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# Application of Point Interpolation Meshless Method for the Numerical Solution of Two-Dimensional Singularly Perturbed Integro-differential Equations

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**Abstract**

Recently, there has been an increasing interest in the study of singular and perturbed systems. In this paper we propose a point interpolation meshless method for solving two-dimensional singularly perturbed integro-differential equations. The results of numerical experiments show that the numerical scheme is very effective and convenient for solving a large number of singularly perturbed problems with high accuracy.

**Keywords**: singularly perturbed problems; Volterra integral equations; Volterra integro-differential equations

## 1- INTRODUCTION

As we know ,much work has been done on developing and analyzing numerical methods for solving one-dimensional integro-differential equation of the second kind, but in two-dimensional cases a small amount of work has been done. In the present work, we consider the two-dimensional singularly perturbed Volterra integro-differential equations (SVIDE)

$$\varepsilon \frac{\partial}{\partial x} u(x,y) = g(x,y,\varepsilon,u(x,y)) + \int_0^x \int_0^y k(x,y,\varepsilon,u(t,s))dsdt \quad \mathrm{x,y} \in \mathrm{I} = [0,\mathrm{x}]\times[0,\mathrm{y}], \tag{1}$$
$$u(0,y) = \alpha$$

where α is a constant and ε is a known perturbation parameter which 0 < ε ≤ 1.

Here, ε is small parameter that given rise to singularly perturbed nature of the problem, the kernel K and the data function g(x) are given smooth functions . under appropriate condition on g and K, for every ε > 0, Eq. 1 has unique continuous solution on [0,x] x [0,y].[1,2,3,10].

The singularly perturbed nature of (1) arises when the properties of the solution with ε > 0 we incompatible with those when perturbed of the ε = 0. for ε > 0, (1) is an integral equation of the second kind which typically is well posed whenever K is sufficiently well behaved. When ε = 0, (1) reduced to an integral equation of the first kind whose solution may well be incompatible with the case for ε >0. The interest here is in those problem which do imply such an incompatibility in the behavior of u near x =0, this suggests the existence of the boundary layer near the origin where the solution undergoes a rapid transition [1,2,3,10].

A point interpolation method(PIM) was proposed to address above two issues[11,12] .the (PIM) seems attractive in several ways, first, its approximation function passes through each node in an influence domain, second, its shape function are simple compared with any other method, third, is shape function and derivatives are easily developed only if basis function are selected,

This paper proposes a point interpolation meshless method based on radial basis function for the solution of two-dimensional singularly perturbed Volterra integro-differential equations. this forms a radial PIM, particularly, multiquadric radial basis function [9] are applied in the radial PIM. Recently, collocation method were developed, e.g in refernces [4,7].

This paper is arranged as follows: in Section 2, the properties of radial (PIM) functions was described . In Section 3, the Legendre-Gauss-Lobatto nodes and weights was introduced .In Section 4, the problem with the proposed method was implemented and in Section 5, our numerical finding and demonstrate the accuracy of the proposed methods were reported. The conclusions are discussed in the final Section.

# 2- RADIAL BASIS FUNCTION

## Definition of radial basis function

Let $R^+ = \{x \in R, x \geq 0\}$ be the non-negative half-line and let $B: R^+ \to R$ be a continuous function with $B(0) \geq 0$. A radial basis functions on $R^d$ is a function of the form $B(\| X - X_i \|)$, where $X, X_i \in R^d$ and $\| . \|$ denotes the Euclidean distance between X and $X_i$. If one chooses N points $\{x\}_{j=1}^{N}$ in $R^d$ then by custom

$$s(x) = \sum_{i=1}^{N} \lambda_i B(\|x - x_i\|), \qquad \lambda_i \in R \tag{2}$$

is called a radial basis functions as well [ 8].

## 2.1 POINT INTERPOLATION BASED ON RADIAL BASIS FUNCTION

Consider an approximation function u(x) in an influence domain that has a set of arbitrarily distributed nodes $P_i$ (x) (i =1,2,…, n). n is the number of nodes in the influence domain of x .Nodal function value is assumed to be $u_i$ at the node( $x_i$) . Radial PIM constructs the approximation function u(x) to pass through all these node points using radial basis function $B_i(x)$ and polynomial basis function $p_j$ (x). [13]

$$u(x) = \sum_{i=1}^{n} a_i B_i(x) + \sum_{j=1}^{m} b_j B_j(x) = B^T(x)a + p^T(x)b \tag{3}$$

where $a_i$ is the coefficient for $B_i$ (x) and $b_j$ the coefficient for $p_j$ (x) (usually, m< n). The vectors are defined a

$$\begin{aligned} a^T &= [a_1, a_2, a_3, ..., a_n] \\ b^T &= [b_1, b_2, b_3, ..., b_m] \\ B^T(x) &= [B_1(x), B_2(x), B_3(x), ..., B_n(x)] \\ P^T(x) &= [P_1(x), P_2(x), P_3(x), ..., P_m(x)] \end{aligned} \tag{4}$$

Basis function are usually the function of co-cordinates $x^T$=[x,y] for two-dimensional problems. A radial basis functions has the following general form;

$$B_i(x) = B_i(r_i) = B_i(x,y)$$

Where $r_i$ is the distance between interpolating point (x,y) and the node $(x_i,y_i)$. This distance in the Euclidean two-dimensional space is expressed as

$$r_i = \left[(x-x_i)^2 + (y-y_i)^2\right]^{1/2}$$

A polynomial basis function has the following monomial terms as:

$$P^T = [\,1, x, y, x^2, xy, y^2, \ldots] \tag{5}$$

The coefficients $a_i$ and $b_j$ in Equation (1) are determined by enforcing the interpolation pass through all n scattered nodal points within the influence domain. The interpolation at the kth point has

$$u_k(x_k,y_k) = \sum_{i=1}^{n} a_i B_i(x_k,y_k) + \sum_{j=1}^{m} b_j B_j(x_k,y_k), \quad \mathrm{k} = 1,2,\ldots,n \tag{6}$$

The polynomial term is an extra-requirement that guarantees unique approximation,[15]. Following constraints are usually imposed:

$$\sum_{i=1}^{n} P_j(x_i,y_i)a_i = 0, \quad \mathrm{j} = 1,2,\ldots,\mathrm{m} \tag{7}$$

It is expressed in matrix form as follows:

$$\begin{bmatrix} B_0 & P_0 \\ B_0^T & 0 \end{bmatrix}\begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} u^e \\ 0 \end{Bmatrix} \quad \text{or} \quad \mathrm{G}\begin{Bmatrix} \mathrm{a} \\ \mathrm{b} \end{Bmatrix} = \begin{Bmatrix} u^e \\ 0 \end{Bmatrix} \tag{8}$$

where the vector for function values is defined as

$$u^e = [u_1, u_2, u_3, \ldots, u_n]^T$$

The coefficient matrix $B_0$ on unknowns $a_i$, i=1,2,…,n is

$$B_0 = \begin{bmatrix} B_1(x_1,y_1) & B_2(x_1,y_1) & B_n(x_1,y_1) \\ B_1(x_2.y_2) & B_2(x_2,y_2) & B_n(x_2,y_2) \\ & & \\ B_1(x_n,y_n) & B_2(x_n,y_n) & B_n(x_n,y_n) \end{bmatrix}_{n\times n} \tag{9}$$

The coefficient matrix $P_0$ on unknowns b is

$$P_0 = \begin{bmatrix} P_1(x_1,y_1) & P_2(x_1,y_1) & P_m(x_1,y_1) \\ P_1(x_2.y_2) & P_2(x_2,y_2) & P_m(x_2,y_2) \\ & & \\ P_1(x_n,y_n) & P_2(x_n,y_n) & P_m(x_n,y_n) \end{bmatrix}_{n\times m} \tag{10}$$

Because the distance is directionless, there is $B_k(x_i,y_i) = B_i(x_k,y_k)$, which means that the matrix $B_0$ is symmetric. Unique solution is obtained if the inverse of matrix $B_0$ exists,

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = G^{-1}\begin{Bmatrix} u^e \\ 0 \end{Bmatrix} = \Lambda$$

The interpolation is finally expressed as

$$u(x,y) = \left[B^T(x,y) \quad \mathrm{P}^{\mathrm{T}}(x,y)\right]G^{-1}\begin{Bmatrix} u^e \\ 0 \end{Bmatrix} = \Phi(x,y)\Lambda \tag{11}$$

where the matrix of shape functions $\Phi(x)$ is defined by

$$\Phi(x,y) = \left[\Phi_1(x,y), \Phi_2(x,y), \ldots, \Phi_n(x,y)\right] \tag{12}$$

in which

$$\Phi_k(x,y) = \sum_{i=1}^{n} B_i(x,y) + \sum_{j=1}^{m} p_j(x) \tag{13}$$

After radial basis functions are determined, shape functions depend only upon the position of scattered nodes. Once the inverse of matrix G is obtained, the derivatives of shape fun- ctions are easily obtained as

$$\frac{\partial \Phi_k}{\partial x} = \sum_{i=1}^{n} \frac{\partial B_i}{\partial x} + \sum_{j=1}^{m} \frac{\partial P_j}{\partial x} \tag{14}$$

$$\frac{\partial \Phi_k}{\partial y} = \sum_{i=1}^{n} \frac{\partial B_i}{\partial y} + \sum_{j=1}^{m} \frac{\partial P_j}{\partial y}$$

The results of this section can be summarized in the following algorithm.

**Algorithm**

The algorithm works in the following manner:

Choose N center poist $\{(x_j, y_j)\}_{j=1}^{N}$ from the domain set [a, b]x[a,b].

1. Approxime $u(x,y)$ as $u_N(x) = \mathbf{\Phi}^T(x,y)\mathbf{\Lambda}$.
2. Substitute $u_{/}(x,y)$ into the main problem and creat residual function $Res(x,y)$.
3. Substitute collocation points $\{(x_j, y_j)\}_{j=1}^{N}$ into the $Res(x,y)$ and create the $N$ equations.
4. Solve the $N$ equations with $N$ unknown coefficients of members of $\mathbf{\Lambda}$ and find the numerical solution.

## 3 -LEGENDER-GAUSS-LOBATTO NODES AND WEIGHTS

Let $H_N[-1, 1]$ denote the space of algebraic polynomials of degree $\leq N$

$$\langle p_i, p_j\rangle = \frac{2}{2j+1}\delta_{ij}$$

Here, $<\ .,.\ >$ represent the usual $L^2[-1, 1]$ inner product and $\{p_i\}, i>0$ are the well-known Legendre polynomials of order $i$ which are orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-1, 1]$, and satisfy the following formulae:

$$P0(x) = 1, \quad P1(x) = x,$$

$$p_{i+1}(x) = (\frac{2i+1}{i+1})xp_i(x) - (\frac{i}{i+1})p_{i-1}(x), \qquad i=1,2,...$$

Next, we let $\{x_j\}_{j=1}^{N}$ as

$$(1 - x_j^2)\,p'(x_j) = 0$$

where P˙ (x) is

$$-1 = x_0 < \mathrm{x}_1 < x_2 < ... < x_N = 1$$

derivative of P (x). No explicit formula for the nodes is known. However, they are computed numerically using existing subroutines[5,6] Now, we assume $f \in H_{2N-1}[-1,1]$, we have

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dydx \cong \sum_{i=1}^{N}\sum_{j=0}^{N} w_i w_j f(x_j, y_j) = I_G(f), \tag{15}$$

Where are $w_j$ the Legendre-Gauss-Lobatto weights given in [11]

$$w_j = \frac{2}{N(N+1)} \times \frac{1}{(p_N(x_j))^2}$$

## 4- SOLUTION OF SVIDEs VIA PIM

In the present method, the closed form PIM approximating function Eq. (11) is first obtained from a set of training points, and its derivative of any order, e.g. *p*th order, can then be calculated in a straightforward manner by differentiating such a closed form DRBF as follows:

$$u(x,y) \cong u_N(x,y) = \sum_{i=1}^{N} \Lambda_i \Phi_i(x,y) = \Phi^T(x,y)\,\Lambda, \tag{16}$$

$$\frac{\partial}{\partial x}u(x,y) = \dot{u}_N(x,y) = \sum_{i=1}^{N} \Lambda_i \Phi_i(x,y) = D\Phi^T(x,y)\Lambda \tag{17}$$

where

$$D\Phi^T(x,y) = \left[\Phi'_1, \Phi'_2, ..., \Phi'_N\right]$$

Then ,from substituting Eq.( 16) and Eq.( 17) into. (1 ), we have

$$\varepsilon D\Phi^T(x,y)\Lambda = g(x,y,\varepsilon,\Phi^T(x,y)\Lambda) + \int_0^{x}\int_0^{y} k(x,y,t,s,\varepsilon,\Phi^T(x,y)\Lambda)\,\mathrm{ds\,dt} \quad (18)$$

$$u_N(0,y) = \alpha$$

We now collocate Eq. (18 ) at point $\{x_j, y_j\}_{j=1}^{N}$ as

$$\varepsilon D\Phi^T(x_j,y_j)\Lambda = g(x_j,y_j,\varepsilon,\Phi^T(x_j,y_j)\Lambda) + \int_0^{x_j}\int_0^{y_j} k(x_j,y_j,t,s,\varepsilon,\Phi^T(t,s)\Lambda)\,\mathrm{ds\,dt} \quad (19)$$

$$u_N(0,y) = \alpha$$

In order to use the Legendre-Gauss-Lobatto integration formula for Eq. (15), we tran- sfer the t-intervals[0, $x_i$] into the η-intervals [−1, 1],by means of the transformations .$\eta = 2(t-1)/x_i$ and transfer the S-intervals [0, $y_i$] into the γ -intervals [−1, 1],by means of the Ttransformations . $\gamma = 2(s-1)/y_i$ Then Eq. ( 19) may be restated as the residual function Res($x_j$,,$y_j$)

$$\begin{aligned}\mathrm{Res}(x_j,y_j) &= -\varepsilon D\Phi^T(x_j)\Lambda + g(x_j,y_j,\varepsilon,\Phi^T(x_j,y_j)\Lambda) \\ &\quad + I_G(K(x_j,y_j,x_j(\eta+1)/2,y_j(\gamma+1)/2,\varepsilon,\Phi^T(x_j(\eta+1)/2,y_j(\gamma+1)/2)\Lambda)\end{aligned} \quad (20)$$

$$\Phi^T(0,y)\Lambda = \alpha \quad (21)$$

The set of equations for obtaining the coefficients $\{\Lambda_j\}_{j=1}^{N}$ come from equalizing Eq. (20) to zero at $N-1$ interpolate nodes $\{\Lambda_j\}_{j=1}^{N}$ plus Eq. (21). behavior of the MQ-RBF method, we applied the following laws

1- The $L_2$ error norm of the solution which is defined by

$$L_2 = \left\| y^{exact}(x,y) - y_N^{implicit}(x,y) \right\|_2 = \left[ \sum_{j=1}^{N} \left( y^{exact}(x_j,y_j) - y_N^{implicit}(x_j,y_j) \right)^2 \right]^{\frac{1}{2}}$$

2-where $\{(x_j,y_j)\}_{j=1}^{N}$ are interpolate nodes which are the zeros of shifted Legendre polynomial $L_N$ (x) ,$0 \le x \le 1$ and y = ( 0,0.05,0.1,…,1);

The $L_\infty$ error norm of the solution which is defined by

$$L_\infty = \left\| y^{exact}(x,y) - y_N^{implicit}(x,y) \right\|_\infty = \max_{0\le j\le N} \left| (y^{exact}(x_j,y_j) - y_N^{implicit}(x_j,y_j)) \right|$$

## 5- NUMERICAL RESULTS

In order to illustrate the performance of radial point interpolation meshless method (PIM) in solving SVIDES and justify the accuracy and efficiency of our method, we consider the following examples. In all examples we use multiquadrics (MQ) RBF.

**Problem 1**

In this problem, we consider the following singularly perturbed Volterra integral equation

$$\varepsilon\, u(x,y) + \int_0^x\int_0^y x^2 y^3 u(t,s)\, ds\, dt = x^2 + y^3 + \frac{x^5 y^4}{\varepsilon} + \frac{x^3 y^7}{\varepsilon} \qquad (22)$$

which has the following exact solution:

$$u(x,y) = (x^2 + y^3)/\varepsilon$$

We applied present method and solved Eq. (22) for different value of N and M . Table 1 shows the $L_2$-error, $L_\infty$-error norms, in some values of $\varepsilon = 2^0, 2^{-1}, 2^{-2}, \ldots, 2^{-4}$ obtained .

Table 1: Error normal 0f MQ Radial (PIM) result with c=0.1 in proplem 1.

| | N=15 & M=5 | | N=15 & M=10 | | N=15 & M=15 | |
|---|---|---|---|---|---|---|
| | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ |
| 2^ -0 | 9.68e-004 | 4.95e-004 | 3.62e-012 | 1.88e-012 | 2.34e-013 | 9.93e-014 |
| 2^ -1 | 3.87e-003 | 1.98e-003 | 1.30e-012 | 5.35e-013 | 5.59e-012 | 2.37e-012 |
| 2^ -2 | 1.55e-002 | 7.91e-003 | 2.88e-011 | 1.40e-011 | 5.93e-012 | 2.52e-012 |
| 2^-3 | 6.19e-002 | 3.16e-002 | 1.24e-010 | 5.96e-011 | 1.15e-012 | 4.94e-013 |
| 2^ -4 | 2.48e-001 | 1.26e-001 | 3.62e-010 | 1.74e-010 | 1.09e-012 | 5.39e-013 |

**Problem 2**

In this problem, we consider the following nonlinear singularly perturbed Volterra integral equation

$$\varepsilon\, u(x,y) - \int_0^x\int_0^y (x-y)\; u^2(t,s)\, ds\, dt = \varepsilon\, x\, y - \frac{x^4 y^3}{12} + \frac{x^3 y^4}{12} \qquad (23)$$

which has the following exact solution:

$$u(x,y) = x\, y$$

We applied present method and solved Eq. (23) for different value of N and M .

Table 2 shows the $L_2$-error, $L_\infty$-error norms, in some values of $\varepsilon = 2^0, 2^{-1}, 2^{-2}, \ldots, 2^{-4}$ obtained.

Table 2: Error normal 0f MQ Radial (PIM) result with c=0.1 in proplem 2.

| | N=10 & M=3 | | N=10 & M=5 | | N=10 & M=8 | |
|---|---|---|---|---|---|---|
| | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ |
| $2^{\wedge -0}$ | $1.28e^{-002}$ | $7.52e^{-003}$ | $2.36e^{-005}$ | $1.77e^{-005}$ | $1.28e^{-005}$ | $8.24e^{-006}$ |
| $2^{\wedge -1}$ | $2.57e^{-002}$ | $1.51e^{-002}$ | $4.61e^{-005}$ | $3.23e^{-005}$ | $7.93e^{-005}$ | $5.15e^{-005}$ |
| $2^{\wedge -2}$ | $4.44e^{-002}$ | $2.62e^{-002}$ | $7.82e^{-005}$ | $5.58e^{-005}$ | $3.53e^{-004}$ | $2.14e^{-004}$ |
| $2^{\wedge -3}$ | $6.29e^{-002}$ | $3.83e^{-002}$ | $1.31e^{-004}$ | $9.55e^{-005}$ | $1.04e^{-003}$ | $6.16e^{-004}$ |
| $2^{\wedge -4}$ | $8.29e^{-002}$ | $4.90e^{-002}$ | $2.33e^{-004}$ | $1.71e^{-004}$ | $2.24e^{-003}$ | $1.31e^{-003}$ |

**Problem** 3

In this problem, we consider the following integro-differentioal singularly perturbed Volterra integral equation

$$\varepsilon \frac{\partial}{\partial x} \mathrm{u}(\mathrm{x},\mathrm{y}) - \int_0^{\mathrm{x}}\int_0^{\mathrm{y}} (\mathrm{t}+\mathrm{s})\, u(t,s)\, \mathrm{ds\, dt} = \frac{(\mathrm{y}+1)}{\varepsilon} - \frac{x^3 y^2}{6} - \frac{x^3 y}{3} - \frac{x^2 y^3}{6} - \frac{t^2 y^2}{2} \quad (24)$$

which has the following exact solution:

$$u(x,y) = \ xy + x$$

We applied present method and solved Eq. (24) for different value of N and M . Table 3 shows the $L_2$-error, $L_\infty$-error norms, in some values of $\varepsilon = 2^0, 2^{-1}, 2^{-2}, \ldots, 2^{-4}$ obtained .

Table 3: Error normal 0f MQ Radial (PIM) result with c=0.1 in proplem 3.

| | N=7 & M=6 | | N=15 & M=6 | | N=20 & M=6 | |
|---|---|---|---|---|---|---|
| | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ |
| $2^{\wedge -0}$ | $1.18e^{-009}$ | $9.89e^{-010}$ | $5.18e^{-010}$ | $2.71e^{-010}$ | $4.88e^{-013}$ | $2.03e^{-013}$ |
| $2^{\wedge -1}$ | $1.99e^{-009}$ | $1.66e^{-009}$ | $1.80e^{-011}$ | $9.08e^{-012}$ | $5.88e^{-013}$ | $2.13e^{-013}$ |
| $2^{\wedge -2}$ | $3.64e^{-010}$ | $2.98e^{-010}$ | $3.96e^{-012}$ | $1.83e^{-012}$ | $9.94e^{-014}$ | $3.19e^{-014}$ |
| $2^{\wedge -3}$ | $7.89e^{-011}$ | $6.16e^{-011}$ | $1.62e^{-012}$ | $6.15e^{-013}$ | $2.09e^{-014}$ | $7.44e^{-015}$ |
| $2^{\wedge -4}$ | $4.44e^{-011}$ | $2.99e^{-011}$ | $6.58e^{-012}$ | $2.46e^{-013}$ | $4.73e^{-015}$ | $1.67e^{-015}$ |

## 6- CONCLUSIONS

In this paper, some of two dimensional integral equations which have the singularly and perturbed properties were discussed. Numerical scheme to solve this equations using collocation points and approximating the solution using the multiquadric (MQ) radial a point interpolation meshless method were proposed. For convenience the solutions we used RBFs with collocation nodes. Additionally, through the comparison with exact solutions. We show that the radial a point interpolation methods (PIM) have good accuracy and efficiency and results obtained using the PIMs method are with low error.

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