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Some Families of Chromatically Unique 5-Partite Graphs

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Abstract: Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H | H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we first characterize certain complete 5-partite graphs G with $5n + i$ vertices for $i = 1, 2, 3$ according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs G with certain star or matching deleted are obtained.

Key Words: Chromatic polynomial, chromatically closed, chromatic uniqueness.

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§1. Introduction

All graphs considered here are simple and finite. For a graph G, let $P(G, \lambda)$ be the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. Many families of χ-unique graphs are known (see [3,4]).

For a graph G, let $V(G)$, $E(G)$, $t(G)$ and $\chi(G)$ be the vertex set, edge set, number of triangles and chromatic number of G, respectively. Let O_n be an edgeless graph with n vertices.

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Let $Q(G)$ and $K(G)$ be the number of induced subgraph C_4 and complete subgraph K_4 in G. Let S be a set of s edges in G. By $G - S$ (or $G - s$) we denote the graph obtained from G by deleting all edges in S, and $\langle S\rangle$ the graph induced by S. For $t \geqslant 2$ and $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, let $K(n_1, n_2, \cdots, n_t)$ be a complete $t-$partite graph with partition sets V_i such that $|V_i| = n_i$ for $i = 1, 2, \cdots, t$. In [2,5-7,9-11,13-15], the authors proved that certain families of complete t-partite graphs ($t = 2, 3, 4, 5$) with a matching or a star deleted are χ-unique. In particular, Zhao et al. [13,14] investigated the chromaticity of complete 5-partite graphs G of $5n$ and $5n+4$ vertices with certain star or matching deleted. As a continuation, in this paper, we characterize certain complete 5-partite graphs G with $5n+i$ vertices for $i = 1, 2, 3$ according to the number of 6-independent partitions of G. Using these results, we investigate the chromaticity of G with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

§2. Some Lemmas and Notations

Let $\mathcal{K}^{-s}(n_1, n_2, \cdots, n_t)$ be the family $\{K(n_1, n_2, \cdots, n_t) - S|\ S \subset E(K(n_1, n_2, \cdots, n_t))$ and $|S| = s\}$. For $n_1 \geqslant s+1$, we denote by $K_{i,j}^{-K_{1,s}}(n_1, n_2, \cdots, n_t)$ (respectively, $K_{i,j}^{-sK_2}(n_1, n_2, \cdots, n_t)$) the graph in $K^{-s}(n_1, n_2, \cdots, n_t)$ where the s edges in S induced a $K_{1,s}$ with center in V_i and all the end vertices in V_j (respectively, a matching with end vertices in V_i and V_j).

For a graph G and a positive integer r, a partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of G if every A_i is independent of G. Let $\alpha(G, r)$ denote the number of r-independent partitions of G. Then, we have $P(G, \lambda) = \sum_{r=1}^{p} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-r+1)$ (see [8]). Therefore, $\alpha(G, r) = \alpha(H, r)$ for each $r = 1, 2, \cdots$, if $G \sim H$.

For a graph G with p vertices, the polynomial $\sigma(G, x) = \sum_{r=1}^{p} \alpha(G, r)x^r$ is called the σ-polynomial of G (see [1]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs G and H.

For disjoint graphs G and H, $G + H$ denotes the disjoint union of G and H. The join of G and H denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer to [12].

Lemma 2.1 (Koh and Teo [3]) *Let G and H be two graphs with $H \sim G$, then $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G, r) = \alpha(H, r)$ for $r = 1, 2, 3, 4, \cdots$, and $2K(G) - Q(G) = 2K(H) - Q(H)$. Note that $\chi(G) = 3$ then $G \sim H$ implies that $Q(G) = Q(H)$.*

Lemma 2.2(Brenti [1]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K(n_1, n_2, \cdots, n_t), x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x)$$

Lemma 2.3(Zhao [13]) *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ and S be a set of some s edges of G. If $H \sim G - S$, then there is a complete graph $F = K(p_1, p_2, p_3, p_4, p_5)$ and a subset S' of $E(F)$ of some s' of F such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s$.*

Let $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant x_5$ be positive integers, $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}\} = \{x_1, x_2, x_3, x_4, x_5\}$. If there exists two elements x_{i_1} and x_{i_2} in $\{x_1, x_2, x_3, x_4, x_5\}$ such that $x_{i_2} - x_{i_1} \geqslant 2$, $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5})$ is called an *improvement* of $H = K(x_1, x_2, x_3, x_4, x_5)$.

Lemma 2.4 (Zhao et al. [13]) *Suppose $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant x_5$ and $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5}\}$ is an improvement of $H = K(x_1, x_2, x_3, x_4, x_5)$, then*

$$\alpha(H, 6) - \alpha(H', 6) = 2^{x_{i_2}-2} - 2^{x_{i_1}-1} \geqslant 2^{x_{i_1}-1}.$$

Let $G = K(n_1, n_2, n_3, n_4, n_5)$. For a graph $H = G - S$, where S is a set of some s edges of G, define $\alpha'(H) = \alpha(H, 6) - \alpha(G, 6)$. Clearly, $\alpha'(H) \geqslant 0$.

Lemma 2.5 (Zhao et al. [13]) *Let $G = K(n_1, n_2, n_3, n_4, n_5)$. Suppose that min $\{n_i | i = 1, 2, 3, 4, 5\} \geqslant s + 1 \geqslant 1$ and $H = G - S$, where S is a set of some s edges of G, then*

$$s \leqslant \alpha'(H) = \alpha(H, 6) - \alpha(G, 6) \leqslant 2^s - 1,$$

and $\alpha'(H) = s$ iff the set of end-vertices of any $r \geqslant 2$ edges in S is not independent in H, and $\alpha'(H) = 2^s - 1$ iff S induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same A_i.

Lemma 2.6(Dong et al. [2]) *Let n_1, n_2 and s be positive integers with $3 \leqslant n_1 \leqslant n_2$, then*

(1) $K_{1,2}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_2 - 2$,

(2) $K_{2,1}^{-K_{1,s}}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_1 - 2$, and

(3) $K^{-sK_2}(n_1, n_2)$ is χ-unique for $1 \leqslant s \leqslant n_1 - 1$.

For a graph $G \in K^{-s}(n_1, n_2, \cdots, n_t)$, we say an induced C_4 subgraph of G is of Type 1 (respectively Type 2 and Type 3) if the vertices of the induced C_4 are in exactly two (respectively three and four) partite sets of $V(G)$. An example of induced C_4 of Types 1, 2 and 3 are shown in Figure 1.

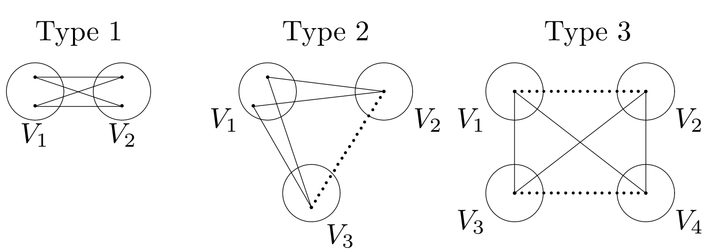

FIGURE 1. Three types of induced C_4

Suppose G is a graph in $K^{-s}(n_1, n_2, \cdots, n_t)$. Let S_{ij} $(1 \leqslant i \leqslant t, 1 \leqslant j \leqslant t)$ be a subset of S such that each edge in S_{ij} has an end-vertex in V_i and another end-vertex in V_j with $|S_{ij}| = s_{ij} \geqslant 0$.

Lemma 2.7 (Lau and Peng [6]) *For integer $t \geqslant 3$, Let $F = K(n_1, n_2, \cdots, n_t)$ be a complete t-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$Q(G) = Q(F) - \sum_{1 \leqslant i < j \leqslant t} (n_i - 1)(n_j - 1)s_{ij} + \binom{s}{2} - \sum_{1 \leqslant i < j < l \leqslant t} s_{ij}s_{il} - \sum_{\substack{1 \leqslant i < j \leqslant t \\ 1 \leqslant k < l \leqslant t \\ i < k}} s_{ij}s_{kl} + \sum_{1 \leqslant i < j \leqslant t} \left[s_{ij} \sum_{k \notin \{i,j\}} \binom{n_k}{2} \right] + \sum_{\substack{1 \leqslant i < j \leqslant t \\ 1 \leqslant i < k < l \leqslant t \\ j \notin \{k,l\}}} s_{ij}s_{kl},$$

and

$$K(G) = K(F) - \sum_{1 \leqslant i < j \leqslant t} \left[s_{ij} \sum_{\substack{1 \leqslant k < l \leqslant t \\ \{i,j\} \cap \{k,l\} = \emptyset}} n_k n_l \right] + \sum_{\substack{1 \leqslant i < j \leqslant t \\ 1 \leqslant i < k < l \leqslant t \\ j \notin \{k,l\}}} s_{ij}s_{kl}.$$

By using Lemma 2.7, we obtain the following.

Lemma 2.8 *Let $F = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph and let $G = F - S$ where S is a set of s edges in F. If S induces a matching in F, then*

$$\begin{aligned} Q(G) = {} & Q(F) - \sum_{1 \leqslant i < j \leqslant 5} (n_i - 1)(n_j - 1)s_{ij} + \binom{s}{2} - s_{12}(s_{13} + s_{14} + s_{15} + s_{23} \\ & + s_{24} + s_{25}) - s_{13}(s_{14} + s_{15} + s_{23} + s_{34} + s_{35}) - s_{14}(s_{15} + s_{24} + s_{34} + s_{45}) \\ & - s_{15}(s_{25} + s_{35} + s_{45}) - s_{23}(s_{24} + s_{25} + s_{34} + s_{35}) - s_{24}(s_{25} + s_{34} + s_{45}) \\ & - s_{25}(s_{35} + s_{45}) - s_{34}(s_{35} + s_{45}) - s_{35}s_{45} + \sum_{1 \leqslant i < j \leqslant 5} \left[s_{ij} \sum_{k \notin \{i,j\}} \binom{n_k}{2} \right], \end{aligned}$$

$$\begin{aligned} K(G) = {} & K(F) - \sum_{1 \leqslant i < j \leqslant 5} \left[s_{ij} \sum_{\substack{1 \leqslant k < l \leqslant 5 \\ \{i,j\} \cap \{k,l\} = \emptyset}} n_k n_l \right] + s_{12}(s_{34} + s_{35} + s_{45}) \\ & + s_{13}(s_{24} + s_{25} + s_{45}) + s_{14}(s_{23} + s_{25} + s_{35}) + s_{15}(s_{23} + s_{24} + s_{34}) + s_{23}s_{45} \\ & + s_{24}s_{35} + s_{25}s_{34}. \end{aligned}$$

Moreover, these equalities hold if and only if each edge in S joins vertices in the same two partite sets of smallest size in F.

§3. Characterization

In this section, we shall characterize certain complete 5-partite graph $G = K(n_1, n_2, n_3, n_4, n_5)$ according to the number of 6-independent partitions of G where $n_5 - n_1 \leqslant 4$.

Theorem 3.1 *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 1$ and $n_5 - n_1 \leqslant 4$. Define $\theta(G) = [\alpha(G, 6) - 2^{n+1} - 2^n + 5]/2^{n-2}$. Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(n, n, n, n, n + 1)$;

(ii) $\theta(G) = 1$ *if and only if* $G = K(n - 1, n, n, n + 1, n + 1)$;

(iii) $\theta(G) = 2$ *if and only if* $G = K(n - 1, n - 1, n + 1, n + 1, n + 1)$;

(iv) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(n - 2, n, n + 1, n + 1, n + 1)$;

(v) $\theta(G) = 3$ *if and only if* $G = K(n - 1, n, n, n, n + 2)$;

(vi) $\theta(G) = 4$ *if and only if* $G = K(n - 1, n - 1, n, n + 1, n + 2)$;

(vii) $\theta(G) = 4\frac{1}{4}$ *if and only if* $G = K(n - 3, n + 1, n + 1, n + 1, n + 1)$;

(viii) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(n - 2, n, n, n + 1, n + 2)$;

(ix) $\theta(G) = 5\frac{1}{2}$ *if and only if* $G = K(n - 2, n - 1, n + 1, n + 1, n + 2)$;

(x) $\theta(G) = 7$ *if and only if* $G = K(n - 1, n - 1, n - 1, n + 2, n + 2)$;

(xi) $\theta(G) = 7\frac{1}{2}$ *if and only if* $G = K(n - 2, n - 1, n, n + 2, n + 2)$;

(xii) $\theta(G) = 9$ *if and only if* $G = K(n - 2, n - 2, n + 1, n + 2, n + 2)$;

(xiii) $\theta(G) = 10$ *if and only if* $G = K(n - 1, n - 1, n, n, n + 3)$;

(xiv) $\theta(G) = 11$ *if and only if* $G = K(n - 1, n - 1, n - 1, n + 1, n + 3)$.

Proof In order to complete the proof of the theorem, we first give a table for the θ-value of various complete 5-partite graphs with $5n + 1$ vertices as shown in Table 1.

(i) G_1 is the improvement of G_2 and G_3 with $\theta(G_2) = 1$ and $\theta(G_3) = 3$;

(ii) G_2 is the improvement of G_3, G_4, G_5, G_6 and G_7 with $\theta(G_3) = 3$, $\theta(G_4) = 2$, $\theta(G_5) = 4$, $\theta(G_6) = 2\frac{1}{2}$ and $\theta(G_7) = 4\frac{1}{2}$;

(iii) G_3 is the improvement of G_5, G_7, G_8 and G_9 with $\theta(G_5) = 4$, $\theta(G_7) = 4\frac{1}{2}$ and $\theta(G_8) = 10$ and $\theta(G_9) = 10\frac{1}{2}$;

(iv) G_4 is the improvement of G_5, G_6 and G_{10} with $\theta(G_5) = 4$, $\theta(G_6) = 2\frac{1}{2}$ and $\theta(G_{10}) = 5\frac{1}{2}$;

(v) G_5 is the improvement of G_7, G_8, G_{10}, G_{11}, G_{12}, G_{13} and G_{14} with $\theta(G_7) = 4\frac{1}{2}$, $\theta(G_8) = 10$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{11}) = 7$, $\theta(G_{12}) = 11$, $\theta(G_{13}) = 7\frac{1}{2}$ and $\theta(G_{14}) = 11\frac{1}{2}$;

(vi) G_6 is the improvement of G_7, G_{10}, G_{15} and G_{16} with $\theta(G_7) = 4\frac{1}{2}$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{15}) = 4\frac{1}{4}$ and $\theta(G_{16}) = 6\frac{1}{4}$;

G_i $(1 \leqslant i \leqslant 21)$	$\theta(G_i)$	G_i $(22 \leqslant i \leqslant 41)$	$\theta(G_i)$
$G_1 = K(n,n,n,n,n+1)$	0	$G_{22} = K(n-2,n-2,n+1,n+2,n+2)$	9
$G_2 = K(n-1,n,n,n+1,n+1)$	1	$G_{23} = K(n-2,n-2,n+1,n+1,n+3)$	13
$G_3 = K(n-1,n,n,n,n+2)$	3	$G_{24} = K(n-3,n-1,n+1,n+2,n+2)$	$9\frac{1}{4}$
$G_4 = K(n-1,n-1,n+1,n+1,n+1)$	2	$G_{25} = K(n-3,n-1,n+1,n+1,n+3)$	$13\frac{1}{4}$
$G_5 = K(n-1,n-1,n,n+1,n+2)$	4	$G_{26} = K(n-2,n-1,n-1,n+2,n+3)$	$14\frac{1}{2}$
$G_6 = K(n-2,n,n+1,n+1,n+1)$	$2\frac{1}{2}$	$G_{27} = K(n-2,n-1,n-1,n+1,n+4)$	$26\frac{1}{2}$
$G_7 = K(n-2,n,n,n+1,n+2)$	$4\frac{1}{2}$	$G_{28} = K(n-2,n-2,n,n+2,n+3)$	15
$G_8 = K(n-1,n-1,n,n,n+3)$	10	$G_{29} = K(n-3,n-1,n,n+2,n+3)$	$15\frac{1}{4}$
$G_9 = K(n-2,n,n,n,n+3)$	$10\frac{1}{2}$	$G_{30} = K(n-4,n+1,n+1,n+1,n+2)$	$8\frac{1}{8}$
$G_{10} = K(n-2,n-1,n+1,n+1,n+2)$	$5\frac{1}{2}$	$G_{31} = K(n-4,n,n+1,n+2,n+2)$	$10\frac{1}{8}$
$G_{11} = K(n-1,n-1,n-1,n+2,n+2)$	7	$G_{32} = K(n-4,n,n+1,n+1,n+3)$	$14\frac{1}{8}$
$G_{12} = K(n-1,n-1,n-1,n+1,n+3)$	11	$G_{33} = K(n-4,n,n,n+2,n+3)$	$16\frac{1}{8}$
$G_{13} = K(n-2,n-1,n,n+2,n+2)$	$7\frac{1}{2}$	$G_{34} = K(n-3,n-2,n+2,n+2,n+2)$	$12\frac{3}{4}$
$G_{14} = K(n-2,n-1,n,n+1,n+3)$	$11\frac{1}{2}$	$G_{35} = K(n-3,n-2,n+1,n+2,n+3)$	$16\frac{3}{4}$
$G_{15} = K(n-3,n+1,n+1,n+1,n+1)$	$4\frac{1}{4}$	$G_{36} = K(n-4,n-1,n+2,n+2,n+2)$	$13\frac{1}{8}$
$G_{16} = K(n-3,n,n+1,n+1,n+2)$	$6\frac{1}{4}$	$G_{37} = K(n-4,n-1,n+1,n+2,n+3)$	$17\frac{1}{8}$
$G_{17} = K(n-3,n,n,n+2,n+2)$	$8\frac{1}{4}$	$G_{38} = K(n-5,n+1,n+1,n+2,n+2)$	$12\frac{1}{16}$
$G_{18} = K(n-3,n,n,n+1,n+3)$	$12\frac{1}{4}$	$G_{39} = K(n-5,n+1,n+1,n+1,n+3)$	$16\frac{1}{16}$
$G_{19} = K(n-1,n-1,n-1,n,n+4)$	25	$G_{40} = K(n-5,n,n+2,n+2,n+2)$	$14\frac{1}{16}$
$G_{20} = K(n-2,n-1,n,n,n+4)$	$25\frac{1}{2}$	$G_{41} = K(n-5,n,n+1,n+2,n+3)$	$18\frac{1}{16}$
$G_{21} = K(n-3,n,n,n,n+4)$	$26\frac{1}{4}$		

Table 1 **Complete 5-partite graphs with** $5n+1$ **vertices.**

By the definition of improvement, we have the followings:

(vii) G_7 is the improvement of G_9, G_{10}, G_{13}, G_{14}, G_{16}, G_{17} and G_{18} with $\theta(G_9) = 10\frac{1}{2}$, $\theta(G_{10}) = 5\frac{1}{2}$, $\theta(G_{13}) = 7\frac{1}{2}$, $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{17}) = 8\frac{1}{4}$and $\theta(G_{18}) = 12\frac{1}{4}$;

(viii) G_8 is the improvement of G_9, G_{12}, G_{14}, G_{19} and G_{20} with $\theta(G_9) = 10\frac{1}{2}$, $\theta(G_{12}) = 11$, $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{19}) = 25$ and $\theta(G_{20}) = 25\frac{1}{2}$;

(ix) G_9 is the improvement of G_{14}, G_{18}, G_{20} and G_{21} with $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{18}) = 12\frac{1}{4}$, $\theta(G_{20}) = 25\frac{1}{2}$ and $\theta(G_{21}) = 26\frac{1}{4}$;

(x) G_{10} is the improvement of G_{13}, G_{14}, G_{16}, G_{22}, G_{23}, G_{24} and G_{25} with $\theta(G_{13}) = 7\frac{1}{2}$, $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{16}) = 6\frac{1}{4}$, $\theta(G_{22}) = 9$, $\theta(G_{23}) = 13$, $\theta(G_{24}) = 9\frac{1}{4}$ and $\theta(G_{25}) = 13\frac{1}{4}$;

(xi) G_{11} is the improvement of G_{12}, G_{13} and G_{26} with $\theta(G_{12}) = 11$, $\theta(G_{13}) = 7\frac{1}{2}$ and $\theta(G_{26}) = 14\frac{1}{2}$;

(xii) G_{12} is the improvement of G_{14}, G_{19}, G_{26} and G_{27} with $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{19}) = 25$, $\theta(G_{26}) = 14\frac{1}{2}$ and $\theta(G_{27}) = 26\frac{1}{2}$;

(xiii) G_{13} is the improvement of G_{14}, G_{17}, G_{22}, G_{24}, G_{26}, G_{28} and G_{29} with $\theta(G_{14}) = 11\frac{1}{2}$, $\theta(G_{17}) = 8\frac{1}{4}$, $\theta(G_{22}) = 9$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{26}) = 14\frac{1}{2}$, $\theta(G_{28}) = 15$ and $\theta(G_{29}) = 15\frac{1}{4}$, ;

(xiv) G_{15} is the improvement of G_{16} and G_{30} with $\theta(G_{16}) = 6\frac{1}{4}$ and $\theta(G_{30}) = 8\frac{1}{8}$;

(xv) G_{16} is the improvement of G_{17}, G_{18}, G_{24}, G_{25}, G_{30}, G_{31} and G_{32} with $\theta(G_{17}) = 8\frac{1}{4}$, $\theta(G_{18}) = 12\frac{1}{4}$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{25}) = 13\frac{1}{4}$, $\theta(G_{30}) = 8\frac{1}{8}$, $\theta(G_{31}) = 10\frac{1}{8}$ and $\theta(G_{32}) = 14\frac{1}{8}$;

(xvi) G_{17} is the improvement of G_{18}, G_{24}, G_{29}, G_{31} and G_{33} with $\theta(G_{18}) = 12\frac{1}{4}$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{29}) = 15\frac{1}{4}$, $\theta(G_{31}) = 10\frac{1}{8}$ and $\theta(G_{33}) = 16\frac{1}{8}$;

(xvii) G_{22} is the improvement of G_{23}, G_{24}, G_{28}, G_{34} and G_{35} with $\theta(G_{23}) = 13$, $\theta(G_{24}) = 9\frac{1}{4}$, $\theta(G_{28}) = 15$, $\theta(G_{34}) = 12\frac{3}{4}$ and $\theta(G_{35}) = 16\frac{3}{4}$;

(xviii) G_{24} is the improvement of G_{25}, G_{29}, G_{31}, G_{34}, G_{35}, G_{36} and G_{37} with $\theta(G_{25}) = 13\frac{1}{4}$, $\theta(G_{29}) = 15\frac{1}{4}$, $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{34}) = 12\frac{3}{4}$, $\theta(G_{35}) = 16\frac{3}{4}$, $\theta(G_{36}) = 13\frac{1}{8}$ and $\theta(G_{37}) = 17\frac{1}{8}$;

(xix) G_{30} is the improvement of G_{31}, G_{32}, G_{38} and G_{39} with $\theta(G_{31}) = 10\frac{1}{8}$, $\theta(G_{32}) = 14\frac{1}{8}$, $\theta(G_{38}) = 12\frac{1}{16}$ and $\theta(G_{39}) = 16\frac{1}{16}$;

(xx) G_{31} is the improvement of G_{32}, G_{33}, G_{36}, G_{37}, G_{38}, G_{40} and G_{41} with $\theta(G_{32}) = 14\frac{1}{8}$, $\theta(G_{33}) = 16\frac{1}{8}$, $\theta(G_{36}) = 13\frac{1}{8}$, $\theta(G_{37}) = 17\frac{1}{8}$, $\theta(G_{38}) = 12\frac{1}{16}$, $\theta(G_{40}) = 14\frac{1}{16}$ and $\theta(G_{41}) = 18\frac{1}{16}$.

Hence, by Lemma 2.4 and the above arguments, we know (i) to (xiv) holds. Thus the proof is completed. □

Similarly to the proof of Theorem 3.1, we can obtain Theorems 3.2 and 3.3.

Theorem 3.2 *Let $G = K(n_1, n_2, n_3, n_4, n_5)$ be a complete 5-partite graph such that $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 2$ and $n_5 - n_1 \leqslant 4$. Define $\theta(G) = [\alpha(G, 6) - 3 \cdot 2^n - 2^{n-1} + 5]/2^{n-2}$. Then*

(i) $\theta(G) = 0$ if and only if $G = K(n, n, n, n+1, n+1)$;

(ii) $\theta(G) = 1$ if and only if $G = K(n-1, n, n+1, n+1, n+1)$;

(iii) $\theta(G) = 2$ if and only if $G = K(n, n, n, n, n+2)$;

(iv) $\theta(G) = 2\frac{1}{2}$ if and only if $G = K(n-2, n+1, n+1, n+1, n+1)$;

(v) $\theta(G) = 3$ if and only if $G = K(n-1, n, n, n+1, n+2)$;

(vi) $\theta(G) = 4$ if and only if $G = K(n-1, n-1, n+1, n+1, n+2)$;

(vii) $\theta(G) = 4\frac{1}{2}$ if and only if $G = K(n-2, n, n+1, n+1, n+2)$;

(viii) $\theta(G) = 6$ if and only if $G = K(n-1, n-1, n, n+2, n+2)$;

(ix) $\theta(G) = 6\frac{1}{2}$ if and only if $G = K(n-2, n, n, n+2, n+2)$;

(x) $\theta(G) = 7\frac{1}{2}$ if and only if $G = K(n-2, n-1, n+1, n+2, n+2)$;

(xi) $\theta(G) = 9$ if and only if $G = K(n-1, n, n, n, n+3)$;

(xii) $\theta(G) = 10$ if and only if $G = K(n-1, n-1, n, n+1, n+3)$;

(xiii) $\theta(G) = 11$ *if and only if* $G = K(n-2, n-2, n+2, n+2, n+2)$;

(xiv) $\theta(G) = 13$ *if and only if* $G = K(n-1, n-1, n-1, n+2, n+3)$.

Theorem 3.3 *Let* $G = K(n_1, n_2, n_3, n_4, n_5)$ *be a complete 5-partite graph such that* $n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 3$ *and* $n_5 - n_1 \leqslant 4$. *Define* $\theta(G) = [\alpha(G, 6) - 2^{n+2} + 5]/2^{n-1}$. *Then*

(i) $\theta(G) = 0$ *if and only if* $G = K(n, n, n+1, n+1, n+1)$;

(ii) $\theta(G) = \frac{1}{2}$ *if and only if* $G = K(n-1, n+1, n+1, n+1, n+1)$;

(iii) $\theta(G) = 1$ *if and only if* $G = K(n, n, n, n+1, n+2)$;

(iv) $\theta(G) = 1\frac{1}{2}$ *if and only if* $G = K(n-1, n, n+1, n+1, n+2)$;

(v) $\theta(G) = 2\frac{1}{4}$ *if and only if* $G = K(n-2, n+1, n+1, n+1, n+2)$;

(vi) $\theta(G) = 2\frac{1}{2}$ *if and only if* $G = K(n-1, n, n, n+2, n+2)$;

(vii) $\theta(G) = 3$ *if and only if* $G = K(n-1, n-1, n+1, n+2, n+2)$;

(viii) $\theta(G) = 3\frac{1}{4}$ *if and only if* $G = K(n-2, n, n+1, n+2, n+2)$;

(ix) $\theta(G) = 4$ *if and only if* $G = K(n, n, n, n, n+3)$;

(x) $\theta(G) = 4\frac{1}{2}$ *if and only if* $G = K(n-1, n, n, n+1, n+3)$;

(xi) $\theta(G) = 4\frac{3}{4}$ *if and only if* $G = K(n-2, n-1, n+2, n+2, n+2)$;

(xii) $\theta(G) = 5$ *if and only if* $G = K(n-1, n-1, n+1, n+1, n+3)$;

(xiii) $\theta(G) = 6$ *if and only if* $G = K(n-1, n-1, n, n+2, n+3)$;

(xiv) $\theta(G) = 9\frac{1}{2}$ *if and only if* $G = K(n-1, n-1, n-1, n+3, n+3)$.

§4. Chromatically Closed 5-Partite Graphs

In this section, we obtained several χ-closed families of graphs from the graphs in Theorem 3.1 to 3.3.

Theorem 4.1 *The family of graphs* $\mathcal{K}^{-s}(n_1, n_2, n_3, n_4, n_5)$ *where* $n_1+n_2+n_3+n_4+n_5 = 5n+1$, $n_5 - n_1 \leqslant 4$ *and* $n_1 \geqslant s+5$ *is* χ*-closed.*

Proof By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 $(i), (ii), \cdots, (xiv)$ by $G_1, G_2, \cdots, G_{14}$, respectively. Suppose $H \sim G_i - S$. It suffices to show that $H \in \{G_i - S\}$. By Lemma 2.3, we know there exists a complete 5-partite graph $F = (p_1, p_2, p_3, p_4, p_5)$ such that $H = F - S'$ with $|S'| = s' = e(F) - e(G) + s \geqslant 0$.

Case 1. Let $G = G_1$ with $n \geqslant s+2$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n,n,n,n,n+1)$. By Lemma 2.5, we have

$$\alpha(G-S,6) = \alpha(G,6) + \alpha'(G-S) \text{ with } s \leqslant \alpha'(G-S) \leqslant 2^s - 1,$$
$$\alpha(F-S',6) = \alpha(F,6) + \alpha'(F-S') \text{ with } 0 \leqslant s' \leqslant \alpha'(F-S').$$

Hence,

$$\alpha(F-S',6) - \alpha(G-S,6) = \alpha(F,6) - \alpha(G,6) + \alpha'(F-S') - \alpha'(G-S).$$

By the definition, $\alpha(F,6) - \alpha(G,6) = 2^{n-2}(\theta(F) - \theta(G))$. By Theorem 3.1, $\theta(F) \geqslant 0$. Suppose $\theta(F) > 0$, then

$$\begin{aligned} \alpha(F-S',6) - \alpha(G-S,6) &\geqslant 2^{n-2} + \alpha'(F-S') - \alpha'(G-S) \\ &\geqslant 2^s + \alpha'(F-S') - 2^s + 1 \geqslant 1, \end{aligned}$$

contradicting $\alpha(F-S',6) = \alpha(G-S,6)$. Hence, $\theta(F) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathcal{K}^{-s}(n,n,n,n,n+1)$.

Case 2. Let $G = G_2$ with $n \geqslant s+3$. In this case, $H \sim F - S \in \mathcal{K}^{-s}(n-1,n,n,n+1,n+1)$. By Lemma 2.5, we have

$$\alpha(G-S,6) = \alpha(G,6) + \alpha'(G-S) \text{ with } s \leqslant \alpha'(G-S) \leqslant 2^s - 1,$$
$$\alpha(F-S',6) = \alpha(F,6) + \alpha'(F-S') \text{ with } 0 \leqslant s' \leqslant \alpha'(F-S').$$

Hence,

$$\alpha(F-S',6) - \alpha(G-S,6) = \alpha(F,6) - \alpha(G,6) + \alpha'(F-S') - \alpha'(G-S).$$

By the definition, $\alpha(F,6) - \alpha(G,6) = 2^{n-2}(\theta(F) - \theta(G))$. Suppose $\theta(F) \neq \theta(G)$. Then, we consider two subcases.

Subcase 2.1 $\theta(F) < \theta(G)$. By Theorem 3.1, $F = G_1$ and $H = G_1 - S' \in \{G_1 - S'\}$. However, $G - S \notin \{G_1 - S'\}$ since by Case (i) above, $\{G_1 - S'\}$ is χ-closed, a contradiction.

Subcase 2.2 $\theta(F) > \theta(G)$. By Theorem 3.1, $\alpha(F,6) - \alpha(G,6) \geqslant 2^{n-2}$. So,

$$\begin{aligned} \alpha(F-S',6) - \alpha(G-S,6) &\geqslant 2^{n-2} + \alpha'(F-S') - \alpha'(G-S) \\ &\geqslant 2^s + \alpha'(F-S') - 2^s + 1 \geqslant 1, \end{aligned}$$

contradicting $\alpha(F-S',6) = \alpha(G-S,6)$. Hence, $\theta(F) - \theta(G) = 0$ and so $F = G$ and $s = s'$. Therefore, $H \in \mathcal{K}^{-s}(n-1,n,n,n+1,n+1)$.

Using Table 1, we can prove (iii) to (xiv) in a similar way. This completes the proof. □

Similarly, we can prove Theorems 4.2 and 4.3.

Theorem 4.2 *The family of graphs $\mathcal{K}^{-s}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5 = 5n+2$, $n_5 - n_1 \leqslant 4$ and $n_1 \geqslant s+6$ is χ-closed.*

Theorem 4.3 *The family of graphs* $\mathcal{K}^{-s}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5=5n+3$, $n_5-n_1\leqslant 4$ *and* $n_1\geqslant s+6$ *is* χ*-closed.*

§5. Chromatically Unique 5-Partite Graphs

The following results give several families of chromatically unique complete 5-partite graphs having $5n+1$ vertices with a set S of s edges deleted where the deleted edges induce a star $K_{1,s}$ and a matching sK_2, respectively.

Theorem 5.1 *The graphs* $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5)$ *where* $n_1+n_2+n_3+n_4+n_5=5n+1$, $n_5-n_1\leqslant 4$ *and* $n_1\geqslant s+5$ *are* χ*-unique for* $1\leqslant i\neq j\leqslant 5$.

Proof By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\cdots,(xiv)$ by $G_1,G_2,\cdots,G_{14}$, respectively. The proof for each graph obtained from G_i $(i=1,2,\cdots,14)$ is similar, so we only give the detail proof for the graphs obtained from G_2 below.

By Lemma 2.5 and Case 2 of Theorem 4.1, we know that $K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ $=\{K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)|(i,j)\in\{(1,2),(2,1),(1,4),(4,1),(2,3),(2,4),(4,2),(4,5)\}$ is χ-closed for $n\geqslant s+3$. Note that

$$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=t(G_2)-s(3n+2) \text{ for } (i,j)\in\{(1,2),(2,1)\},$$
$$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=t(G_2)-s(3n+1) \text{ for } (i,j)\in\{(1,4),(4,1),(2,3)\},$$
$$t(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=t(G_2)-3sn \text{ for } (i,j)\in\{(2,4),(4,2)\},$$
$$t(K_{4,5}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=t(G_2)-s(3n-1).$$

By Lemmas 2.2 and 2.6, we conclude that $\sigma(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))\neq\sigma(K_{j,i}^{-K_{1,s}}(n-1,n,n,n+1,n+1))$ for each $(i,j)\in\{(1,2),(1,4),(2,4)\}$. We now show that $K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ and $K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ for $(i,j)\in\{(1,4),(4,1)\}$ are not χ-equivalent. We have

$$Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=Q(G_2)-s(n-1)^2+\binom{s}{2}+s\left[\binom{n-1}{2}+2\binom{n+1}{2}\right],$$

$$Q(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))=Q(G_2)-sn(n-2)+\binom{s}{2}+s\left[2\binom{n}{2}+\binom{n+1}{2}\right]$$

for $(i,j)\in\{(1,4),(4,1)\}$ with

$$Q\Big(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)\Big)-Q\Big(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)\Big)=0$$

since $s_{ij}=0$ if $(i,j)\neq\{(1,4),(4,1),(2,3)\}$. We also obtain

$$\begin{aligned}K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)) &= K(G_2)-s(3n^2+2n-1);\\ K(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)) &= K(G_2)-s(3n^2+2n)\end{aligned}$$

for $(i,j)\in\{(1,4),(4,1)\}$ with

$$K\Big(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1)\Big)-K\Big(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)\Big)=s$$

since $s_{ij}=0$ if $(i,j)\neq\{(1,4),(4,1),(2,3)\}$. This means that $2K(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))-Q(K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1))\neq 2K(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1))-Q(K_{2,3}^{-K_{1,s}}(n-1,n,n,n+1,n+1))$ for $(i,j)\in\{(1,4),(4,1)\}$, contradicting Lemma 2.1. Hence, $K_{i,j}^{-K_{1,s}}(n-1,n,n,n+1,n+1)$ is χ-unique where $n\geqslant s+3$ for $1\leqslant i\neq j\leqslant 5$. The proof is thus complete. □

Theorem 5.2 *The graphs $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5=5n+1$, $n_5-n_1\leqslant 4$ and $n_1\geqslant s+5$ are χ-unique.*

Proof By Theorem 3.1, there are 14 cases to consider. Denote each graph in Theorem 3.1 $(i),(ii),\cdots,(xiv)$ by $G_1,G_2,\cdots,G_{14}$, respectively. For a graph $K(p_1,p_2,p_3,p_4,p_5)$, let $S=\{e_1,e_2,\cdots,e_s\}$ be the set of s edges in $E(K(p_1,p_2,p_3,p_4,p_5))$ and let $t(e_i)$ denote the number of triangles containing e_i in $K(p_1,p_2,p_3,p_4,p_5)$. The proofs for each graph obtained from G_i $(i=1,2,\cdots,14)$ are similar, so we only give the proof of the graph obtained from G_1 and G_2 as follows.

Suppose $H\sim G=K_{1,2}^{-sK_2}(n,n,n,n,n+1)$ for $n\geqslant s+2$. By Theorem 4.1 and Lemma 2.1, $H\in\mathcal{K}^{-s}(n,n,n,n,n+1)$ and $\alpha'(H)=\alpha'(G)=s$. Let $H=F-S$ where $F=K(n,n,n,n,n+1)$. Clearly, $t(e_i)\leqslant 3n+1$ for each $e_i\in S$. So,

$$t(H)\geqslant t(F)-s(3n+1),$$

with equality holds only if $t(e_i)=3n+1$ for all $e_i\in S$. Since $t(H)=t(G)=t(F)-s(3n+1)$, the equality above holds with $t(e_i)=3n+1$ for all $e_i\in S$. Therefore each edge in S has an end-vertex in V_i and another end-vertex in V_j $(1\leqslant i<j\leqslant 4)$. Moreover, S must induce a matching in F. Otherwise, equality does not hold or $\alpha'(H)>s$. By Lemma 2.8, we obtain

$$Q(G)=Q(F)-s(n-1)^2+\binom{s}{2}+s\left[2\binom{n}{2}+\binom{n+1}{2}\right]$$

whereas

$$\begin{aligned}Q(H) &= Q(F)-s(n-1)^2+\binom{s}{2}-s_{12}(s_{13}+s_{14}+s_{23}+s_{24}+s_{34})\\ &\quad -s_{13}(s_{14}+s_{23}+s_{24}+s_{34})-s_{14}(s_{23}+s_{24}+s_{34})-s_{23}(s_{24}+s_{34})-s_{24}s_{34}\\ &\quad +s\left[2\binom{n}{2}+\binom{n+1}{2}\right]+s_{12}s_{34}+s_{13}s_{24}+s_{14}s_{23}\\ &= Q(G)-s_{12}(s_{13}+s_{14}+s_{23}+s_{24})-s_{13}(s_{14}+s_{23}+s_{34})-s_{14}(s_{24}+s_{34})\\ &\quad -s_{23}(s_{24}+s_{34})-s_{24}s_{34}.\end{aligned}$$

Moreover, $K(G)=K(F)-s(3n^2+2n)$ whereas

$$\begin{aligned}K(H) &= K(F)-s(3n^2+2n)+s_{12}s_{34}+s_{13}s_{24}+s_{14}s_{23}\\ &= K(G)+s_{12}s_{34}+s_{13}s_{24}+s_{14}s_{23}.\end{aligned}$$

Hence,

$$\begin{aligned}2K(H)-Q(H) &= 2K(G)-Q(G)+2(s_{12}s_{34}+s_{13}s_{24}+s_{14}s_{23})+\\ &\quad s_{12}(s_{13}+s_{14}+s_{23}+s_{24})+s_{13}(s_{14}+s_{23}+s_{34})+s_{14}(s_{24}+s_{34})+\\ &\quad s_{23}(s_{24}+s_{34})+s_{24}s_{34},\end{aligned}$$

and that $2K(H)-Q(H)=2K(G)-Q(G)$ if and only if $s=s_{ij}$ for $1\leqslant i<j\leqslant 4$. Therefore, we have $\langle S\rangle\cong sK_2$ with $H\cong G$.

Suppose $H\sim G=K_{1,2}^{-sK_2}(n-1,n,n,n+1,n+1)$ for $n\geqslant s+3$. By Theorem 4.1 and Lemma 2.1, $H\in\mathcal{K}^{-s}(n-1,n,n,n+1,n+1)$ and $\alpha'(H)=\alpha'(G)=s$. Let $H=F-S$ where $F=K(n-1,n,n,n+1,n+1)$. Clearly, $t(e_i)\leqslant 3n+2$ for each $e_i\in S$. So,

$$t(H)\geqslant t(F)-s(3n+2),$$

with equality holds only if $t(e_i)=3n+2$ for all $e_i\in S$. Since $t(H)=t(G)=t(F)-s(3n+2)$, the equality above holds with $t(e_i)=3n+2$ for all $e_i\in S$. Therefore each edge in S has an end-vertex in V_1 and another end-vertex in V_j $(2\leqslant j\leqslant 3)$. Moreover, S must induce a matching in F. Otherwise, equality does not hold or $\alpha'(H)>s$. By Lemma 2.8, we obtain

$$Q(G)=Q(F)-s(n-1)(n-2)+\binom{s}{2}+s\left[\binom{n}{2}+2\binom{n+1}{2}\right]$$

whereas

$$\begin{aligned}Q(H) &= Q(F)-s(n-1)(n-2)+\binom{s}{2}-s_{12}s_{13}+s\left[\binom{n}{2}+2\binom{n+1}{2}\right]\\ &\leqslant Q(G),\end{aligned}$$

and the equality holds if and only if $s=s_{1j}$ $(2\leqslant j\leqslant 3)$. Moreover, $K(G)=K(H)=K(F)-s(3n^2+4n+1)$. Hence, $2K(G)-Q(G)\neq 2K(H)-Q(H)$ and the equality holds if and only if $\langle S\rangle\cong sK_2$ with $H\cong G$. Thus the proof is complete. □

Similarly to the proofs of Theorems 5.1 and 5.2, we can prove Theorems 5.3 to 5.6 following.

Theorem 5.3 *The graphs $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5=5n+2$, $n_5-n_1\leqslant 4$ and $n_1\geqslant s+6$ are χ-unique for $1\leqslant i\neq j\leqslant 5$.*

Theorem 5.4 *The graphs $K_{i,j}^{-K_{1,s}}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5=5n+3$, $n_5-n_1\leqslant 4$ and $n_1\geqslant s+6$ are χ-unique for $1\leqslant i\neq j\leqslant 5$.*

Theorem 5.5 *The graphs $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5=5n+2$, $n_5-n_1\leqslant 4$ and $n_1\geqslant s+6$ are χ-unique.*

Theorem 5.6 *The graphs $K_{1,2}^{-sK_2}(n_1,n_2,n_3,n_4,n_5)$ where $n_1+n_2+n_3+n_4+n_5=5n+3$, $n_5-n_1\leqslant 4$ and $n_1\geqslant s+6$ are χ-unique.*

Remark 5.7 This paper generalized the results and solved the open problems in [9,10,11].

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