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A Point Interpolation Meshless Method for the Numerical Solution of the Singularly Perturbed Integral and Integro-differential Equations

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Abstract

Recently, there has been an increasing interest in the study of singular and perturbed systems. In this paper we propose a point interpolation mesh less method for solving singularly perturbed integro-differential integral equations. The method is based upon radial basis functions, using zeros of the shifted Legendre polynomial as the collocation points. The results of numerical experiments are compared with the exact solution in illustrative examples to confirm the accuracy and efficiency of the presented scheme.

Keywords: singularly perturbed problems; Volterra integral equations; Volterra integro-differential equations; radial basis functions; Legendre polynomials

1- Introduction

In the present work, we consider the singularly perturbed Volterra integro-differential equations (SVIDE)

$$\varepsilon \frac{d}{dx} y(x) = u\big(x, \varepsilon, y(x)\big) + \int_0^x K\big(x, t, \varepsilon, y(t)\big)\, dt\,, \quad x \in I = [0, x] \tag{1}$$

$$y(0) = \alpha$$

Where α is a constant and ε is a known perturbation parameter which $0 < \varepsilon \ll 1$. Smoothness assumptions on u and Imply existence of a unique solution of Eq. (1) for $\varepsilon > 0$.

By substituting $\varepsilon = 0$ in Eq. (1), we obtain the reduced equation

$$0 = u\big(x, y(x)\big) + \int_0^x K\big(x, t, y(t)\big) dt, \quad x \in I = [0, x] \tag{2}$$

which is a Volterra integral equation (VIE) of the second kind. The singularly perturbed nature of Eq. (1) occurs when the properties of the solution with $\varepsilon > 0$ are incompatible with those when $\varepsilon = 0$. The interest here is in those problems which do imply such an incompatibility in the behavior of $y(x)$ near $x = 0$. This suggests the existence of a boundary layer near the origin where the solution undergoes a rapid transition. Singularly perturbed Volterra integro-differential equations arise in many physical and biological problems. Among these are diffusion-dissipation processes, epidemic dynamics, synchronous control systems, renewal processes and filament stretching. For a comprehensive review, see [5,6,26]. Finding the solutions of these problems has been widely studied by researchers in the last decade. Implicit Runge- Kutta methods were presented for singularly perturbed integro-differential-algebraic equations in [22] and for singular- ly perturbed integro-differential systems in [21]. In [30], Orsi applied a Petrov-Galerkin method to singularly perturbed integro-differential-algebraic equations. El-Gendi [12] applied spectral methods to obtain solution of singularly perturbed differential, integral and integro-differential equations. Hu [17] and Horvat et al. [16] solved the SVIDEs by using the spline collocation methods. Recently, in [35] a numerical procedure based on finite difference was presented for solving a class of SVIDEs. More recently, Ramos [34] applied Piecewise-quasilinearization techniques to obtain solution of SVIDEs. For more references about SVIDEs see [4,2,3].

In recent decades, the so-called meshless methods have been extensively used to find approximate solutions of various types of linear and nonlinear equations such as differential equations (DEs) and integral equations (IEs). Unlike the other methods which were used to mesh the domain of the problem, meshless method don't require a structured grid and only make use of a scattered set of collocation points regardless of the connectivity information between the collocation points. For the last years, the radial basis functions (RBFs) method was known as a powerful tool for the scattered data interpola- tion problem. The main advantage of numerical methods which use radial basis functions is the meshless characteristic of these methods. The use of radial basis functions as a meshless method for the numerical solution of ordinary differential equations (ODEs) and partial differential equations (PDEs) is based on the collocation method. One of the domain-type meshless methods is given in [20] in 1990, which directly collocates radial basis functions, particularly the multiquadric (MQ), to find an approximate solution of linear and nonlinear DEs. Kansa's method has recently received a great deal of attention from researchers [10,28,37,39,41].

Name of function	Definition
Multiquadrics (MQ)	$\sqrt{r + c^2}$
Inverse Multiquadrics (IMQ)	$1/(\sqrt{r + c^2})$
Gaussian (GA)	exp^{-cr}
Hyperbolic secant (sech)	$sech(c\sqrt{r})$

Table 1: Some well-known radial basis functions ($r = \|\| x - xi \|\| = r_i$), $c > 0$.

Recently, Kansa's method was extended to solve various ordinary and partial differential equations including the non- linear Klein-Gordon equation [1], regularized long wave (RLW) equation [18], high order ordinary differential equations [29], the case of heat transfer equations [31], Hirota-Satsuma coupled KdV equations [25], second-order parabolic equa- tion with nonlocal boundary conditions [11], Volterra's Population model [32], steady flow of a third-grade fluid in a porous half space [24], Fokker-Planck equation [23], Second-order hyperbolic telegraph equation [9] and so on.

All of the radial basis functions have global support, and in fact many of them, such as multiquadrics (MQ), do not even have isolated zeros [10, 18, 7]. The RBFs can be compactly and globally supported, infinitely differentiable, and contain a free parameter *c*, called the shape parameter [18,7,36]. The interested reader is referred to the recent books and paper by Buhmann [7, 8] and Wendland [40] for more basic details about RBFs, compactly and globally supported and convergence rate of the radial basis functions. There are two basic approaches for obtaining basis functions from RBFs, namely direct approach (DRBF) based on a differential process [19] and indirect approach (IRBF) based on an integration process [27,28,29,].
In this paper, we use the multiquadrics direct radial basis function for finding the solution of SVIDEs. The MQ was ranked as the best based on its accuracy, visual aspect, sensitivity to parameters, execution time, storage requirements, and ease of implementation.For convenience the solution we use RBFs with $\{x\}_{j=1}^{N}$ nodes which are the zeros of the shifted Legendre polynomial $L_N(x)$, $0 \le x \le 1$. The shifted Legendre polynomials $Li(x)$ are defined on the interval [0, 1] and satisfy the following formulae [15]:

$$L_0(x) = 1 \quad , \quad L_1(x) = 2x - 1 ,$$

$$L_{i+1}(x) = \frac{2i}{i+1}(2x-1)L_i(x) - \frac{i}{i+1}L_{i-1}(x) \qquad i = 1,2,3, \ldots$$

This paper is arranged as follows: in Section 2, we describe the properties of radial (PIM) functions. In Section 3, we introduce the Legendre-Gauss-Lobatto nodes and weights. In Section 4 we implement the problem with the proposed method and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed methods. The conclusions are discussed in the final Section.

2-Radial basis functions

2.1 Definition of radial basis function

Let $R^+ = \{x \in R, x \ge 0\}$ be the non-negative half-line and let $B : R+ \rightarrow R$ be a continuous function with $B(0) \ge 0$. A radial basis functions on R^d is a function of the form $B(\|X - Xi\|)$, where $X, Xi \in R^d$ and $\|.\|$ denotes the Euclidean distance between *X* and *X i*s. If one chooses *N* points $\{x\}_{j=1}^{N}$ in R^d then by custom

$$s(X) = \sum_{i=1}^{N} \lambda i \, B(\| X - Xi \|) \qquad ; \lambda i \in R ,$$

is called a radial basis functions as well [14].

2.2Point Interpolation based on radial basis function

Consider an approximation function y(x) in an influence domain that has a set of arbitrarily distributed nodes P_i (x) (i =1; 2;:::; n). n is the number of nodes in the influence domain of x. Nodal function value is assumed to be ui at the node xi . Radial PIM constructs the approximation function y(x) to pass through all these node points using radial basis function $B_i(x)$ and polynomial basis function p_j (x) [33]

$$y(x) = \sum_{i=1}^{n} B_i(x)\, a_i + \sum_{j=1}^{m} P_j(x) b_j = B^T(x)a + P^T(x)b \tag{3}$$

where ai is the coefficient for Bi (x) and bj the coefficient for pi (x) (usually, m< n). The vectors are defined a

$$\begin{aligned} a^T &= [a_1, a_2, \dots, a_n] \\ b^T &= [b_1, b_2, \dots, b_m] \\ B^T &= [B_1(x) \quad B_2(x) \quad \dots \quad B_n(x)] \\ P^T &= [P_1(x) \quad P_2(x) \quad \dots \quad P_m(x)] \end{aligned} \tag{4}$$

A polynomial basis function has the following monomial terms as:

$$P^T(x) = [1, x, x^2, x^3, \dots] \tag{5}$$

The coefficients a_i and b_j in Equation (1) are determined by enforcing the interpolation pass through all n scattered nodal points within the influence domain. The interpolation at the kth point has

$$y_k(x) = \sum_{i=1}^{n} a_i\, B_i(x_k) + \sum_{j=1}^{m} b_j\, P_j(x_k) \quad k = 1,2, \dots, n \tag{6}$$

The polynomial term is an extra-requirement that guarantees unique approximation [15]. Following constraints are usually imposed:

It is expressed in matrix form as follows:

$$\begin{bmatrix} B_0 & P_0 \\ P_0^T & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} y^e \\ 0 \end{Bmatrix} \text{or} \quad G \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} y^e \\ 0 \end{Bmatrix} \tag{7}$$

where the vector for function values is defined as

$$y^e = [y_1, y_2, \dots, y_n]^T \tag{8}$$

The coefficient matrix B_0 on unknowns a is

$$B_0 = \begin{pmatrix} B_1(x_1) & B_2(x_1) & \dots & B_n(x_1) \\ B_1(x_2) & B_2(x_2) & \dots & B_n(x_2) \\ & & & \\ B_1(x_1) & B_2(x_1) & \dots & B_n(x_1) \end{pmatrix} \quad (9)$$

The coefficient matrix P_0 on unknowns b is

$$P_0 = \begin{pmatrix} P_1(x_1) & P_2(x_1) & \dots & P_n(x_1) \\ P_1(x_2) & P_2(x_2) & \dots & P_n(x_2) \\ & & & \\ P_1(x_1) & P_2(x_1) & \dots & P_n(x_1) \end{pmatrix} \quad (10)$$

Because the distance is directionless, there is $B_k(x_i) = B_i(x_k)$, which means that the matrix $B0$ is symmetric. Unique solution is obtained if the inverse of matrix $B0$ exists,

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = G^{-1} \begin{Bmatrix} y^e \\ 0 \end{Bmatrix}$$

The interpolation is finally expressed as

$$y(x) = [B^T(x) \quad P^T(x)]G^{-1} \begin{Bmatrix} y^e \\ 0 \end{Bmatrix} = \quad \phi(x)\,\Lambda \quad (11)$$

where the matrix of shape functions $\phi(x)$ is defined by

$$\phi(x) = [\phi_1(x), \phi_2(x), \dots, \phi_n(x)] \quad (12)$$

in which

$$\phi_k(x) = \sum_{i=1}^{n} B_i(x)G^{-}_{i,k} + \sum_{j=1}^{m} P_j(x)G^{-}_{n+j,k} \quad (13)$$

where $G^{-}_{i,k}$ is the $(i;k)$ element of matrix G^{-1}. After radial basis functions are determined, shape functions depend only upon the position of scattered nodes. Once the inverse of matrix G is obtained, the derivatives of shape functions are easily obtained as

$$\frac{\partial \Phi_k}{\partial x} = \sum_{i=1}^{n} \frac{\partial B_i}{\partial x} G_{i,k}^{-} + \sum_{j=1}^{m} \frac{\partial P_j}{\partial x} G_{j,k}^{-} \qquad (14)$$

The results of this section can be summarized in the following algorithm.

Algorithm

The algorithm works in the following manner:

Choose N center poist $\{x\}_{j=1}^{N}$ from the domain set [a, b].

1. Approxime $y(x)$ as $y_N(x) = \mathbf{\Phi}^T(x)\mathbf{\Lambda}$.
2. Substitute $y_N(x)$ into the main problem and creat residual function $Res(x)$.
3. Substitute collocation points $\{x\}_{j=1}^{N}$ into the $Res(x)$ and create the N equations.
4. Solve the N equations with N unknown coefficients of members of $\mathbf{\Lambda}$ and find the numerical solution

3 Legendre-Gauss-Lobatto nodes and weights

Let $HN[-1, 1]$ denote the space of algebraic polynomials of degree $\leq N$

$$< Pi, Pj > \frac{2}{2j+1}\delta_{ij}$$

Here, $< ., . >$ represent the usual $L^2[-1, 1]$ inner product and $\{x\}_{j=1}^{N}$ are the well-known Legendre polynomials of order i which are orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-1, 1]$, and satisfy the following formulae:

$$P_0(x) = 1, \quad P_1(x) = x,$$

$$P_{i+1}(x) = \frac{2i+1}{i+1} x P_i(x) - \frac{i}{i+1} P_{i-1}(x) \qquad i = 1,2,3, \ldots$$

Next, we let $\{x\}_{j=1}^{N}$ as

$$(1 - x2)P'(xj) = 0;$$

$$-1 = x_0 < x_1 < x_2 < \ldots < x_N = 1$$

where $\dot{P}(x)$ is derivative of $P(x)$. No explicit formula for the nodes $\{x\}_{j=1}^{N}$ is known. However, they are computed numerically using existing subroutines [13,14]. Now, we assume $f \in H_{2N-1}[-1, 1]$,we have

$$\int_{-1}^{1} f(x) \approx \sum_{j=1}^{N} w_j f(x_j) = I_G(f) \qquad (15)$$

Where w_j are the Legendre-Gauss-Lobatto weights given in [39]

$$W_j = \frac{2}{N(N+1)} \times \frac{1}{(P_N(x_j))^2}$$

4 Solution of SVIDEs via PIM

In the present method, the closed form PIM approximating function Eq. (3) is first obtained from a set of training points, and its derivative of any order, e.g. *p*th order, can then be calculated in a straightforward manner by differentiating such a closed form DRBF as follows:

$$y(x) \simeq y_N(x) = \sum_{i=1}^{N} \lambda_i \phi_i(x) = \Phi^T(x)\Lambda, \tag{16}$$

$$\frac{d}{dx} y(x) \approx \dot{y_N}(x) = \sum_{i=1}^{N} \lambda_i \dot{\Phi_i}(x) = D\Phi^T(x) \tag{17}$$

Where $D\Phi^T(x) = [\dot{\Phi_1}, \dot{\Phi_2}, \ldots, \dot{\Phi_N}]$

Then, from substituting Eq. (8) and Eq. (9) into Eq. (1), we have

$$\varepsilon D\Phi T(x)\Lambda = u(x, \varepsilon, \Phi T(x)\Lambda) + \int_0^x k(x, t, \varepsilon, \Phi^T(x)\Lambda)dt \tag{18}$$

$$y_N(0) = \alpha$$

We now collocate Eq. (10) at points $\{x_j\}_{j=1}^{N}$ as

$$\varepsilon D\Phi T(x_i)\Lambda = u(x_i, \varepsilon, \Phi T(x_i)\Lambda) + \int_0^x K(x_i, t, \varepsilon, \Phi T(t)\Lambda)dt, \tag{19}$$

$$\Phi^T(0)\Lambda = \alpha.$$

In order to use the Legendre-Gauss-Lobatto integration formula for Eq. (11), we transfer the t-intervals $[0, x_i]$ into the η-intervals $[-1, 1]$, by means of the transformations $\eta = \frac{2}{x_i} t - 1$. Then Eq. (11) may be restated as the residual function Res(x)

$$Res(x_i) = -\varepsilon D\Phi T(x_i)\Lambda + u(x_i, \varepsilon, \Phi T(x_i)\Lambda) + I_G(K(x_i, x_i/2(\eta + 1), \varepsilon, \Phi T(x_i/2(\eta + 1))\Lambda)), \tag{20}$$

$$\Phi^T(0)\Lambda = \alpha. \tag{21}$$

The set of equations for obtaining the coefficients $\{\boldsymbol{\lambda}\}^{N}_{i=1}$ come from equalizing Eq. (20) to zero at $N - 1$ interpolate nodes $\{\boldsymbol{\lambda}\}^{N}_{i=1}$ plus Eq. (21).behavior of the MQ-RBF method, we applied the following laws

1- The $L2$ error norm of the solution which is defined by

$$L_2 = \| y^{exact}(x) - y_N^{implicit}(x) \|_2 = \left[\sum_{j=1}^{N}\left(y^{exact}(x_j) - y_N^{implicit}(x_j)\right)^2\right]^{1/2} \quad (22)$$

2-where$\{x\}_{j=1}^{N}$ are interpolate nodes which are the zeros of shifted Legendre polynomial $L_N(x), 0 \le x \le 1$

The $L\infty$ error norm of the solution which is defined by

$$L_\infty = \| y^{exact}(x) - y_N^{implicit}(x) \|_\infty = \max_{0\le j\le N}\left|\left(y^{exact}(x_j) - y_N^{implicit}(x_j)\right)\right| \quad (23)$$

5- Numerical results

In order to illustrate the performance of radial point interpolation meshless method (PIM) in solving SVIDES and justify the accuracy and efficiency of our method, we consider the following examples. In all examples we use multiquadrics (MQ) RBF.

5.1 Problem 1

In this problem, we consider the following singularly perturbed Volterra integral equation [2,3]

$$\epsilon\, y(x) = \int_0^x (1 + t + y(t))\, dt \quad (24)$$

which has the following exact solution:

$$y(x) = x + 1 - e^{-x/\epsilon} - \epsilon\left(1 - e^{-x/\epsilon}\right)$$

We applied present method and solved Eq. (24) for different value of N . Table 2 shows the L2-error, L∞-error norms, in some values of ε obtained for N = 16 and M=3,5,7. Then the figure 1 the exact and (PIM) solution for N=16 , M=7 and $\varepsilon = 2^0, 2^{-1}, 2^{-2}, ..., 2^{-5}$ is represented.

Table 2: Error normal 0f MQ Radial (PIM) result with c=0.1 in proplem 1.

	N=16 & M=3		N=16 & M=5		N=16 & M=7	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
2^-0	2.28e-012	9.48e-013	3.91e-012	1.69e-012	2.74e-012	1.49e-012
2^-1	5.20e-006	2.88e-006	1.78e-007	8.89e-008	2.28e-009	1.12e-009
2^-2	8.71e-005	4.99e-005	8.65e-006	4.66e-006	3.99e-007	2.14e-007
2^-3	1.12e-003	6.57e-004	2.38e-004	1.38e-004	3.29e-005	1.91e-005
2^-4	4.48e-003	2.57e-002	4.63e-003	2.65e-003	1.33e-003	7.62e-004
2^-5	7.20e-002	3.56e-003	2.00e-002	1.03e-002	2.82e-002	1.45e-002

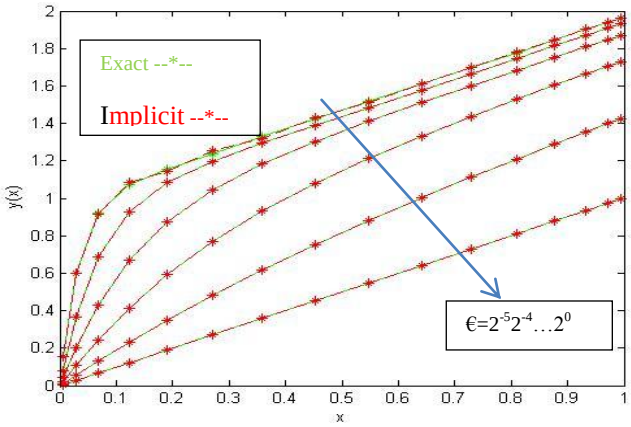

Figure 1: comparison between (PIM) approximation of y(x) and exact solution for different of $\boldsymbol{\varepsilon}$ in problem 1

5.2 Problem 2

Similar to previous problem, in this problem we consider the following singularly perturbed Volterra integro-differential equation [4]

$$\epsilon \frac{d}{dx} y(x) + y(x) + \int_0^x t y(t) dt = (2 + 9x + \epsilon x + 11x + x^2) e^{-1} \qquad (25)$$

$$-10(\epsilon x + \epsilon^2) e^{-x/\epsilon} + 5x^2 + 10\,\epsilon^2 - 2.$$

With the initial condition $y(0) = 10$. The exact solution is

$$y(x) = 10 - (10 + x) e^{-x} + 10 e^{-x/\epsilon}.$$

We applied present method and solved Eq. (25) for different value of N . Table 2 shows the L2-error, L∞-error norms, in some values of ε obtained for N = 16 and M=3,5,7. Then in Figure 4 the exact and (PIM) solutions for N =16 and M=7. $\varepsilon = 2^0, 2^{-1}, 2^{-2}, ..., 2^{-5}$ is represented.

Table 5: Error normal 0f MQ Radial (PIM) result with c=0.1 in problem 2.

	N=16 & M=3		N=16 & M=5		N=16 & M=7	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
2^-0	1.09e-002	5.27e-003	2.39e-004	1.18e-004	1.04e-008	5.28e-009
2^-1	9.41e-002	4.27e-002	4.96e-003	2.29e-003	1.96e-006	9.37e-007
2^-2	2.13e-001	9.56e-002	3.30e-002	1.49e-002	1.71e-004	7.21e-005
2^-3	3.12e-001	1.47e-001	1.18e-001	5.56e-002	5.86e-003	2,56e-003
2^-4	3.71e-001	1.64e-001	2.26e-001	1.05e-001	5.61e-002	2.56e-003
2^-5	6.56e-001	4.97e001	4.95e-001	3.69e-001	2.93e-001	2.22e-001

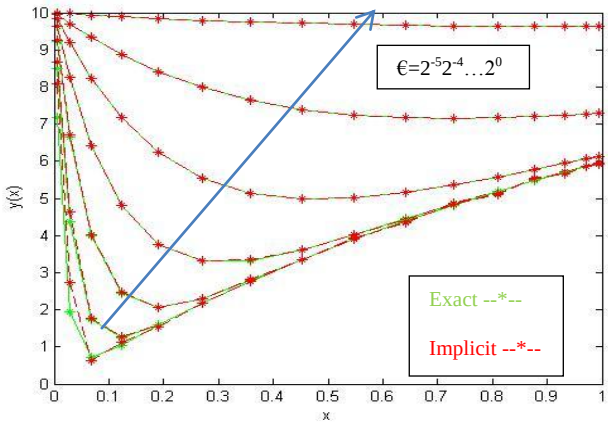

Figure 4: comparison between(PIM) approximation of y(x) and exact solution for different of ε in problem 2

6-Conclusion

In this paper, we discussed the some of integral equations which have the singularly and perturbed properties. We proposed a numerical scheme to solve this equations using collocation points and approximating the solution using the multiquadric (MQ) radial a point interpolation meshless method. For convenience the solutions we used RBFs with collocation nodes which are the zeros of the shifted Legendre polynomial. Additionally, through the comparison with exact solutions. We show that the radial a point interpolation methods (PIM) have good accuracy and efficiency and results obtained using the PIMs method are with low error.

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