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On direct theorems for best polynomial approximation

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Abstract This paper is to obtain similarity for the best approximation degree of functions which are unbounded in $L_{p,\alpha}(A = [0,1])$which called weighted space by algebraic polynomials $E_n^H(f)_{p,\alpha}$ and the best approximation degree in the same space on the interval $[0,2\pi]$ by trigonometric polynomials $E_n^T(f)_{p,\alpha}$ of direct wellknown theorems in forms the average modules

1 Introduction

Recently several papers have been devoted to study polynomial approximation with constraints In particular [16] polynomial approximation was considered Although several problems concerning algebraic approximation or trigonometric approximation have been studied the theory as well as developed as the one classical polynomial approximation In this paper we present some algebraic polynomial and trigonometric polynomials to approximate unbounded functions in terms average modules in weighted space $L_{p,\alpha}([0,1])$ and $L_{p,\alpha}([0,2\pi], (1 \leq p < \infty)$ respectivly Approximation of functions have integrable bounded derivative $f^{(k)}$ on closed intervals well be considered in other papers Let $A = [0,1]$ in algebraic polynomials $A = [0,2\pi]$ in trigonometric polynomials approximation W be the suitable set of all weight functions on $A \ni |f(x)| \leq M\alpha(x)\ \forall x \in A\ M$ positive real number $\alpha: A \to \mathbb{R}^+$weight function and denoted by $L_{p,\alpha}(A)$ the space of unbounded functions $f: A \to \mathbb{R}, (1 \leq p < \infty)$ with the norm defined by:

$$\|f\|_{p,\alpha} = (\int_A \left|\frac{f(x)}{\alpha(x)}\right|^p dx)^{1/p} \tag{1}$$

The kth average module is used in this paper

$$\tau_k(f,\delta)_{p,\alpha} = \|\omega_k(f, \cdot, \delta)\|_{p,\alpha} \tag{2}$$

Where

$$\omega_k(f, \cdot, \delta) = \sup\{\left|\Delta_h^k f(x)\right|; t, t+h \in [x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2},]\} \tag{3}$$

$$\Delta_h^k f(x) = \begin{cases} \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} f(x+ih) & if \quad x, x+h \in X \\ 0 & otherwise \end{cases} \tag{4}$$

And

$$\omega_k(f,\delta)_{p,\alpha} = \left\|\Delta_h^k f(\cdot)\right\|_{p,\alpha} \tag{5}$$

As far as we know average modules of this type were first used by Sendov and Popov (1998) use this modules for $p = 1$ for Housdorff approximation by pointwise monotonic functions and established several of its basic properties
Combine all algebraic polynomials on $A = [0,1]$ of degree equals n or less in a set say H_n The best approximation degree of f in the space $L_{p,\alpha}(A)$ is defined as:

$$E_n^H(f)_{p,\alpha} = \inf\left\{\|f - P_n\|_{p,\alpha};\ P_n \in H_n\right\} \tag{6}$$

Also define the set T_n which contains every trigonometric polynomials of order n or less Then the best approximation degree of $f \in L_{p,\alpha}(A)$ is defined by

$$E_n^T(f)_{p,\alpha} = \inf\left\{\|f - t_n\|_{p,\alpha};\ t_n \in T_n\right\} \tag{7}$$

2 Auxiliary Results

We shall use the following supplementary results:

2.1 Lemma [4]
Let f be an element in $L_{p,\alpha}(A), (1 \le p < \infty)$ Then

$$\tau(f, \lambda\delta)_{p,\alpha} \le (\lambda + 1)\tau(f, \delta)_{p,\alpha}$$

2.2 Lemma
Let f be an element in $L_{p,\alpha}(A), (1 \le p < \infty)$ Then

$$\tau_k(f, \lambda\delta)_{p,\alpha} \le (\lambda + 1)^k\ \tau_k(f, \delta)_{p,\alpha} \qquad \lambda > 0$$

Proof

We have $\omega_k(f, \lambda\delta) \le (\lambda + 1)^k\ \omega_k(f, \delta)$ see [4]
So from (2) $\tau_k(f, \lambda\delta)_{p,\alpha} = \|\ \omega_k\ (f\ ,\ \cdot\ , \lambda\delta)\|_{p,\alpha}$

$$\le \left\|(1 + \lambda)^k \omega_k(f,\ \cdot\ , \lambda\delta)\right\|_{p,\alpha}$$

$$= \left|(1 + \lambda)^k\right| \|\omega_k(f,\ \cdot\ , \lambda\delta)\|_{p,\alpha}$$

$$= (1 + \lambda)^k \tau_k(f, \lambda)_{p,\alpha}$$

2.3 Lemma
Let f be an element in $L_{p,\alpha}(A), (1 \le p < \infty)$
Then $\tau_k(f, \delta)_{p,\alpha} \le \delta\|f'\|_{p,\alpha}$

Proof

From (3) we have:

$$\omega(f, x, \delta) = \sup\{|f(x_1) - f(x_2)|, where\ x_1, x_2 \in [x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2}]\}$$

$$= \sup\{\left|\int_{x_1}^{x_2} f'(u)du\right|, where\ x_1, x_2 \in [x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2}]\}$$

$$\le \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} f(t)dt = \int_0^\delta f'(x - t)dt$$

Also equation (2) yields:

$$\tau_k(f, \delta)_{p,\alpha} = \|\omega_k(f\ ,\ \cdot\ ,\ \delta)\|_{p,\alpha}$$

$$\leq \left\| \int_0^\delta f'(x-t)dt \right\|_{p,\alpha}$$

$$\leq \int_0^\delta \|f'(x-t)\|_{p,\alpha} dt$$

$$= \delta \|f'\|_{p,\alpha}$$

More generally the kth average modules can be estimated by means of norm in weighted space of the kth derivative of the function f

$\tau_k(f,\delta)_{p,\alpha} \leq c(k)\delta^k \left\|f^{(k)}\right\|_{p,\alpha}$

Where c is absolute constant depended on k

24 Lemma

Let $A = [0,1]$, f be an element in , $L_{p,\alpha}(A), (1 \leq p < \infty)$ Then for any natural number n the following is true:

$$E_n^H(f)_{p,\alpha} \leq (n+1)E_{n-1}^n(f')_{p,\alpha}$$

Proof

Let $P_n \in H_n$, $f(0) = 0$, and from (6) we have

$$E_{n-1}(f')_{p,\alpha} = \inf\{\|f' - P_n\|_{p,\alpha},\ P_n \in H_n\}$$

For $x \in A$ we consider the polynomial where c is absolute constant

$$q_n(x) = \{\int_0^x p_n(u)du + \sum_{i=0}^{k-n-1} h_i \int_0^x g_i(u)du\} \geq 0$$

Set $p_n(x) = 0$ for $x < 0$ Then

$$\left(E_n(f)_{p,\alpha}(u)\right)^p \leq \|f - q_n\|_{p,\alpha}^p = \int_A \left|\frac{f(x)-q_n(x)}{\alpha(x)}\right|^p dx$$

$$= \int_A \left|\frac{\int_0^x (f'(u)-p_n(u))du - \sum_{i=0}^{x-n-1} h_i \int_0^x g(u)du}{\alpha(x)}\right|^p dx$$

$$\leq \int_A \left|\frac{\int_{x-n-1}^x (f'(u)-p_n(u))du}{\alpha(x)}\right|^p dx$$

$$= \int_0^{n+1} (\int_A \left|\frac{(f'(x-(n+1)+u)-p_n(x-(n+1)+u)}{\alpha(x)}\right|^p dx)du$$

$$\leq \int_0^{n+1} \left(E_{n-1}^H(f')_{p,\alpha}\right)^p du = (n+1)E_{n-1}^n(f')_{p,\alpha}$$

3 Main results

We consider the approximation of unbounded function $f(x)$ defined on [0,1] by algebraic polynomials

3.1Theorem

If f unbounded function defined on $A = [0,1]$ then

$$E_n^H(f)_{p,\alpha} \leq C(p)\tau(f,\delta)_{p,\alpha}\ ;\ (1 \leq p < \infty)$$

Where C is constant depends on p

Proof

Assume

$$g(x) = \sup_{u\in A} f(u) \tag{8}$$

It's clear that $g_n \in H_n$
From (3) we get

$$\omega(f,x,\delta) = \sup_{x\in A}\{|f(x) - g_n(x)|\}$$

So by using (1)(2)(3) and (6)

$$E_n^H(f)_{p,\alpha} = \inf\{\|f - g_n\|_{p,\alpha} \ , g_n \in H_n\}$$

$$\le \left[\int_A \left|\frac{f(t) - g_n(t)}{\alpha(t)}\right|^p dt\right]^{1/p}$$

$$\le \left[\int_A \sup\{\left|\frac{f(t)-g_n(t)}{\alpha(t)}\right|^p\}dt\right]^{1/p}$$

$$= \left[\int_A \{\left|\frac{\sup\{f(t)-g_n(t)\}}{\alpha(t)}\right|^p\}dt\right]^{1/p}$$

$$= \left[\int_A \{\left|\frac{\omega(f,t,\delta)}{\alpha(t)}\right|^p\}dt\right]^{1/p}$$

$$= \|\omega(f,t,\delta)\|_{p,\alpha}$$

$$\le C(p)\tau(f,\delta)_{p,\alpha}$$

3.2 Theorem

Let f be an element in $L_{p,\alpha}(A), A = [0,1]$, $(1 \le p < \infty)$ with nth derivative $f^{(n)}$ Then

$$E_n^H(f)_{p,\alpha} \le C(n+1)\tau(f^{(n)},\delta)_{p,\alpha}$$

Where C is absolute constant depended on p

Proof

By (23) Lemma and 31 Theorem we obtain

$$E_n^H(f)_{p,\alpha} \le (n+1)E_{n-1}^H(f')_{p,\alpha} \le (n+1)\cdot nE_{n-2}^H(f'')_{p,\alpha} \le \cdots \le (n+1)!\,E_0^H\left(f^{(n)}\right)_{p,\alpha}$$

$$\le C(p)(n+1)!\,\tau\left(f^{(n)},\delta\right)_{p,\alpha} \le C(p)\tau(f(u))_{p,\alpha}$$

Where

$$C = \max\{C(p)(n+1)!, C(p)\}. \quad \blacksquare$$

Now we can state the following theorems by trigonometric polynomials on $[0,2\pi]$

3.3 Theorem

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Let f be an element in $L_{p,\alpha}(A), A = [0,2\pi]$, $(1 \le p < \infty)$ then:

$$E_n^T(f)_{p,\alpha} \le K\tau\left(f, 1/n\right)_{p,\alpha}$$

Where K is constant

Proof

Put $$\boldsymbol{a_i = \frac{i\pi}{n}, \quad i = 0, 1, 2, \dots, 2n.}$$

$$b_i = \frac{a_i + a_{in}}{2}, \quad i = 0,1,2,\dots,2n$$

$$b_{2n+1} = b_1$$

We define the trigonometric polynomial t_n as following:

$$t_n(x) = \begin{cases} \sup\limits_{u \in [a_{i-1}, a_i]} \{f(x)\}\ for\ a_i = b_i, i = 0,1,2,\dots,2n \\ \max\{t_n(b_i), t_n(b_{i+1}) for\ x = a_i\ , i = 0,1,2,\dots,2n\} \\ t_n(0) = t_n(2\pi) \\ linear\ and\ continuous\ for\ x \epsilon [a_{i-1}, b_i\ ], i = 0,1,2,\dots,2n \end{cases}$$

It is clear that $t_n(x) \ge f(x)$ for $x \epsilon [0,2\pi] = X$ and $t_n \in T_n$
The derivative $t_n'(x)$ exist at every point in A
If $x \in [a_{i-1}, b_i\ ]$ then t_n is linear and continuous from definition of t_n

$$\text{So} \quad |f(x) - t_n(x)| \ \le \ \frac{2n}{\pi}\ \omega(f(x), \tfrac{4\pi}{n})$$

From (7) (21) Lemma

$$E_n^T(f)_{p,\alpha} \le \|f - t_n\|_{p,\alpha}$$

$$\le \sup\left(\int_A \left|\frac{\{f(t) - t_n(t)\}}{\alpha(t)}\right|^p dt\right)^{1/p}$$

$$\le \left(\int_A \left|\frac{\sup\{f(t) - t_n(t)\}}{\alpha(t)}\right|^p dt\right)^{1/p}$$

$$\le \left(\int_A \left|\frac{\omega(f, t, \frac{4\pi}{n}\}}{\alpha(t)}\right|^p dt\right)^{1/p}$$

$$\le \left\|\omega(f, \cdot, \tfrac{4\pi}{n}\right\|_{p,\alpha}$$

$$\le (4\pi + 1)\tau\left(f, 1/n\right)_{p,\alpha}$$

$$\le C\tau\left(f, 1/n\right)_{p,\alpha} \quad \blacksquare$$

3.4Theorem:

Let f be an element in $L_{p,\alpha}(A), A = [0,2\pi]$, $(1 \le p < \infty)$ then:

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$$E_n^T(f)_{p,\alpha} \le (\tfrac{c}{n})^k \tau\big(f^{(k)},\delta\big)_{p,\alpha} where\ \delta = \tfrac{1}{n}$$

C is absolute constant

Proof:

From (33) Theorem (21) Lemmas and (22) we get

$$E_n^T(f)_{p,\alpha} \le (\tfrac{c}{n})^k\ \tau\left(f,\frac{1}{n}\right)_{p,\alpha}$$

$$\le (\tfrac{c}{n})^k \left\|f^{(k)}\right\|_{p,\alpha}$$

$$\le (\tfrac{c}{n})^k \underbrace{\sup}_{0<h<\delta} (\textstyle\int_A \{\left|\frac{f^{(k)}(t)-f^{(k)}(t+h)}{\alpha(t)}\right|^p\}dt)^{1/p}$$

$$\le (\tfrac{c}{n})^k (\textstyle\int_A \{\left|\frac{\sup\{f^{(k)}(t)-f^{(k)}(t+h)\}}{\alpha(t)}\right|^p\}dt)^{1/p}$$

$$\le (\tfrac{c}{n})^k (\textstyle\int_A \{\left|\frac{\omega(f^{(k)},t,\delta)}{\alpha(t)}\right|^p\}dt)^{1/p}$$

$$\le \left(\frac{c}{n}\right)^k \tau\big(f^{(k)},\delta\big)_{p,\alpha} \quad \blacksquare$$

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