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*Article*

# Solution of System of Viscous Burgers' Equation via Collocation Method

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**Abstract:** This paper presents a new approach to solve one dimensional system viscous Burgers' equation with boundary conditions Dirichlet type using collocation method based on cubic trigonometric B-spline. The usual finite difference scheme is applied to discretize the time derivative. Cubic Trigonometric B-spline basis functions are used as an interpolating function in the space dimension. Two test problems are presented to confirm the accuracy and efficiency of the new scheme and to show the performance of trigonometric basis functions. The numerical results are found to be in good agreement with known exact solutions and also with earlier studies.

**Keywords:** PDE, Burgers' equation, One dimensional coupled viscous Burgers' equation, cubic trigonometric B-spline basis functions, cubic trigonometric B-spline collocation method, stability

**Mathematics Subject Classification**: 35A, 35B, 35C, 65M, 34D

## 1. Introduction

Partial differential equations (PDEs) have numerous essential applications in various fields of science and engineering such as fluid mechanic, thermodynamics, heat transfer and physics. Most of these equations are nonlinear partial differential equations. It is difficult to handle nonlinear part of these equations. Although most of scientists applied numerical methods to find the solution of these equations, solving such equations analytically is of fundamental importance since the existent numerical methods which approximate the solution of PDE don't result in such an exact and analytical solution which is obtained by analytical methods.

In this paper, we are discussing the numerical solutions of one dimensional system of Burgers' equations, proposed by Esipov [1]. This system is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids under the effect of gravity [2]. The coupled Burgers' equation is given by

$$\begin{aligned} &u_t + \alpha u u_x + \beta (uv)_x - u_{xx} = 0 \qquad x \in [a,b],\ 0 \le t \le T \\ &v_t + \alpha v v_x + \eta (uv)_x - v_{xx} = 0 \qquad x \in [a,b],\ 0 \le t \le T \end{aligned} \tag{1}$$

with initial conditions

$$u(x,0) = u_0(x)\ , v(x,0) = v_0(x) \tag{2}$$

and boundary conditions

$$\begin{cases} u(a,t) = f_1(t) & ,\ u(b,t) = f_2(t) \\ v(a,t) = g_1(t) & ,\ v(b,t) = g_2(t) \end{cases} \tag{3}$$

where $\alpha, \beta$ and $\eta$ are constant, and subscripts $x$ and $t$ denote differentiation of distance and time respectively. The coupled Burgers equations belong to an important class of basic flow equations [3], Ersoy and Idris solved nonlinear coupled Burger Equation by Exponential Cubic B-spline Finite Element [4]. Kutluay and Ucar solved coupled Burgers' equation by the Galerkin quadratic B-spline finite element method [5]. Vineet et al. used the fully implicit Finite-difference to solve one dimensional Coupled Nonlinear Burgers' equations [6]. Kaya solved the coupled viscous Burgers equation by the decomposition method [7]. Mittal and Tripathi used the Collocation Method for solved Coupled Burgers' equations [8]. Mittal and Arora solved the coupled viscous Burgers' equations using cubic B-spline collocation scheme on the uniform mesh points based on Crank–Nicolson formulation for time integration and cubic B-spline functions for space integration [9]. Mittal et al., in [10] used Haar wavelet-based numerical investigation to solve coupled viscous Burgers' equation. Ghotbi et al. employed the homotopy perturbation method [11]. Rashid and Ismail used the Fourier pseudo-spectral method for finding the approximate solutions of the coupled Burgers' equation [12]. Abazari and Borhanifar obtained both numerical and analytical solutions of the Burgers' and coupled Burgers' equations using the differential transformation method [13]. Zhang et al. have extended the local discontinuous Galerkin method to solve Burgers' and coupled Burgers' equations [14]. Siraj-ul-Islam et al., in [15] solved coupled Burgers' equation numerically by a simple classical RBFs collocation (Kansa) method without using a mesh to discretize the problem domain. Khater [16] and Rashid [17] solved the Burgers-type equations using a Chebyshev spectral collocation method and Chebyshev–Legendre Pseudo-Spectral method respectively.

In our paper, a numerical collocation finite difference technique based on cubic trigonometric B-spline is presented for the solution of system of viscous Burgers' equation (1) with initial conditions in equation (2) and boundary conditions in equations (3). A usual finite difference scheme is applied to discretize the time derivative while cubic trigonometric B-spline is utilized as an interpolating function in the space dimension.

The outline of this paper is as follows: In section 2, cubic trigonometric B-spline scheme is explained. In section 3, described the method and applied to the system of viscous Burgers' equation. In section 4, stability of the method is discussed. In section 5, the accuracy and efficiency of suggested method are illustrated by examples. Conclusion is given in section 6.

## 2. Cubic Trigonometric B-Spline Functions

In this section, we define the cubic trigonometric basis function as follows [7, 8].

$$T_j^4(x)=\frac{1}{z}\begin{cases} q^3(x_j), & x\in[x_j,x_{j+1}) \\ q(x_j)\big(q(x_j)p(x_{j+2})+p(x_{j+3})q(x_{j+1})\big)+p(x_{j+4})p^2(x_{j+1}), & x\in[x_{j+1},x_{j+2}) \\ p(x_{j+4})\big(p(x_{j+1})q(x_{j+3})+q(x_{j+4})p(x_{j+2})\big)+p(x_j)q^2(x_{j+3}), & x\in[x_{j+2},x_{j+3}) \\ p^3(x_{j+4}), & x\in[x_{j+3},x_{j+4}] \end{cases} \tag{4}$$

where,

$$q(x_j)=\sin\left(\frac{x-x_j}{2}\right),\ p(x_j)=\sin\left(\frac{x_j-x}{2}\right),\ z=\sin\left(\frac{h}{2}\right)\sin\left(h\right)\sin\left(\frac{3h}{2}\right)$$

where $h=(b-a)/n$ and $T_j^4(x)$ is a piecewise cubic trigonometric function with some geometric properties like $C^2$ continuity, non-negativity and partition of unity [18,19]. The values of $T_j^4(x)$ and its derivatives at nodal points are required and these derivatives are tabulated in Table 1. Secondly, we discuss the cubic trigonometric B-spline collocation method (CuTBSM) for the solving numerically the system of viscous Burgers' system (1).

**Table 1:** Values $T_j^4(x)$ and its derivatives

| $x$ | $x_j$ | $x_{j+1}$ | $x_{j+2}$ | $x_{j+3}$ | $x_{j+4}$ |
|---|---|---|---|---|---|
| $T_j$ | 0 | $b_1$ | $b_2$ | $b_1$ | 0 |
| $T_j'$ | 0 | $b_3$ | 0 | $b_4$ | 0 |
| $T_j''$ | 0 | $b_5$ | $b_6$ | $b_5$ | 0 |

where

$$b_1 = \frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)},$$

$$b_2 = \frac{2}{1+2\cos(h)},$$

$$b_3 = -\frac{3}{4\sin\left(\frac{3h}{2}\right)},$$

$$b_4 = \frac{3}{4\sin\left(\frac{3h}{2}\right)},$$

$$b_5 = \frac{3(1+3\cos(h))}{16\sin^2\left(\frac{h}{2}\right)\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)},$$

$$b_6 = -\frac{3\cos^2\left(\frac{h}{2}\right)}{\sin^2\left(\frac{h}{2}\right)(2+4\cos(h))}.$$

## 3. Description of Numerical Method

This section discusses the cubic trigonometric B-spline collocation method for solving the system of viscous Burgers' equation (1). The domain $a \le x \le b$ is equally divided by knots $x_j$ into $N$ subintervals, $[x_j, x_{j+1}]$, $j = 0,1,2,...,N-1$ where $a = x_0 < x_1 < ... < x_N = b$ .

Our approach for system of Burgers' equation (1) using cubic trigonometric B-spline is to seek an approximate solution as:

$$\begin{aligned} U_j(x,t) &= \sum_{j=-3}^{N-1} C_j(t) T_j^4(x) \\ V_j(x,t) &= \sum_{j=-3}^{N-1} D_j(t) T_j^4(x) \end{aligned} \tag{5}$$

where $C_j(t)$ and $D_j(\mathrm{t})$ are to be determined for the approximated solutions $V_j(x,t), U_j(x,t)$ to the exact solutions $u_{exc}(x,t), v_{exc}(x,t)$ at the point $(x_j, t_i)$ .

The approximations $U_j^i, V_j^i$ at the point $(x_j, t_i)$ over subinterval $[x_j, x_{j+1}]$ can be defined as:

$$\begin{aligned} U_j^i &= \sum_{k=j-3}^{j-1} C_k^i T_k^4(x) \\ V_j^i &= \sum_{k=i-3}^{i-1} D_k^j T_k^4(x) \end{aligned} \tag{6}$$

where $j = 0,1,2,...,N$ . So as to get the approximations to the solution, the values of $T_j^4(x)$ and its derivatives at nodal points are required and these derivatives are tabulated using approximate functions (4) and (6), the values at the knots of $U_j^i, V_j^i$ and their derivatives up to second orders are:

$$\begin{cases} (U)_j^i = b_1C_{j-3}^i + b_2C_{j-2}^i + b_1C_{j-1}^i, \\ \left(\dfrac{\partial U}{\partial x}\right)_j^i = b_3C_{j-3}^i + b_4C_{j-1}^i \\ \left(\dfrac{\partial^2 U}{\partial x^2}\right)_j^i = b_5C_{j-3}^i + b_6C_{j-2}^i + b_5C_{j-1}^i \\ (V)_j^i = b_1D_{j-3}^i + b_2D_{j-2}^i + b_1D_{j-1}^i, \\ \left(\dfrac{\partial V}{\partial x}\right)_j^i = b_3D_{j-3}^i + b_4D_{j-1}^i, \\ \left(\dfrac{\partial^2 V}{\partial x^2}\right)_j^i = b_5C_{j-3}^i + b_6C_{j-2}^i + b_5C_{j-1}^i, \end{cases} \tag{7}$$

The approximations for the solutions for system of viscous Burgers' equation (1) at $t_{j+1}$ th time level can be given as:

$$\begin{aligned} (U_t)_j^i + \theta A_j^{i+1} + (1-\theta)A_j^{i+1} = 0 \\ (V_t)_j^i + \theta B_j^{i+1} + (1-\theta)B_j^{i+1} = 0 \end{aligned} \tag{8}$$

where $A_j^i = \alpha(UU_x)_j^i + \beta((UV)_x)_j^i - (U_{xx})_j^i$ and $B_j^i = \alpha(VV_x)_j^i + \eta((UV)_x)_j^i - (V_{xx})_j^i$ the subscripts $j$ and $j+1$ are successive time levels, $j = 0,1,2,3,...$ Discretizing the time derivatives in the usual finite difference way and rearranging the equations, we get:

$$\begin{aligned} U_j^{i+1} + \Delta t\theta A_j^{i+1} = U_j^i - \Delta t(1-\theta)A_j^i \\ V_j^{i+1} + \Delta\theta t B_j^{i+1} = V_j^j - \Delta t(1-\theta)\,\mathrm{B}_j^i \end{aligned} \tag{9}$$

where $\Delta t$ is the time step size. The nonlinear term $(UU_x)_i^{j+1}, (VV_x)_i^{j+1}$ and $((UV)_x)_i^{j+1}$ in equation (9) is linearized by using the following form [20]:

$$\begin{aligned} ((UV)_x)^{i+1} &= (VU_x)^{i+1} + (UV_x)^{i+1} \\ (UU_x)^{j+1} &= U^{i+1}U_x^i + U^iU_x^{i+1} - U^{i+1}U_x^i \\ (VV_x)^{i+1} &= V^{i+1}V_x^i + V^iV_x^{i+1} - V^{i+1}V_x^i \end{aligned} \tag{10}$$

Substituting equation (10) into (9) and for Crank-Nicolson scheme [21] we set $\theta = 0.5$, in this paper. The equation (10) yields the following

$$\begin{aligned}
&(1+\frac{\Delta t}{2}\alpha U_x^i + \frac{\Delta t}{2}\beta V_x^i)U^{i+1} + (\frac{\Delta t}{2}\alpha U^i + \frac{\Delta t}{2}\beta V^i)U_x^{i+1} + \frac{\Delta t}{2}\beta U^i V_x^{i+1} + \frac{\Delta t}{2}\beta V^{i+1}U_x^i - \frac{\Delta t}{2}U_{xx}^{i+1} \\
&\qquad = U^i + U_{xx}^i \\
&(1+\frac{\Delta t}{2}\alpha V_x^i + \frac{\Delta t}{2}\beta U_x^i)V^{i+1} + (\frac{\Delta t}{2}\alpha V^i + \frac{\Delta t}{2}\eta U^i)V_x^{i+1} + \frac{\Delta t}{2}\eta V^i U_x^{i+1} + \frac{\Delta t}{2}\eta U^{i+1}V_x^i - \frac{\Delta t}{2}V_{xx}^{i+1} \\
&\qquad = V^i + V_{xx}^i
\end{aligned} \tag{11}$$

After simplifying (11) and using (7), the system consists $2(N+1)$ linear equations known with $2(N+3)$ unknowns $C_{-3}, C_{-2}, ..., C_{N-1}, D_{-3}, D_{-2}, ..., D_{N-1}$ at the time level $t = t_{j+1}$.

The boundary conditions given in (3) are applied for four additional linear equations to get a unique solution of the resulting system.

$$\begin{aligned}
(U)_0^{i+1} &= f_1(t_{j+1}) \\
(U)_N^{i+1} &= f_2(t_{j+1}) \\
(V)_0^{i+1} &= g_1(t_{j+1}) \\
(V)_N^{i+1} &= g_2(t_{j+1})
\end{aligned} \tag{12}$$

Thus, the system becomes a matrix system of dimension $2(N+3)\times 2(N+3)$ which is a tri-diagonal system that can be solved by the Thomas Algorithm [18-19]. The system (11) can be written in the matrix form as follows:

$$MF^{i+1} = NF^i + b \tag{13}$$

where

$F^i = [C_{-3}^i, C_{-2}^i, ..., C_{N-1}^i, D_{-3}^i, D_{-2}^i, ..., D_{N-1}^i]^T$

$b = [\, f_1(t_{j+1}), 0, 0, 0, ..f_2(t_{j+1}), g_1(t_{j+1}), 0, 0, 0, ..g_2(t_{j+1})]^T, \;\; j = 0,12,..$

and $M$ is an $2(N+3)\times 2(N+3)$ dimensional matrix given by:

$$M=\begin{bmatrix}
b_1 & b_2 & b_1 & 0 & . & . & 0 & | & 0 & 0 & 0 & . & . & . & 0\\
a & b & c & 0 & . & . & 0 & & d & e & f & . & . & . & 0\\
0 & a & b & c & . & . & 0 & & 0 & 0 & 0 & . & . & . & 0\\
. & . & . & . & . & . & . & | & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & & . & . & . & . & . & . & .\\
0 & 0 & . & . & a & b & c & | & 0 & 0 & 0 & . & d & e & f\\
0 & . & . & . & b_1 & b_2 & b_1 & & 0 & 0 & 0 & . & . & . & 0\\
- & - & - & - & - & - & - & | & - & - & - & - & - & - & -\\
0 & 0 & . & . & . & . & 0 & & b_1 & b_2 & b_1 & 0 & . & . & 0\\
d_1 & e_1 & f_1 & . & . & . & 0 & & a_1 & k & c_1 & 0 & . & . & 0\\
. & . & . & . & . & . & 0 & | & 0 & 0 & . & . & . & . & .\\
. & . & . & . & . & , & . & & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & | & . & . & . & . & . & . & .\\
0 & 0 & . & . & d_1 & e_1 & f_1 & & 0 & 0 & . & . & a_1 & k & c_1\\
0 & 0 & . & . & . & 0 & 0 & | & 0 & 0 & . & . & b_1 & b_2 & b_1
\end{bmatrix}.$$

Also $N$ is an $2(N+3)\times 2(N+3)$ dimensional matrix given by:

$$N=\begin{bmatrix}
0 & 0 & 0 & 0 & . & . & 0 & | & 0 & 0 & 0 & . & . & . & 0\\
p_1 & q_1 & z_1 & 0 & . & . & 0 & & 0 & 0 & 0 & . & . & . & 0\\
0 & p_1 & q_1 & z_1 & . & . & 0 & & 0 & 0 & 0 & . & . & . & 0\\
. & . & . & . & . & . & . & | & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & & . & . & . & . & . & . & .\\
0 & 0 & . & . & p_1 & q_1 & z_1 & | & 0 & 0 & 0 & . & 0 & 0 & 0\\
0 & . & . & . & 0 & 0 & 0 & & 0 & 0 & 0 & . & . & . & 0\\
- & - & - & - & - & - & - & | & - & - & - & - & - & - & -\\
0 & 0 & . & . & . & . & 0 & & 0 & 0 & 0 & 0 & . & . & 0\\
0 & 0 & 0 & . & . & . & 0 & & p_2 & q_2 & z_2 & 0 & . & . & 0\\
. & . & . & . & . & . & 0 & | & 0 & 0 & . & . & . & . & .\\
. & . & . & . & . & , & . & & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & | & . & . & . & . & . & . & .\\
0 & 0 & . & . & 0 & 0 & 0 & & 0 & 0 & . & . & p_2 & q_2 & z_2\\
0 & 0 & . & . & . & 0 & 0 & | & 0 & 0 & . & . & 0 & 0 & 0
\end{bmatrix}.$$

where

$$a=(1+\frac{\Delta t}{2}\alpha u_x^i+\frac{\Delta t}{2}\beta v_x^i)b_1+(\frac{\Delta t}{2}\alpha u^i+\frac{\Delta t}{2}\beta v^i)b_3-\frac{\Delta t}{2}b_5,$$

$$b=(1+\frac{\Delta t}{2}\alpha u_x^i+\frac{\Delta t}{2}\beta v_x^i)b_2-\frac{\Delta t}{2}b_6\ ,$$

$$c=(1+\frac{\Delta t}{2}\alpha u_x^i+\frac{\Delta t}{2}\beta v_x^i)b_1+(\frac{\Delta t}{2}\alpha u^i+\frac{\Delta t}{2}\beta v^i)b_4-\frac{\Delta t}{2}b_5,$$

$$d=\frac{\Delta t}{2}\beta u_x^i b_1+\frac{\Delta t}{2}\beta u^i b_3,$$

$$e=\frac{\Delta t}{2}\beta u_x^i b_2,$$

$$f=\frac{\Delta t}{2}\beta u_x^i b_1+\frac{\Delta t}{2}\beta u^i b_4,$$

$$p_1=b_1+\frac{\Delta t}{2}b_5,$$

$$q_1=b_2+\frac{\Delta t}{2}b_6,$$

$$z_1=b_1+\frac{\Delta t}{2}b_5,$$

$$a_1=(1+\frac{\Delta t}{2}\alpha v_x^i+\frac{\Delta t}{2}\eta u_x^i)b_1+(\frac{\Delta t}{2}\alpha v^i+\frac{\Delta t}{2}\eta u^i)b_3-\frac{\Delta t}{2}b_5,$$

$$k=(1+\frac{\Delta t}{2}\alpha v_x^i+\frac{\Delta t}{2}\eta u_x^i)b_2-\frac{\Delta t}{2}b_6\ ,$$

$$c_1=(1+\frac{\Delta t}{2}\alpha v_x^i+\frac{\Delta t}{2}\eta u_x^i)b_1+(\frac{\Delta t}{2}\eta u^i+\frac{\Delta t}{2}\alpha v^i)b_4-\frac{\Delta t}{2}b_5,$$

$$d_1=\frac{\Delta t}{2}\eta v_x^i b_1+\frac{\Delta t}{2}\eta u^i b_3,$$

$$e_1=\frac{\Delta t}{2}\eta v_x^i b_2,$$

$$f_1=\frac{\Delta t}{2}\eta v_x^i b_1+\frac{\Delta t}{2}\eta u^i b_4,$$

$$p_2=b_1+\frac{\Delta t}{2}b_5,$$

$$q_2=b_2+\frac{\Delta t}{2}b_6,$$

$$z_2=b_1+\frac{\Delta t}{2}b_5.$$

### *3.1. Initial State*

The initial vectors $D^0$ and $\mathrm{C}^0$ are computed from the initial conditions, the approximate solution $U_j^{i+1}$ and $V_j^{i+1}$ at a particular time can be calculated repeatedly the recurrence relation. $D^0, \mathrm{C}^0$ can be provided from initial condition and boundary values of the derivatives as follows:

$$\begin{cases} (U_j^0)_x = u_0'(x_j) & j = 0 \\ U_j^0 = u_0(x_j) & j = 0,1,\ldots,N \\ (U_j^0)_x = u_0'(x_j) & j = N \end{cases} \tag{14}$$

Also to approximate another solution $V_j^{i+1}$

$$\begin{aligned} &(V_j^0)_x = v_0'(x_j) && j = 0 \\ &V_j^0 = v_0(x_j) && j = 0,1,\ldots,N \\ &(V_j^0)_x = v_0'(x_j) && j = N \end{aligned} \tag{15}$$

Thus, equations (14) and (15) provided a $2(N+3)\times 2(N+3)$ matrix system, of the form:

$$AF^0 = d$$

Where

$$A = \left[\begin{array}{ccccccc|ccccccc}
b_3 & 0 & b_4 & 0 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0 \\
b_1 & b_2 & b_1 & 0 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0 \\
0 & b_1 & b_2 & b_1 & . & . & 0 & 0 & 0 & 0 & . & . & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & . & . & b_1 & b_2 & b_1 & 0 & 0 & 0 & . & . & . & 0 \\
0 & . & . & . & b_3 & 0 & b_4 & 0 & 0 & 0 & . & . & . & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & 0 & . & . & . & . & 0 & b_3 & 0 & b_4 & 0 & . & . & 0 \\
0 & 0 & . & . & . & . & 0 & b_1 & b_2 & b_1 & 0 & . & . & 0 \\
. & . & . & . & . & . & 0 & 0 & 0 & . & . & . & . & . \\
. & . & . & . & . & , & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & . & . & . & 0 & 0 & 0 & 0 & . & . & b_1 & b_2 & b_1 \\
0 & 0 & . & . & . & 0 & 0 & 0 & 0 & . & . & b_3 & 0 & b_4
\end{array}\right].$$

$$F^0 = [C_{-3}^0, C_{-2}^0, \ldots, C_{N-1}^0, D_{-3}^0, D_{-2}^0, \ldots, D_{N-1}^0]^T$$

$$d=[u_0'(x_0),u_0(x_0),u_0(x_1),...u_0(x_N),u_0'(x_N),v_0'(x_0),v_0(x_0),v_0(x_1),...v_0(x_N),v_0'(x_N)]^T$$

## 4. Stability Analysis

A solution of numerical method is said to be unstable, if errors introduced at some stage in the calculations (for example, from erroneous boundary conditions or local truncation or round-off errors) are propagated without bound throughout subsequent calculations. Thus a method is stable if the difference between the theoretical and numerical solutions remains bounded at a given t, as time and space steps tend to zero or time step remains fixed at every level and $t \to \infty$.

Following, suppose the errors are given by:

$E_{um}^{2n+1} = U_{exact}(x_m) - U_{approx.}(x_m)$

And $E_{vm}^{2n+1} = V_{exact}(x_m) - V_{approx.}(x_m)$ (16)

$U_{exact}(x_m)$ and $V_{exact}(x_m)$ is the exact solution and $U_{approx.}(x_m)$ and $V_{approx.}$ is the solution with suggested method of the scheme so that $U_{approx.}(x_m)$ and $V_{approx.}$ contains approximate errors.

We suggest the following condition:

$$\lim_{n\to\infty} \| E_m^{2n+1} \| = 0$$

### 4.1. Conditioning

A small change to the differential equation or initial or boundary condition, results in a small change to the solution must be considered, a problem has this property is said to be well-conditioned. Otherwise, the problem is said to be ill-conditioned.

Consider the problem:

$u_{xx} = f(x, t, u_x, v_x, u_t, v_t)$

$v_{xx} = g(x, t, u_x, v_x, u_t, v_t)$ (17)

For a well-posed problem we now make the following assumptions:

1) Problem (17) has an approximate solution with this solution and $\rho > 0$, associate the spheres:

$S\rho(U_{approx.}(x)) =\{ U \in IR^n: | U_{approx.}(x) - U_{exact}(x) | \le \rho \}$

$S\rho(V_{approx.}(x)) = \{ V \in IR^n: | V_{approx.}(x) - V_{exact}(x) | \le \rho \}$

2) $f(x, t, u_x, v_x, u_t, v_t)$ is continuously differentiable with respect to u, v and $\frac{\partial f(x,t,u_x,v_x,u_t,v_t)}{\partial u}$, $\frac{\partial g(x,t,u_x,v_x,u_t,v_t)}{\partial v}$, are continuous.

This property is important due to the error associated with approximate solutions to the problem. Depending on the suggested method, approximate solution may exactly satisfy the perturbed:

$u_{xx\ approx.} = f(x, t, u_x, v_x, u_t, v_t) + r(x);$

$v_{xx\ approx.} = g(x, t, u_x, v_x, u_t, v_t) + s(x);$ (18)

where r, s: R → $R^m$.

If $u_{\text{xx approx.}}$ and $v_{\text{xx approx.}}$ is a reasonably good approximate solution to (1), then $\|r(x)\|$ and $\|s(x)\|$ is small. However, this may not imply that the approximate is close to the exact solution. A measure of conditioning that relate $\|r(x)\|$ and $\|s(x)\|$ to the error in the approximate solution can be determined.

### *4.2. Error / Defect Weights*

Every known BVP software package reports an estimate of either the relative error or the maximum relative defect. The weights used to scale either the error or the maximum defect differs among BVP software. Therefore, the BVP component of pythODE allows users to select the weights they wish to use. The default weights depend on whether an estimate of the error or maximum defect is being used. If the error is being estimated, then the BVP component of pythODE uses. In this paper we modify this package to consist our problem with named "pythCuTBSM", which defined as:

$$\frac{\left\|u_{\text{xx approx.}} - f(x,t,u_x,v_x,u_t,v_t)\right\|_{\infty}}{1 + \|f(x,t,u_x,v_x,u_t,v_t)\|_{\infty}}$$

$$\frac{\left\|v_{\text{xx approx.}} - g(x,t,u_x,v_x,u_t,v_t)\right\|_{\infty}}{1 + \|g(x,t,u_x,v_x,u_t,v_t)\|_{\infty}}$$

## 5. Numerical Illustrations

To illustrate the accuracy and efficiency of suggested method, two examples are solved in this section with $L_\infty$ and relative $L_2$ error norms are calculated by:

$$L_\infty = \max_i \left|U_{exci} - U_i\right|, \quad L_2 = \frac{\sqrt{\sum_i^N \left|U_{exci} - U_i\right|^2}}{\sqrt{\sum_i^N \left|U_{exci}\right|^2}}$$

The numerical order of convergence $p$ for numerical solution $U(x,t)$ and $V(x,t)$ is obtained by using the formula [9, 19]

$$p = \frac{Log\left(L_\infty(N)/L_\infty(2N)\right)}{Log\left(2N/N\right)}$$

where $L_\infty(N)$ and $L_\infty(2N)$ are the errors at number of partitions $n$ and $2n$ respectively. We compare the numerical solutions obtained by cubic trigonometric B-spline collocation method for system of viscous Burgers' equation (1) with known exact solutions and those numerical methods which were exiting in literature. Numerical results are computed by cubic trigonometric B-spline collocation method for system of viscous Burgers' equation (1) at different time levels which are tabulated and depicted in different Tables and Figures respectively. The feasibility of the method is shown by test problems and the approximated solutions are found to be in good agreement with the exact solutions. The proposed method is superior to Mittal and Arora [9], Rashid et al. [12], Khater et al. [16] and Rashid et al. [17].

**Problem 1**

Consider the one dimensional system of viscous Burgers' equation (1) with $\alpha = \beta = 1.0$ and $\eta = -2$ which leads equation (1) – (2) as [9, 12, 17]:

$$\begin{cases} u_t - u_{xx} - 2uu_x + (uv)_x = 0 \\ v_t - v_{xx} - 2vv_x + (uv)_x = 0 \end{cases}$$

with the initial conditions given by

$$u_0(x) = v_0(x) = \sin\left(x\right), \; -\pi \le x \le \pi$$

and boundary conditions as follows:

$$f_1(t) = f_2(t) = 0, \; g_1(t) = g_2(t) = 0, \quad 0 < t \le T$$

The known solutions of this problem is $U_{exc}(x,t) = V_{exc}(x,t) = e^{-t}\sin\left(x\right)$. The proposed method is applied to calculate the numerical solutions for system of viscous Burgers equation (1)-(3) by taking domain $-\pi \le x \le \pi$ with $\Delta t = 0.001$. The absolute errors at different time levels and different number of partitions are reported in Table 2. The ratio in absolute errors $L_\infty$ and order of convergence of the proposed method at different time levels and different number of partitions which are tabulated in Table 3 and it shows that the method has an approximately two order of convergence. Figure 1 depicts the graphs of comparison between exact and numerical solutions at different time levels with $N = 200, \Delta t = 0.001$. Figure 2 shows the space-time graph of exact and approximate solutions at $T = 1.0$ with $h = 1/200, \Delta t = 0.001$. Due to symmetric initial and boundary conditions, the numerical results are similar for $V(x,t)$. The numerical solution of suggested method gives more accurate than Mittal and Arora [9] and Rashid et al [12, 17].

**Table 2:** Relative errors and maximum errors of Problem 1 for $U(x,t)$ with $\Delta t = 0.001$

| | CuTBSM (Proposed) | | | | CuBSM[9] | | | |
|---|---|---|---|---|---|---|---|---|
| time | $L_2$ (N=200) | $L_\infty$ | $L_2$ (N=400) | $L_\infty$ | $L_2$ (N=200) | $L_\infty$ | $L_2$ (N=400) | $L_\infty$ |
| 0.1 | 1.23E-05 | 6.96E-06 | 1.91E-06 | 1.73E-06 | 8.21E-06 | 7.45E-06 | 2.05E-06 | 1.86E-06 |
| 0.5 | 3.85E-05 | 2.33E-05 | 9.59E-06 | 5.82E-06 | 2.49E-05 | 4.10E-05 | 1.02E-05 | 6.22E-06 |
| 1.0 | 7.70E-05 | 2.83E-05 | 1.91E-05 | 7.06E-06 | 3.00E-05 | 8.21E-05 | 2.04E-05 | 7.56E-06 |

| | Rashid [12] | | | | Rashid [17] | | | |
|---|---|---|---|---|---|---|---|---|
| time | $L_2$ (N=200) | $L_\infty$ | $L_2$ (N=400) | $L_\infty$ | $L_2$ (N=200) | $L_\infty$ | $L_2$ (N=200) | $L_\infty$ |
| 0.1 | No data | No data | No data | No data | No data | No data | No data | No data |
| 0.5 | | | | | | | | |
| 1.0 | 2.88E-05 | 1.16E-05 | | | 2.77E-05 | 1.05E-05 | | |

**Table 3:** $L_\infty$ errors, ratio and order of convergence of Problem 1 for $U(x,t)$ at different time

| Method | $N$ | $L_\infty(t=0.1)$ | Ratio | Order of Conv. | $L_\infty(0.5)$ | Ratio | Order of Conv. |
|---|---|---|---|---|---|---|---|
| CuTBSM | 32 | 2.7336E-04 | -------- | ------- | 9.1674E-04 | -------- | ------- |
| | 64 | 6.8117E-05 | 4.0090 | 2.0034 | 2.2854E-04 | 4.0113 | 2.0440 |
| | 128 | 1.7029E-05 | 4.0036 | 2.0013 | 5.7076E-05 | 4.0040 | 2.0015 |
| | 256 | 4.2510E-06 | 4.0058 | 2.0021 | 1.4247E-05 | 4.0061 | 2.0022 |
| | 512 | 1.0570E-06 | 4.0215 | 2.0077 | 3.5428E-06 | 4.0216 | 2.0077 |
| CuBSM[9] | 32 | 2.9104E-04 | -------- | ------- | 9.7478E-04 | -------- | ------- |
| | 64 | 7.2704E-05 | 4.0030 | 2.001 | 2.4361E-04 | 4.0014 | 2.005 |
| | 128 | 1.8178E-05 | 3.9996 | 1.999 | 6.0896E-05 | 4.0004 | 2.001 |
| | 256 | 4.5497E-05 | 3.9953 | 1.998 | 1.5223E-05 | 4.0003 | 2.001 |
| | 512 | 1.1430E-06 | 3.9806 | 1.993 | 3.8052E-05 | 4.0006 | 2.002 |

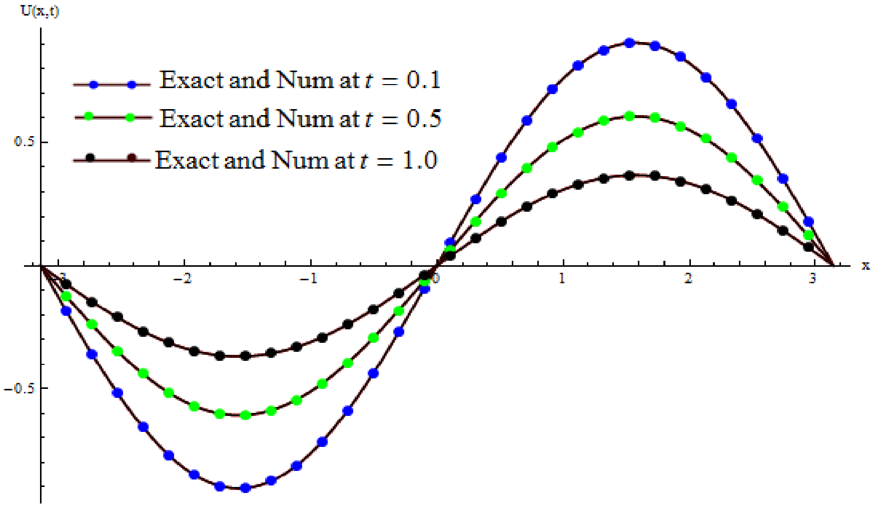

**Figure 1:** A comparison between numerical and exact solutions of $U(x,t)$ for Problem 1

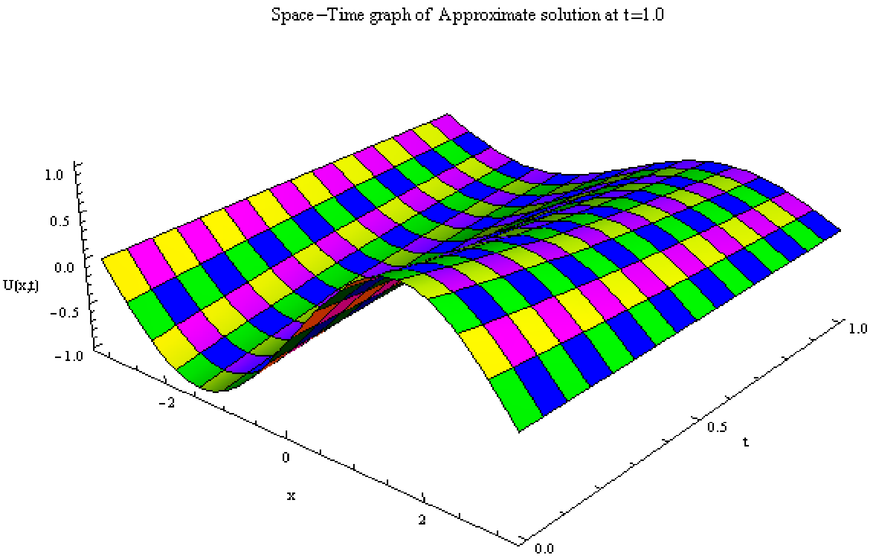

**Figure 2:** Space-time graph of approximate solution $U(x,t)$ for Problem 1 at $t = 1.0$ and $\Delta t = 0.001$

**Problem 2**

Consider the one dimensional system of viscous Burgers' equation (1) for different values of $\alpha, \beta$ and $\eta = 2$ which leads equation (1) – (2) as [9, 12, 16-17]:

$$\begin{cases} u_t - u_{xx} + 2uu_x + \alpha(uv)_x = 0 \\ v_t - v_{xx} + 2vv_x + \beta(uv)_x = 0 \end{cases}$$

with the initial conditions given by

$$\begin{cases} u_0(x) = a_0\left(1 - \tanh\left(\lambda x\right)\right) \\ v_0(x) = a_0\left(\left(\dfrac{2\beta - 1}{2\alpha - 1}\right) - \tanh\left(\lambda x\right)\right), \end{cases} \quad -10 \le x \le 10$$

and boundary conditions as follows:

$$\begin{cases} f_1(t) = a_0\left(1 - \tanh\left(\lambda(-10 - 2\lambda t)\right)\right) \\ f_2(t) = a_0\left(1 - \tanh\left(\lambda(10 - 2\lambda t)\right)\right) \end{cases} \quad 0 < t \le T$$

and

$$\begin{cases} g_1(t) = a_0\left(\left(\dfrac{2\beta - 1}{2\alpha - 1}\right) - \tanh\left(\lambda(-10 - 2\lambda t)\right)\right) \\ g_2(t) = a_0\left(\left(\dfrac{2\beta - 1}{2\alpha - 1}\right) - \tanh\left(\lambda(10 - 2\lambda t)\right)\right) \end{cases} \quad 0 < t \le T$$

where $a_0 = 0.05$ and $\lambda = \dfrac{a_0}{2}\left(\dfrac{4\alpha\beta - 1}{2\alpha - 1}\right)$. The known solutions of this problem are $U_{exc}(x,t) = a_0\left(1 - \tanh\left(\lambda(x - 2\lambda t)\right)\right)$, $V_{exc}(x,t) = a_0\left(\left(\dfrac{2\beta - 1}{2\alpha - 1}\right) - \tanh\left(\lambda(x - 2\lambda t)\right)\right)$. The proposed method is used to calculate the numerical solutions of the system equation (1)-(3) over the domain $-10 \le x \le 10$ with $\Delta t = 0.01,\ N = 100$. The absolute errors at different time levels and different values of $\alpha,\ \beta$ for $U(x,t)$ and $V(x,t)$ are tabulated in Table 4 and Table 5 respectively. Figures 3 and 4 show the space-time graph of exact and approximate solutions $U(x,t)$ and $V(x,t)$ at $T = 1.0$ with $h = 0.01, \Delta t = 0.01$. The numerical results of this problem are more accurate than Mittal and Arora [9], Rashid et al [12, 17] and Khater [16].

**Table 4:** Relative errors and maximum errors of Problem 2 for $U(x,t)$ with $\Delta t = 0.01$

| t | $\alpha$ | $\beta$ | CuTBSM<br>$L_2$ | $L_\infty$ | CuBSM[9]<br>$L_2$ | $L_\infty$ | Khater[16]<br>$L_2$ | $L_\infty$ | Rashid[12]<br>$L_2$ | $L_\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.10 | 0.30 | 3.21E-05 | 1.21E-05 | 6.74E-04 | 4.17E-05 | 1.44E-03 | 4.38E-05 | 3.25E-05 | 9.62E-04 |
| | 0.30 | 0.03 | 1.98E-05 | 6.67E-05 | 7.33E-04 | 4.59E-05 | 6.68E-03 | 4.58E-05 | 2.73E-05 | 4.31E-04 |
| 1.0 | 0.10 | 0.30 | 6.26E-05 | 2.33E-05 | 1.33E-03 | 8.26E-05 | 1.27E-03 | 8.66E-05 | 2.40E-05 | 1.15E-03 |
| | 0.30 | 0.03 | 3.89E-04 | 1.32E-05 | 1.45E-03 | 9.18E-05 | 1.30E-03 | 9.16E-05 | 2.83E-05 | 1.27E-03 |
| | | | | | | | | | | Rashid[21] |
| 0.5 | 0.10 | 0.10 | | | | | | | 1.27E-05 | 3.26E-05 |
| | 0.30 | 0.30 | | | | | | | 1.16E-05 | 3.23E-05 |
| 1.0 | 0.10 | 0.10 | | | | | | | 1.14E-05 | 2.27E-05 |
| | 0.30 | 0.30 | | | | | | | 1.13E-05 | 2.17E-05 |

**Table 5:** Relative errors and maximum errors of Problem 2 for $V(x,t)$ with $\Delta t = 0.01$

| t | $\alpha$ | $\beta$ | CuTBSM<br>$L_2$ | $L_\infty$ | CuBSM[9]<br>$L_2$ | $L_\infty$ | Khater[16]<br>$L_2$ | $L_\infty$ | Rashid[12]<br>$L_2$ | $L_\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.10 | 0.30 | 5.75E-05 | 1.66E-05 | 9.05E-04 | 1.48E-04 | 5.42E-04 | 4.99E-05 | 2.74E-05 | 3.33E-04 |
| | 0.30 | 0.03 | 2.44E-05 | 9.65E-05 | 1.59E-03 | 5.73E-04 | 1.20E-03 | 1.81E-04 | 2.45E-04 | 1.15E-03 |
| 1.0 | 0.10 | 0.30 | 1.13E-05 | 3.28E-05 | 1.25E-03 | 4.77E-05 | 1.29E-03 | 9.92E-05 | 3.74E-05 | 1.16E-03 |
| | 0.30 | 0.03 | 4.73E-04 | 1.86E-05 | 2.25E-03 | 3.62E-04 | 2.35E-03 | 3.62E-04 | 4.52E-04 | 1.64E-03 |
| | | | | | | | | | Rashid[17] | |
| 0.5 | 0.10 | 0.30 | | | | | | | 1.12E-05 | 1.27E-05 |
| | 0.30 | 0.03 | | | | | | | 1.17E-05 | 1.26E-05 |
| 1.0 | 0.10 | 0.30 | | | | | | | 1.12E-05 | 1.21E-05 |
| | 0.30 | 0.03 | | | | | | | 1.13E-05 | 1.22E-05 |

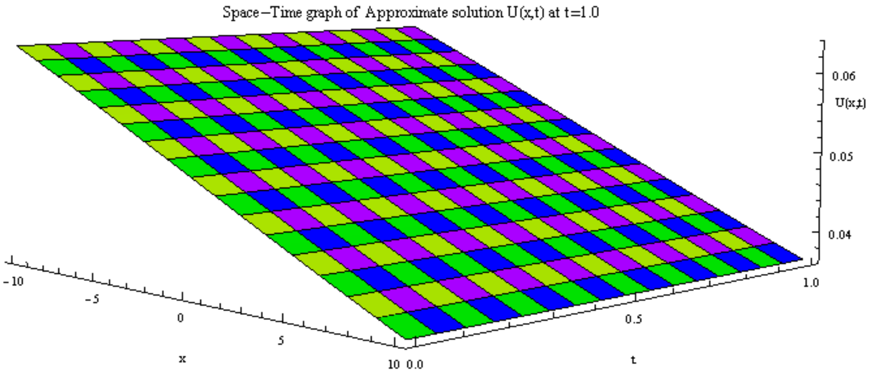

**Figure 3:** Space-time graph of approximate solution $U(x,t)$ for Problem 2 at $t=1.0$ and $\Delta t=0.01$

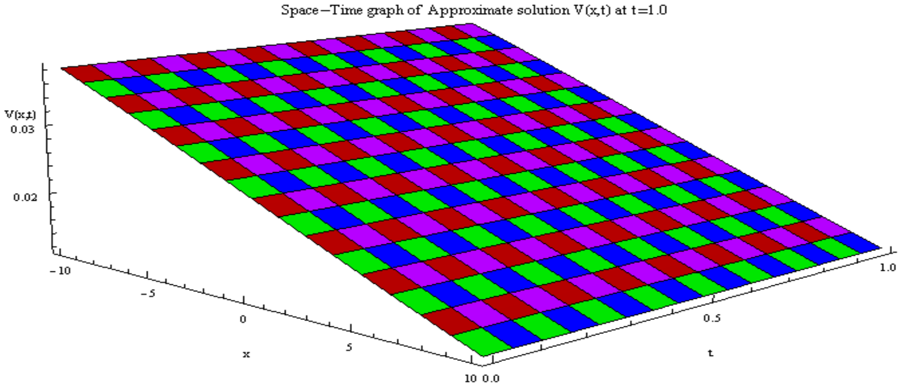

**Figure 4:** Space-time graph of approximate solution $V(x,t)$ for Problem 2 at $t=1.0$ and $\Delta t=0.01$

## 6. Conclusions

This paper suggest the cubic trigonometric B-spline collocation method to find the numerical solution of the one dimensional system of viscous Burgers equation with initial - Dirichlet boundary conditions. A usual finite difference approach is used to discretize the time derivatives. The cubic trigonometric B-spline is used for interpolating the solutions at each time. The numerical results shown in Tables (2 – 5) and Figures (1- 4) indicate the reliability of results obtained. The obtained solution for various time levels has been compared with the exact solution and existing methods by calculating $L_\infty$ and $L_2$. It is found that suggested method has provided more accurate results as compared to Mittal and Arora [9], Rashid et al [12, 17] and Khater [16].

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