using conserved laws and L_∞ and L_2 error norms and applying the Von-Neumann stability analysis shows to be unconditionally stable.

Keyword: Quartic trigonometric B-spline, finite difference, modified equal width equation, Von-Neumann.

1. Introduction

The modified equal width equation single wave is considered here has normalized form.

$$\frac{\partial v}{\partial t}+\varepsilon v^2\frac{\partial v}{\partial x}-\mu\frac{\partial v}{\partial xxt}=0 \qquad x\in[a,b],t\in[0,T] \tag{1}$$

With collocation boundary conditions (BCs)

$$v(a,t)=0 \ , v(b,t)=0$$

$$\begin{aligned} &v_x(a,t)=v_x(b,t)=0,\\ &v_{xx}(a,t)=v_{xx}(b,t)=0, \end{aligned} \tag{2}$$

with initial condition (IC)

$$v(x,t)=f(x) \quad ,a\le x\le b] \tag{3}$$

The μ is a positive parameter and ε is an arbitrary constant, $f(x)$ is a localized disturbance inside the interval [a, b] (IslamTirmizi, I2010). Solved the (MEW) equation by using quartic B-spline collocation method, and obtained a high efficient and accuracy through comparing their solution with the exact solution. (Saka, 2007) used quintic B-spline collection method for numerical solution for the (MEW) equation through algorithm which is based on Crank-Nicolson formulation. (Karakoç & Geyikli, 2012) Solved the (MEW) equation by using lumped galerkin method which based on cubic b-spline finite element method. The numerical results they found are equivalent with the exact solution through computing the numerical conserved laws and $L_\infty and\, L_2$ error norms. (Zaki, 2000) Obtained a remarkably successful numerical solution for the (MEW) equation by applying septic b-spline finite element method on the motion of a single solitary wave (Evans & Raslan, 2005). used quadratic B-spline method for solving (MEW) equation. (Esen & Kutluay, 2008) Solved modified equal width (MEW) equation by linerizing numerical scheme based on finite difference method. (Shazalina 2016) solved (MEW) equation numerically via a petrow-galerkin method by using quinitic b-spline finite elements.

2- Quartic Trigonometric B-Spline Collocation Method

In this section, Firstly, the quartic trigonometric basis function is defined as follows.

$$TB_j^5(x)=\frac{1}{w}\begin{cases} a^4(x_j), & x\in[x_j,x_{j+1}) \\ a^3(x_j)\,b(x_{j+2})+a^2(x_j)\,b(x_{j+3}) & x\in[x_{i+1},x_{i+2}) \\ +a(x_j)\,b(x_{j+4})\,a^2(x_{j+1})+b(x_{j+5})\,a^3(x_{j+1}), & \\ a^2(x_j)\,b^2(x_{j+3})+a(x_j)\,b(x_{j+4})\,a(x_{j+1})\,b(x_{j+3}) & \\ +a(x_j)\,b^2(x_{j+4})\,a(x_{j+2})+b(x_{j+5})\,a^2(x_{j+1})\,b(x_{j+3}) & x\in[x_{i+2},x_{i+3}) \\ +b(x_{j+5})\,a(x_{j+1})\,b(x_{j+4})\,a(x_{j+2})+b^2(x_{j+5})\,a^2(x_{j+2}), & \\ a(x_j)\,b^3(x_{j+4})+b(x_j+5)\,a(x_{j+1})\,b^2(x_{j+4}) & x\in[x_{i+3},x_{i+4}] \\ +b^2(x_{j+5})\,a(x_{j+2})\,b(x_{j+4})+b^3(x_{j+5})\,a(x_{j+3}), & \\ b^4(x_{j+5}), & x\in[x_{i+4},x_{i+5}] \\ 0 & otherwise \end{cases} \quad (4)$$

where $a(x_j)=\sin\left(\frac{x-x_j}{2}\right),\ b(x_j)=\sin\left(\frac{x_j-x}{2}\right),\ w=\sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\sin(2h)$

Due to local support properties of B-spline ,there are only four nonzero basis functions,

$B_{5,j-4}(x_j), B_{5,j-3}(x_j), B_{5,j-2}(x_j)\, and\, B_{5,j-1}(x_j)$ included over subinterval $[x_j,x_{j+1}]$.

Where $h=(b-a)/n$ and values of $TB_i^5(x)$ and its derivatives at nodal points are required and these derivatives are tabulated in Table 1. (Shazalina 2016). Secondly, It is discussed the quartic trigonometric B-spline collocation method (QuTBSM) for solving the modified equal width equation (1).

Table 1: Values of $TB_i^4(x)$ and its derivatives

x	x_{j-4}	x_{j-3}	x_{j-2}	x_{j-1}	x_j
TB_j	q_1	q_2	q_2	q_1	0
TB'_j	q_3	q_4	q_4	q_3	0
TB''_j	q_5	q_5	q_5	q_5	0
TB'''_j	q_6	q_7	q_7	q_6	0

where

$$q_1=\frac{\sin^3(\frac{h}{2})}{\sin(\mathrm{h})\sin(\frac{3h}{2})\sin(2\,\mathrm{h})}\quad ,q_2=\frac{5+6\cos(\mathrm{h})}{8\cos^2(\mathrm{h})\cos(\mathrm{h})(1+2\cos(\mathrm{h}))}\quad ,q_3=\frac{1}{2\sin(\mathrm{h})\cos(\mathrm{h})(1+2\cos(\mathrm{h}))}\quad ,$$

$$q_4=-\frac{1}{\sin(2\,\mathrm{h})}\ ,\qquad q_5=\frac{1}{\sin(\mathrm{h})\sin(2\,\mathrm{h})}$$

$$q_6=\frac{\cos(\frac{h}{2})(1-4\cos(\mathrm{h}))}{\sin(\mathrm{h})\sin(\frac{3h}{2})\sin(2\,\mathrm{h})}\ ,\qquad q_7=\frac{1+2\cos(\mathrm{h})}{2\sin^2(\frac{h}{2})\sin(2\,\mathrm{h})}$$

The solution domain $a \le x \le b$ is equally divided by knots x_i into n subintervals $[x_i, x_{i+1}]$, $i = 0,1,2,...,n-1$ where $a = x_0 < x_1 < ... < x_n = b$. Our approach for (MEW) equation using quartic trigonometric B-spline is to seek an approximate solution as [8]:

$$V_j^i=\sum_{k=j-3}^{j-1} D_k^i TB_k^4(x) \qquad (5)$$

where $D_j(t)$ is a time dependent unknown to be determined where $j = 0,1,2,...,n$. So as to get the approximations to the solution, the values of $B_{5,j}(x)$ and its derivatives at nodal points are required and these derivatives are tabulated using approximate functions (4) and (5), the values at the knots of V_i^j and their derivatives up to three orders are

$$\begin{cases}(v)_j^i = q_1D_{j-4}^i + q_2D_{j-3}^i + q_2D_{j-2}^i + q_1D_{j-1}^i,\\ (v_x)_j^i = q_3D_{j-4}^i + q_4D_{j-3}^i - q_4D_{j-2}^i - q_3D_{j-1}^i\\ (v_{xx})_j^i = q_5D_{j-4}^i - q_5D_{j-3}^i - q_5D_{j-2}^i + q_5D_{j-1}^i\\ (v_{xxx})_j^i = q_6D_{j-4}^i + q_7D_{j-3}^i - q_7D_{j-2}^i - q_6D_{j-1}^i\end{cases} \qquad (6)$$

The approximations for the solutions of (MEW) equation (1) at t_{j+1} th time level can be given as

$$[\frac{(v-v_{xx})^{n+1}-(v-v_{xx})^{n}}{\Delta t}]+\varepsilon[\frac{(v^2v_x)^{n+1}+(v^2v_x)^{n}}{2}]=0 \qquad (7)$$

where $n=0,1,2,..$ and Δt is the time step. The nonlinear term in equation (7) approximated using the Taylor series [1]:

$$(v^2)^{n+1}v_x^{\,n+1}=(v^n)^2v_x^{\,n+1}+2v^nv_x^nv^{n+1}-2(v^n)^2v_x^n \qquad (8)$$

The equation (7) with putting the values of nodal values v and derivatives using (6) becomes the following difference equation with variable D_j , $j=-3,...,n-1$ and noted the equation a Crank-Nicolson when $\theta=\frac{1}{2}$

$$a_1D_{j-4}^{n+1}+a_2D_{j-3}^{n+1}+a_3D_{j-2}^{n+1}+a_4D_{j-1}^{n+1}=b_1D_{j-4}^{n}+b_2D_{j-3}^{n}+b_3D_{j-2}^{n}+b_4D_{j-1}^{n} \qquad (9)$$

where

$$\begin{aligned}
a_1&=((2+2\Delta t\,\varepsilon\,v^nv_x^n)q_1+\Delta t\varepsilon(v^n)^2q_3-2\mu q_5)\ , &\quad b_1&=(2q_1+\Delta t\,\varepsilon(v^n)^2q_3-2\mu q_5)\\
a_2&=((2+2\Delta t\,\varepsilon\,v^nv_x^n)q_2+\Delta t\varepsilon(v^n)^2q_4+2\mu q_5)\ , &\quad b_2&=(2q_2+\Delta t\,\varepsilon(v^n)^2q_4+2\mu q_5)\\
a_3&=((2+2\Delta t\,\varepsilon\,v^nv_x^n)q_2-\Delta t\varepsilon(v^n)^2q_4+2\mu q_5)\ , &\quad b_3&=(2q_2+\Delta t\,\varepsilon(v^n)^2q_4+2\mu q_5)\\
a_4&=((2+2\Delta t\,\varepsilon\,v^nv_x^n)q_1-\Delta t\varepsilon(v^n)^2q_3-2\mu q_5)\ , &\quad b_4&=(2q_1-\Delta t\,\varepsilon(v^n)^2q_3-2\mu q_5)
\end{aligned} \qquad (10)$$

when simplifing (9) the system, It will consist of $(N+1)$ linear equation in $(N+4)$ unknown $D^i=[D_{j-3}^n,...,D_{N-1}^n]$ at the time level $t=t_{i+1}$. In order to get the unique solution to the system, these three equations are added by getting it from boundary conditions; the system consists $(N+4)\times(N+4)$ in the following form:

$$A_{(N+4)\times(N+4)}\,D_{1xN+4}^{n+1}=B_{(N+4)\times(N+4)}\,D_{1xN+4}^{n}$$

From the initial conditions and its derivatives, lt is computed the initial vector by using this approximate solution

$$\begin{cases}
(v_j^0)_x = f_0^{'}(x_j) & j=0\\
(v_j^0)_{xx} = f^{''}(x_j) & j=0\\
v_j^0 = f_0(x_j) & j=0,1,..N\\
(v_j^0)_x = f_n^{'}(x_j) & j=N
\end{cases} \qquad (11)$$

From equation (11) obtain system consist $(N+4)\times(N+4)$ of the form

$$AF^0=d$$

3.Stability analysis

In this section apply the von- Neumann method the nonlinear term εv^2v_x is linearized by take v^2 as constant value λ .the linearized form of proposed scheme as followed:

$$v_t+\varepsilon\lambda v_x-\mu v_{xxt}=0$$

The linearized form of proposed scheme as followed:

$$a_1C_{j-4}^{n+1}+a_2C_{j-3}^{n+1}+a_3C_{j-2}^{n+1}+a_4C_{j-1}^{n+1}=b_1C_{j-4}^{n}+b_2C_{j-3}^{n}+b_3C_{j-2}^{n}+b_4C_{j-1}^{n} \qquad (12)$$

where

$$a_1 = p_1 + \frac{\Delta t}{2}(\alpha+\delta)p_3 - (1+\frac{\lambda\,\Delta t}{2})p_5\ , \qquad b_1 = p_1 - \frac{\Delta t}{2}(\alpha+\delta)p_3 + (-1+\frac{\lambda\,\Delta t}{2})p_5$$
$$a_2 = p_2 + \frac{\Delta t}{2}(\alpha+\delta)p_4 + (1+\frac{\lambda\,\Delta t}{2})p_5, \qquad b_2 = p_2 - \frac{\Delta t}{2}(\alpha+\delta)p_4 + (-1+\frac{\lambda\,\Delta t}{2})p_5$$
$$a_3 = p_2 - \frac{\Delta t}{2}(\alpha+\delta)p_4 + (1+\frac{\lambda\,\Delta t}{2})p_5\ , \qquad b_3 = p_2 + \frac{\Delta t}{2}(\alpha+\delta)p_4 + (1-\frac{\lambda\,\Delta t}{2})p_5$$
$$a_4 = p_1 - \frac{\Delta t}{2}(\alpha+\delta)p_3 - (1+\frac{\lambda\,\Delta t}{2})p_5, \qquad b_4 = p_1 + \frac{\Delta t}{2}(\alpha+\delta)p_3 + (-1+\frac{\lambda\,\Delta t}{2})p_5,$$

Substitution of $D_j^n = \zeta^n e^{(\text{im}\eta h)}$ $,i=\sqrt{-1}$ into equation (12) after simplifying the equation, It is obtained

$$\zeta^n = \frac{X_2 + iY_2}{X_1 + iY_1}$$

where

$$X_1 = (2q_1 + \Delta t\varepsilon\lambda q_3 - 2\mu q_5)\cos(2\eta\,\mathrm{h}) + (2(q_2+q_1) + \Delta t\varepsilon\lambda(q_4-q_3))\cos(\eta\,\mathrm{h}) + (2q_2 - \Delta t\varepsilon\lambda q_4 + 2\mu q_5)$$
$$Y_1 = -((2q_1 + \Delta t\varepsilon\lambda q_3 - 2\mu q_5)\sin(2\eta\,\mathrm{h}) + (2(q_2-q_1) + \Delta t\varepsilon\lambda(q_4+q_3) + 4\mu q_5)\sin(\eta\,\mathrm{h})))$$
$$X_2 = (2q_1 - \Delta t\varepsilon\lambda q_3 - 2\mu q_5)\cos(2\eta\,\mathrm{h}) + (2(q_2+q_1) + \Delta t\varepsilon\lambda(q_3-q_4))\cos(\eta\,\mathrm{h}) + (2q_2 + \Delta t\varepsilon\lambda q_4 + 2\mu q_5)$$
$$Y_2 = -((2q_1 - \Delta t\varepsilon\lambda q_3 - 2\mu q_5)\sin(2\eta\,\mathrm{h}) + (2(q_2-q_1) - \Delta t\varepsilon\lambda(q_4+q_3) + 4\mu q_5)\sin(\eta\,\mathrm{h}))$$

Therefore the linearized numerical scheme for modified equal width equation is unconditionally stable.

4. Numerical Example and discussion

In this section, two examples are given in this section with L_∞ and L_2 error norms are calculated by $L_\infty = \max_i |v_i - V_i|$ and $L_2 = \sqrt{h\left(\sum_i^n |v_i - V_i|^2\right)}$.

The conservation laws apple on equation (1) as follows [11]

$$C_1 = \int_a^b \mathrm{v(x,t)\,dx},$$

$$C_2 = \int_a^b \mathrm{v(x,t)^2\,dx},$$

$$C_3 = \int_a^b [\mathrm{v(x,t)^2} + \frac{1}{3}\mathrm{v(x,t)^3}]\,\mathrm{dx},$$

Where $C_1, \mathrm{C}_2, \mathrm{C}_3$ correspond mass, momentum and energy, respectively.

Then, by comparing the numerical solutions, It is obtained by testing the quartic trigonometric B-spline collocation method for (MEW) equation (1) with the exact solutions and those numerical methods which were exiting in previous literature. Numerical results are computed at different time levels.

Example 4.1

The initial condition for the (MEW) equation is given by $v(\mathrm{x},0) = \mathrm{Asech(k(x-x_0))}$ and boundary conditions $v(0,\mathrm{t}) = 0, \mathrm{v}(80,\mathrm{t}) = 0$ taken from the exact solution $v(\mathrm{x,t}) = \mathrm{Asech(k(x-ct-x_0))}$ with the parameters c is the wave velocity, $c = \frac{\varepsilon A^2}{6}$ and $k = \sqrt{\frac{1}{\mu}}$ and a=0,b=80 . table 2. Shows the error norms L_∞ and L_2 errors at different time levels and C_1, C_2, C_3 with $\Delta t = 0.2$, $\Delta x = 0.1$, A=0.25, and $x_0 = 30$ and compare the result that get with (Saka, 2007) and (Esen & Kutluay, 2008) and t=20 found more accurate for quartic trigonometric B-spline. Figures 1 shows the single solitary wave solutions at t=0,20 and A=0.25.Figuers 2 at different value to A=0.25,0.5,0.75 and 1. shows the single solitary wave solutions at t=20.

Table 2 **Error norms and invariants for single wave at different times.**

T	Δt	C_1	C_2	C_3	$L_\infty \times 10^4$	$L_2 \times 10^4$
0	0.2	0.785398	0.166666	0.005208	0	0
5		0.785417	0.166709	0.005210	0.005687	0.008376
10		0.784639	0.166478	0.005194	0.011401	0.016708
15		0.783098	0.165993	0.005162	0.017107	0.025068
20		0.780851	0.165285	0.005115	0.022814	0.033396
20[2]		0.781668	0.166434	0.005194	1.744330	1.958879
20[6]		0.785398	0.166474	0.005208	2.576377	2.701647
0	0.05	0.785398	0.166666	0.005208	0	0
5		0.785002	0.166573	0.005202	0.0056111	0.008384
10		0.784008	0.166207	0.005177	0.0112608	0.016765
15		0.782158	0.165590	0.005137	0.0169151	0.025148
20		0.779612	0.164751	0.005082	0.0225604	0.033538
20[6]		0.785398	0.166474	0.005208	2.569972	2.692812

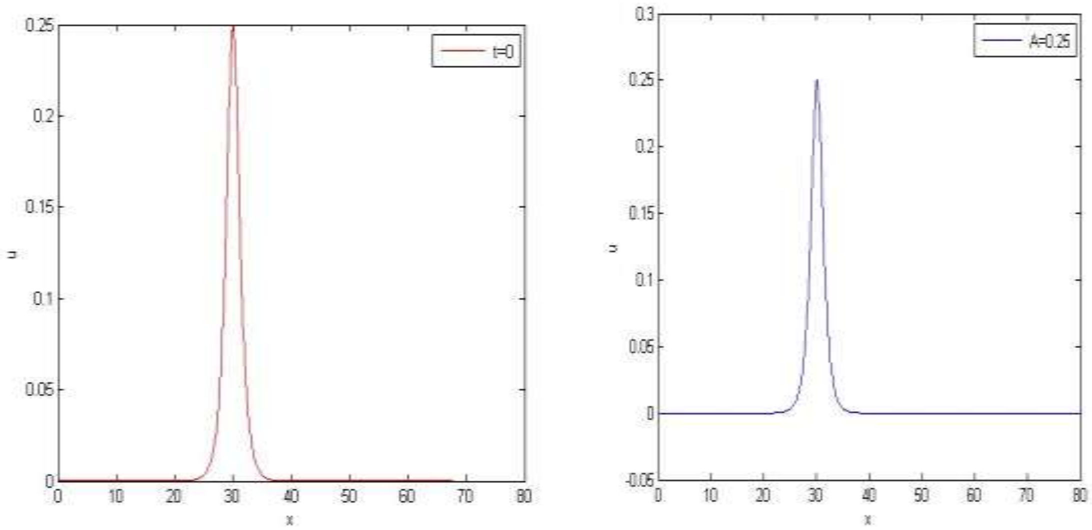

Figure 1 .single ssolitary wave solutions at t=0,20 and A=0.25.

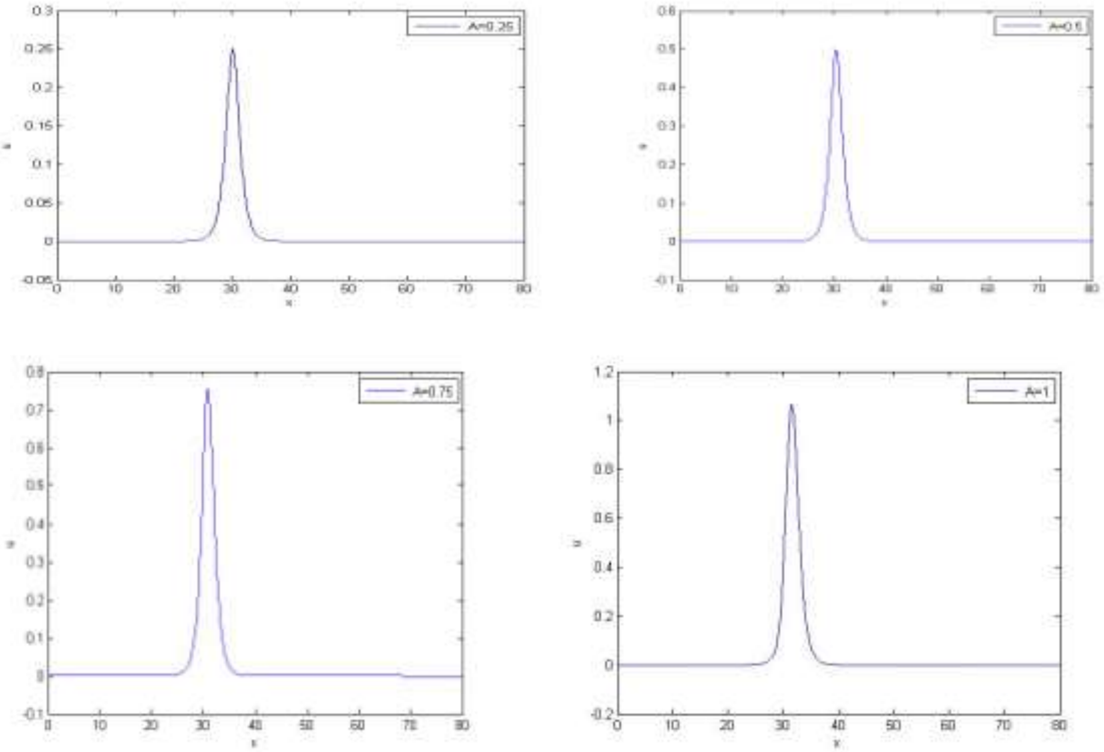

Figure 2 .Single solitary wave solutions at t=20 for different values of A.

Example 4.2.

The exact solution $v_t + 3v^2 v_x - v_{xxt} = 0$

With initial condition

$$v(\mathrm{x},0) = \mathrm{Asech}(\mathrm{k}(\mathrm{x}-\mathrm{x}_0))$$

And boundary conditions are obtained from exact solution with parameter a=0,b=70 , with $\Delta t = 0.05$, $\Delta x = 0.1$, $x_0 = 30$ table 2. Shows the error norms L_∞ and L_2 errors at different time and various value to A and calculating conservation properties of the MEW eq. To mass, momentum and energy $C_1, \mathrm{C}_2, \mathrm{C}_3$.

Table 3 Error norms and invariants for single wave at different times and various values to A**.**

t	**A**	C_1	C_2	C_3	L_2	L_∞
0	0.25	0.785398	0.166666	0.005208	0.0	0.0
5		0.785100	0.166573	0.005202	0.008384	0.005611
10		0.784008	0.166207	0.005177	0.016765	0.011260
15		0.782158	0.169151	0.005137	0.025148	0.165590
20		0.796125	0.164751	0.005082	0.033538	0.022560
20[2]		0.785398	0.166667	0.005208	0.275368	0.328530
20[5]		0.784954	0.166476	0.005199	0.290516	0.249825

0	0.5	1.570796	0.666667	0.083333	0.0	0.0
5		1.561702	0.661130	0.081850	0.066792	0.045629
10		1.532771	0.643763	0.077356	0.133671	0.089389
15		1.494190	0.623506	0.072255	0.202022	0.130250
20		1.453792	0.605354	0.067785	0.272280	0.175605
20[2]		1.570796	0.666666	0.083333	0.640123	0.920874
0	1.0	3.141592	2.666666	1.333333	0.0	0.0
5		2.975453	2.540387	1.193472	0.517206	0.328195
10		2.772068	2.451067	1.092882	1.027194	0.615067
15		2.696522	2.462922	1.096435	1.422060	0.809032
20		2.697919	2.529398	1.155016	1.677453	0.909200
20[4]		3.141592	2.666666	1.333333	1.446540	0.836017

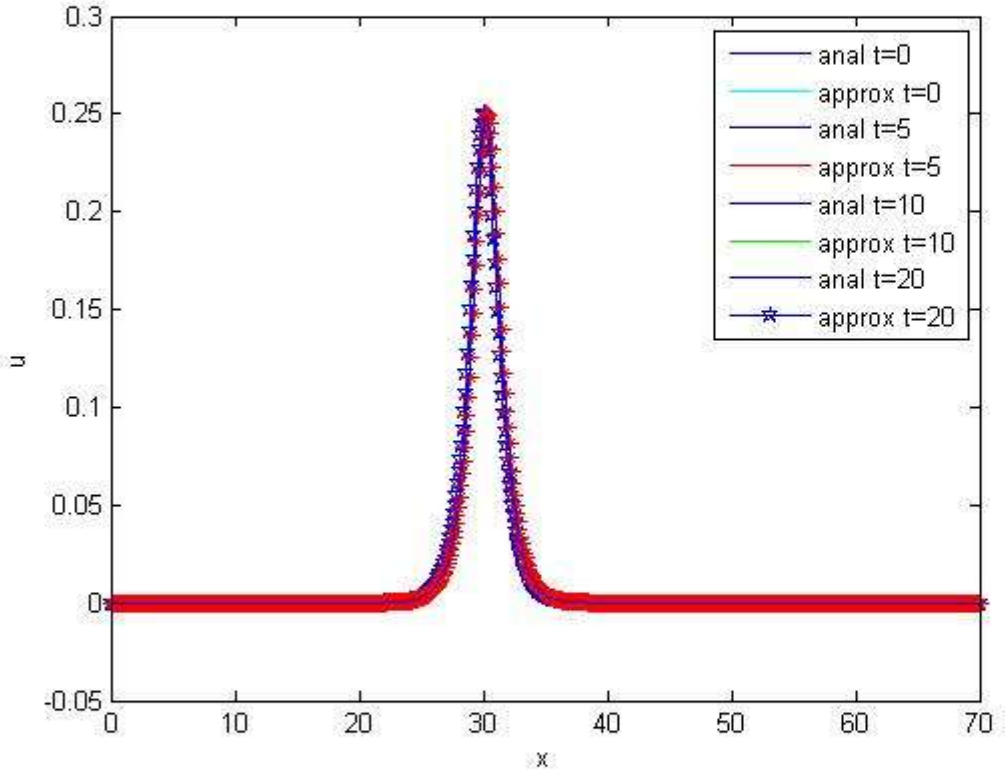

Figure 3. Approximate solution for Single solitary wave and exact solution at $0 \le t \le 20$ and A=0.25

5-Conclusion

Quartic trigonometric B-spline method are presented for solving MEW equation .Two test problems are used to show efficiency and accuracy performance of the method through comparing the results that got it and some other published numerical methods. This method is obtained to be unconditionally stable by Appling the von Neumann method.

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