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Solving Modified Regularized Long Wave Equation Using Collocation Method

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Abstract. In this paper, we suggest collocation method depending on Cubic trigonometric B-spline (CuTBS) approach based on finite difference scheme to solve the modified regularized long wave equation. The single solitary wave motion was studied using the proposed method; thus the accuracy and efficiency of the suggested method were computed from the L_2 , L_∞ norms. Also, the von-Neumann method was used to study the linear stability analysis. The obtained results through the tested two problems exhibited that, the method is an effective numerical scheme to solve Modified Regularized Long Wave equation (MRLW).

Key word: Cubic Trigonometric B-spline method, Finite Difference, Regularized Long Wave Equation ,Von-Neumann method.

1. Introduction

Nonlinear partial differential equations can be effectively employed to express an assortment of physical and applied mathematical concerns. A non-linear partial differential equation is defined as a partial differential equation that comes with nonlinear terms. The numerical solutions derived from nonlinear partial differential equations are particularly useful for deciphering solitary waves of pulses or wave packets [1]. A model of the MRLW eq. can be utilized to symbolize the nonlinear PDEs

$$z_t + z_x + \varepsilon z^2 z_x - \mu z_{xxt} = 0 \qquad (1)$$

with boundary conditions (BC)

$$z(a,t) = 0 \quad , z(b,t) = 0 \qquad (2)$$

and the initial condition(IC)

$$z(a,0) = f(\mathrm{x}) \quad a \le x \le b \qquad (3)$$

in which ε and μ represent positive parameters. Recognized as one of the foremost equations for nonlinear dispersive waves, the MRLW equation is applicable for a wide variety of issues. This includes phonon packets in nonlinear crystals, ion-acoustic waves in plasma and magneto hydrodynamic, ion-acoustic waves in plasma, pressure waves in liquid-gas bubble mixtures and down a tube, and transverse

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waves in shallow water. The lumped galerkin technique based on cubic B-spline was employed to realize a solution to the MRLW equation. Karakoc et al. employed a numerical approach derived from a Petrow-Galerkin procedure to scrutinize the movement of a solitary wave. Their investigation, which delivered precise results, involved the use of quadratic weight functions and cubic B-spline finite elements [2-4]. The arrival to a numerical solution of the MRLW equation can be realized by way of the extended cubic B-spline procedure [5]. When it comes to ascertaining the value of parameter k, the approximation derived through the extended cubic B-spline algorithm has proven to be more exact than that of the cubic B-spline. Several methods have been introduced for realizing a numerical solution to the RLW equation for a solitary wave movement. Dag et al. [6-10] employed the collection method based on quartic B-spline. Other methods include interpolation functions, the quintic B-spline Galerkin finite element method, cubic B-spline functions, the least squares quadratic B-spline finite element method, and the least squares finite element method based on cubic B-spline. The results obtained through these methods were found to be more precise than those previously recorded in related literature.

B-splines are frequently employed for solutions to linear or nonlinear partial differential equations in the domains of engineering and science. The trigonometric B-spline based function is an option to the famous polynomial B-spline based function [11]. While the structure of the former is based on trigonometric functions, the structure of the latter is based on polynomial functions [12]. This study focuses on the arrival at a solution to the MRLW equation through trigonometric B-spline collocation procedures. The cubic trigonometric B-spline (TCuBS) was employed to realize a numerical solution to the non-linear Benjamin-Bona-Mahony-Burger equation, the coupled viscous Burgers' equation [13-14], as well as the generalized nonlinear Klein-Gordon wave equation [15-16].

2. Numerical Solution of MRLW equation.

The standard finite difference formula is applied for estimating the time derivative.

$$\frac{\partial z^n}{\partial t} = \frac{z^{n+1} - z^n}{\Delta t} \tag{4}$$

The use of (4) transforms equation (1) into

$$\frac{z^{n+1} - z^n}{\Delta t} - \mu \frac{(z_{xx})^{n+1} - (z_{xx})^n}{\Delta t} + z_x^n + \varepsilon (z^2 z_x)^n = 0 \tag{5}$$

And the use of the θ weighted technique facilitates the expression of the space derivatives of the MRLW equation (1) as

$$\frac{z^{n+1} - z^n}{\Delta t} - \mu \frac{(z_{xx})^{n+1} - (z_{xx})^n}{\Delta t} + \theta (z_x^{n+1} + \varepsilon (z^2 z_x)^{n+1}) + (1-\theta)(z_x^n + \varepsilon (z^2 z_x)^n) = 0 \tag{6}$$

The use of the rule [20]

$$(z^2)^{n+1} z_x{}^{n+1} = (z^n)^2 z_x{}^{n+1} + 2 z^n z_x^n z^{n+1} - 2(z^n)^2 z_x^n$$

Allows for equation (6) to be generated as

$$\begin{aligned} &z^{n+1} - \mu z_{xx}^{n+1} + \Delta t \theta z_x^{n+1} + \Delta t \theta (z^n)^2 z_x{}^{n+1} + \Delta t \theta 2 z^n z_x^n z^{n+1} \\ &= z^n - \mu z_{xx}^n - \Delta t (1-\theta) z_x^n + \Delta t (1-\theta)(z^n)^2 z_x^n \end{aligned} \tag{7}$$

The system is regarded an explicit scheme if $\theta = 0$, a completely implicit scheme if $\theta = 1$ and a Crank-Nicolson scheme if $\theta = \frac{1}{2}$. As this study opted for the Crank-Nicolson procedure, equation (7) is converted into

$$\begin{aligned} &z^{n+1} - \mu z_{xx}^{n+1} + \frac{\Delta t}{2} z_x^{n+1} + \frac{\Delta t}{2} (z^n)^2 z_x^{\ n+1} + \Delta t z^n z_x z^{n+1} \\ &= z^n - \mu z_{xx}^n - \frac{\Delta t}{2} z_x^n + \frac{\Delta t}{2} (z^n)^2 z_x^n \end{aligned} \tag{8}$$

3.1. CuTBS for Solving MRLW Equation

The basis function of the CuTBS is expressed as:

$$TB_{4,\mathrm{j}}(x) = \frac{1}{z} \begin{cases} a^3(x_j), & x \in [x_j, x_{j+1}) \\ \mathrm{a}(x_j)\left(\mathrm{a}(x_j)\,\mathrm{b}(x_{j+2}) + b(x_{j+3})\,\mathrm{a}(x_{j+1})\right) + b(x_{j+4})\,\mathrm{a}^2(x_{j+1}), & x \in [x_{j+1}, x_{j+2}) \\ \mathrm{b}(x_{j+4})\left(\mathrm{a}(x_{j+1})\,\mathrm{b}(x_{j+3}) + b(x_{j+4})\,\mathrm{a}(x_{j+2})\right) + a(x_j)\,\mathrm{b}^2(x_{j+3}), & x \in [x_{j+2}, x_{j+3}) \\ b^3(x_{j+4}), & x \in [x_{j+3}, x_{j+4}] \end{cases} \tag{9}$$

The values of $TB_{4,\mathrm{j}}(\mathrm{x}), TB'_{4,\mathrm{j}}(\mathrm{x})$ and $TB''_{4,\mathrm{j}}(\mathrm{x})$ at the knots x_j were calculated through equation (9) and registered in (Table 1).

Table 1: $TB_{4,i}(x)$ values and their derivatives.

x	x_j	x_{j+1}	x_{j+2}	x_{j+3}	x_{j+4}
$TB_{4,\mathrm{j}}$	0	p_1	p_2	p_1	0
$TB'_{4,\mathrm{j}}$	0	p_3	0	p_4	0
$TB''_{4,\mathrm{j}}$	0	p_5	p_6	p_5	0

where

$$p_1 = \frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)}, p_2 = \frac{2}{1+2\cos(h)}, p_3 = -\frac{3}{4\sin\left(\frac{3h}{2}\right)}, p_4 = \frac{3}{4\sin\left(\frac{3h}{2}\right)},$$

$$p_5 = \frac{3\left(1+3\cos(h)\right)}{16\sin^2\left(\frac{h}{2}\right)\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}, p_6 = -\frac{3\cos^2\left(\frac{h}{2}\right)}{\sin^2\left(\frac{h}{2}\right)\left(2+4\cos(h)\right)}$$

In keeping with the proposed procedure, the approximation algorithm for solving equation (1) is

$$z_j(x,t) = \sum_{j=-3}^{N-1} C_j(t)\, TB_{4,j}(x) \tag{10}$$

in which $C_j(\mathrm{t})$ are unidentified time dependents that need to be ascertained, $TB_{4,j}(x)$ is a CuTBS. The assessment process at each x_{j} involves only three non-zero basis functions. This circumstance is attributed to the local support properties of basis function. As such, the approximate solution calls for the values of

$TB_{4,j}(x)$ and its derivatives at nodal points to be identified. Approximate functions (9) and (10) were utilized for tabulating these derivatives. Here, the values at the knots Z_j^n of and their derivatives until the second order are:

$$\begin{cases} (z)_j^n = p_1 C_{j-3}^n + p_2 C_{j-2}^n + p_1 C_{j-1}^n, \\ \left(\dfrac{\partial z}{\partial x}\right)_j^n = p_3 C_{j-3}^n + p_4 C_{j-1}^n \\ \left(\dfrac{\partial^2 z}{\partial x^2}\right)_j^n = p_5 C_{j-3}^n + p_6 C_{j-2}^n + p_5 C_{j-1}^n \end{cases} \tag{11}$$

Equation (8) with the nodal values of w and its derivatives uses (11) to realize the difference equation below with the variables C_j , $\mathrm{j} = -3,\ldots,\mathrm{N}-1$.

$$a_1 C_{j-3}^{n+1} + a_2 C_{j-2}^{n+1} + a_3 C_{j-1}^{n+1} = b_1 C_{j-3}^n + b_2 C_{j-2}^n + b_3 C_{j-1}^n \tag{12}$$

Here

$$\begin{aligned} a_1 &= (1+\varepsilon\Delta t z^n z_x^n) p_1 + \left(\frac{\Delta t}{2} + \frac{\Delta t\varepsilon}{2}(\mathrm{z}^n)^2\right) p_3 - \mu p_5 \quad , b_1 = \left(p_1 + \left(\frac{\Delta t\varepsilon}{2}(\mathrm{z}^n)^2 - \frac{\Delta t}{2}\right) p_3 - \mu p_5 \right. \\ a_2 &= (1+\varepsilon\Delta t z^n z_x^n) p_2 - \mu p_6 \quad , b_2 = p_2 - \mu p_6 \\ a_3 &= (1+\varepsilon\Delta t z^n z_x^n) p_1 + \left(\frac{\Delta t}{2} + \frac{\Delta t\varepsilon}{2}(\mathrm{z}^n)^2\right) p_4 - \mu p_5 \quad , b_3 = p_1 + \left(\frac{\Delta t\varepsilon}{2}(\mathrm{z}^n)^2 - \frac{\Delta t}{2}\right) p_4 - \mu p_5 \end{aligned}$$

the simplification of (12) gives rise to a system comprising the $(N+1)$ linear equation in the $(N+3)$ unknown $C^n = [\mathrm{C}_{j-3}^n,\ldots,\mathrm{C}_{N-1}^n]$ at the time level $t = t_{n+1}$, Equation (10), which is applied for acquiring the solution to the boundary conditions (2) is expressed as:

$$\begin{aligned} p_1 C_{-3}^{n+1} + p_2 C_{-2}^{n+1} + p_1 C_{-1}^{n+1} &= 0, \qquad j = 0, \\ p_1 C_{j-3}^{n+1} + p_2 C_{j-2}^{n+1} + p_1 C_{j-1}^{n+1} &= 0, \qquad j = N \end{aligned} \tag{13}$$

For equations (12) and (13), the system comprising $(N+3)\times(N+3)$ is expressed as:

$$M_{(N+3)\times(N+3)}\, C_{(N+3)\times 1}^{n+1} = Z_{(N+3)\times(N+3)}\, C_{(N+3)\times 1}^n$$

Initial state

Based on the initial conditions will calculate the initial vector C^0 . The values of C^0 were obtained using the initial conditions and boundary values of the derivatives for the initial condition as below

$$\begin{cases} (z_j^0)_x = g'(\mathrm{x}_j) & j = 0 \\ z_j^0 = g(\mathrm{x}_j) & j = 0,1,\ldots,N \\ (z_j^0)_x = g'(\mathrm{x}_j) & j = N \end{cases} \tag{14}$$

This generates the following $(N+3)\times(N+3)$ tridiagonal matrix system as follows.

$$A_{(N+3)\times(N+3)}K^{0}_{(N+3)\times 1}=d_{(N+3)\times 1}$$

where

$$K^{0}=[\mathrm{C}^{0}_{-3},\mathrm{C}^{0}_{-2},...,\mathrm{C}^{0}_{N-1}]^{T},\ d=[g^{'}(\mathrm{x}_0),g(\mathrm{x}_0),g(\mathrm{x}_1),...,g(\mathrm{x}_{N-1}),g(\mathrm{x}_N),g^{'}(\mathrm{x}_N)]^{T}$$

4. Stability Analysis

The Fourier method is introduced for an evaluation on the stability of the trigonometric cube B-spline method. For this purpose, Equation (1) is linearized by the assumption that quantity z^2 in the nonlinear term is unvarying β . The linearized configuration of the recommended procedure is expressed as

$$w_1C^{k+1}_{j-3}+w_2C^{k+1}_{j-2}+w_3C^{k+1}_{j-1}=u_1C^{k}_{j-3}+u_2C^{k}_{j-2}+u_3C^{k}_{j-1} \tag{15}$$

in which

$$\begin{aligned}
w_1&=2\,\mathrm{p}_1+\Delta t(1+\varepsilon\beta)\,\mathrm{p}_3-2\mu\,\mathrm{p}_5 &, u_1&=2\,\mathrm{p}_1-\Delta t(1+\varepsilon\beta)\,\mathrm{p}_3-2\mu\,\mathrm{p}_5\\
w_2&=2\,\mathrm{p}_2+0-2\mu\,\mathrm{p}_6 &, u_2&=2\,\mathrm{p}_2-0-2\mu\,\mathrm{p}_6\\
w_3&=2\,\mathrm{p}_1+\Delta t(1+\varepsilon\beta)\,\mathrm{p}_4-2\mu\,\mathrm{p}_5 &, u_3&=2\,\mathrm{p}_1-\Delta t(1+\varepsilon\beta)\,\mathrm{p}_4-2\mu\,\mathrm{p}_5
\end{aligned}$$

the replaced Fourier mode $C^{k}_{j}=\varsigma^{k}e^{(\mathrm{im}\eta h)}$, $i=\sqrt{-1}$ in Equation(15) gives rise to

$$\varsigma=\frac{X-iY}{X+iY} \tag{16}$$

Here,

$$\begin{aligned}
\mathrm{X}&=(4\,\mathrm{p}_1-4\mu\,\mathrm{p}_5)\cos(\eta h)+(2\,\mathrm{p}_2-2\mu\,\mathrm{p}_6)\\
Y&=(2(1+\varepsilon\beta)\,\mathrm{p}_4)\sin(\eta\,\mathrm{h})
\end{aligned}$$

Thus, the stability condition $|\varsigma|\leq 1$, the modulus of Eq. (16) yields $|\varsigma|=1$, and as result the scheme will be unconditionally stable.

5. Numerical Experiments

This segment focuses on the computation for the L_2 and L_∞ error norms by way of the formula below:

$$L_\infty=\max|z^{exact}_i-z^{num}_i|,$$

$$L_2=\sqrt{h(\sum_{i}^{n}|z^{exact}_i-z^{num}_i|^2}$$

Also computed are the conservation laws through the formula [19]. In this formula C_1 represents the mass, C_2 the momentum and C_3 the energy.

$$C_1 = \int_a^b z(x,t)\,\mathrm{d}\,x,$$

$$C_2 = \int_a^b z(x,t)^2\,\mathrm{d}\,x,$$

$$C_3 = \int_a^b [z(x,t)^2 + \frac{1}{3}z(x,t)^3]\,\mathrm{d}\,x,$$

Example 1

The MRLW equation has an precise solution $\mathrm{z}(x,t) = 3c\sec h^2\left(\frac{1}{2}\sqrt{\frac{c}{1+c}}(x-(1+c)t-x_0)\right)$, initial conditions $\mathrm{z}(x,0) = 3c\sec h^2\left(\frac{1}{2}\sqrt{\frac{c}{1+c}}(x-x_0)\right)$ and boundary conditions $\mathrm{z}(0,t)=0, \mathrm{z}(100,t)=0$. $v = 1+c$ represents wave velocity and $x_0 = 40$ [1]. The calculation for numerical answers to this problem was derived through the cubic trigonometric B-spline procedure. The norms errors as well as the conservation laws at distinct time levels with the parameters $\varepsilon = \mu = 1$, c=0.3.0.09, $\Delta t = 0.025$ and $\Delta x = 0.2$ were computed and registered Tables 2 and 3 respectively .The computations of L_∞ and L_2 errors at various times revealed that the margin of error escalated in tandem with the increase in time. This gave rise to minor disparities in the C_1 , C_2 and C_3 values which amounted to below 10^{-3} ,10^{-6} and 10^{-7} respectively . The space–time graph for the approximation and exact solutions at t=5 and t=10 and c=0.3 is displayed in Figure 1.The table and figure show that the of L_∞ and L_2 errors are increasing as time increases

Table 2: c=0.3 , $\Delta t = 0.025$, $\Delta x = 0.2$, $0 \le x \le 100$ cubic trigonometric b-spline

T	L_2	L_∞	C_3	C_2	C_1
5	0.007226	0.001888	0.008865	0.127301	2.107066
10	0.014416	0.003845	0.008863	0.127302	2.106733
15	0.021575	0.005848	0.008858	0.127303	2.106135
20	0.028686	0.007867	0.008851	0.127304	2.104627

Table 3: $h = 0.2, \Delta t = 0.1, x_0 = 40, c = 0.09, 0 \le x \le 100$ cubic trigonometric b-spline

T	L_2	L_∞	C_3	C_2	C_1
5	0.072146	0.024671	0.142305	0.688049	3.759237
10	0.142238	0.049605	0.141782	0.688360	3.759894
15	0.208868	0.072388	0.140899	0.688638	3.760450
20	0.271397	0.092359	0.139735	0.688889	3.760882

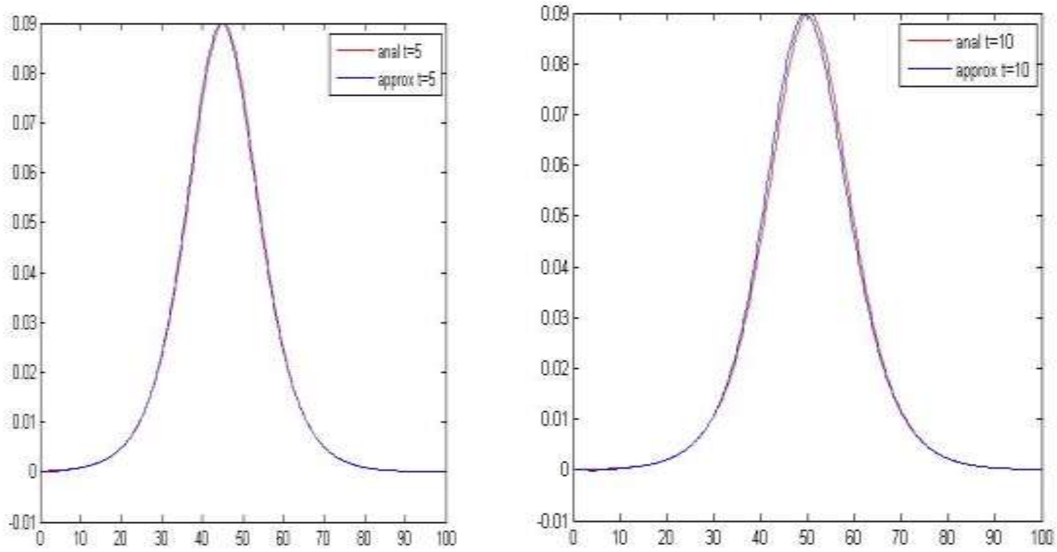

Figure 1: Approximation and exact solution by CuTBS at c=0.3 at different time levels

Example 2
We consider the motion of single wave equation (1) has exact solution in the form [21-23] $z(x,t) = b + 3c\sec h^2\left(k(x - x_0 - (b+c)t)\right)$ where b,c are constant and $k = \frac{1}{2}\sqrt{\frac{c}{\mu(\text{b}+\text{c})}}$ with initial condition $z(x,t) = b + 3c\sec h^2\left(\text{k}(x - x_0)\right)$.We choose $\varepsilon = \mu = 1, x_0 = 40, c = 0.3, b = 1$ and $0 \le x \le 80$ the interval divided element of equal length $\Delta x = 0.2$ at time T =1 and $\Delta t = 0.01$. In Tables 4 lists the calculations of L_2 error at different time as observed the error increased when the time increased. This led to trivial differences in the C_1, C_2 and C_3 values. These differences did not exceed 10^{-2}.

Table 4: $c = 0.3, \varepsilon = \mu = 1, x_0 = 40, h = 0.2, \Delta t = 0.01, 0 \le x \le 80, T = 1$

T	L_2	C_3	C_2	C_1
0.1	1.099427	115.094176	97.940786	85.500228
0.2	1.102576	115.093482	97.952850	85.496447
0.3	1.107873	115.087847	97.971810	85.489544
0.4	1.115286	115.077166	97.998092	85.479538
0.5	1.124764	115.061282	98.032325	85.466455
0.6	1.136247	115.039984	98.075382	85.450329
0.7	1.149666	115.012988	98.128423	85.431207
0.8	1.164942	114.979925	98.192978	85.409143
0.9	1.181990	114.940312	98.271044	85.384209
1.0	1.200722	114.893519	98.365247	85.356491

6-Conclusion

The primary objective of this study is to confirm the effectiveness of the CuTBS procedure. This is in the context of realizing a solution to the unique kind of PDE by way of a one-dimensional nonlinear adapted regularized long wave equation. Two cases in point were investigated. Tables 2, 3 and 4 portray the errors acquired through the application of the proposed procedure on the MRLW equation. It was discovered that the errors increased in tandem with the rise in time. The applicability of this procedure is further enhanced by its elevated level of stability. This was verified through the Von Neumann stability analysis.

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