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An Application of Subclasses of Goodman-Salagean-Type Harmonic Univalent Functions Involving Hypergeometric Function

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Abstract: The main aim of this paper is to create ties between different kinds of documents. By adding a certain convolution operator involving hypergeometric functions, subclasses of harmonic univalent functions. In the open unit disk $\mathcal{L}$, we examine certain relations with Goodman-Salagean-Type harmonic univalent functions.

Key words: Univalent functions, Starlike functions, convex functions, linear operator, Hypergeometric functions, Hadamard product.

INTRODUCTION

Let $\mathcal{L} = \{w \in \mathbb{C} : |w| < 1\}$ be an open unit disc in $\mathbb{C}$. Let $H(\mathcal{L})$ be the analytic functions class in $\mathcal{L}$ and let $\mathcal{L}[a, i]$ be the subclass of $H(\mathcal{L})$ of the form

$$g(w) = a + a_\iota w^\iota + a_{\iota+1} w^{\iota+1} + \cdots ,$$

where $a \in \mathbb{C}$ and $\iota \in \mathbb{N} = \{1,2,\ldots\}$ with $H_0 \equiv H[0,1]$ and $H \equiv H[1,1]$. Let Ψ denote the class of form's analytic functions:

$$g(w) = w + \sum_{\iota=2}^{\infty} a_\iota w^\iota, \quad (a_\iota \geq 0, \iota \in \mathbb{N} = \{1,2,3,\ldots\}, w \in \mathcal{L}). \tag{1}$$

The convolution operator $\mathrm{B}(a, c;\ d)$ has been introduced by Hohlov [12]: $\Psi \to \Psi$ defined by

$$\mathrm{B}(a, c;\ d) g(w) = w\mathrm{E}(a, c; d;\ w) * g(w),$$

where $\mathrm{E}(a, c; d;\ w)$ is a well known Gaussian hypergeometric function and defined by

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$$\mathrm{E}(a,c;d;\,w)=\sum_{\iota=0}^{\infty}\frac{(a)_\iota\,(c)_\iota}{(d)_\iota\,(1)_\iota}w^\iota\,,(a,c,d\in\mathbb{C}\ \text{such that}\ \ d\neq 0,-1,-2,\dots)$$

In $\mathcal{L}$, a hypergeometric function $\mathrm{E}(a,c;d;\,w)$ is analytical and plays an important role in the theory of geometric functions. See Branges[8], Ahuja[2], Carleson and Shaffer[6], Owa and Srivastava[16], Miller and Mocanu[15], Ruscheweyh and Singh[17], Srivastava and Manocha[18], and Swaminathan[19] for their studies.

For a function $\mathcal{g}\in\Psi$ given by Eqn. (1) and $\mathrm{I}\in\Psi$ defined by

$$\phi(w)=w+\sum_{\iota=2}^{\infty}c_\iota\,w^\iota\,,\qquad (w\in\mathcal{L}),\tag{2}$$

we define the Hadamard product of $\mathcal{g}$ and ϕ by

$$(\mathcal{g}*\phi)(w)=w+\sum_{\iota=2}^{\infty}a_\iota\,c_\iota\,w^\iota\,,\qquad (w\in\mathcal{L}).\tag{3}$$

Let F be the family of all harmonic functions $\mathcal{g}=h+\bar{\phi}$, where

$$h(w)=w+\sum_{\iota=2}^{\infty}A_\iota\,w^\iota\ ,\quad \phi(w)=\sum_{\iota=1}^{\infty}C_\iota\,w^\iota\,,\qquad (|C_1|<1,\ \ w\in\mathcal{L})\tag{4}$$

are in class Ψ. We define the functions $\psi_1=w\mathrm{E}(a_1,c_1;d_1;\,w)$ and $\psi_2=w\mathrm{E}(a_2,c_2;d_2;\,w)$ for complex parameters a_1,c_1,d_1,a_2,c_2,d_2 ($d_1\,,d_2\neq 0,-1,-2,\dots$).

We consider the following convolution operator to fit these functions

$$\mho=\mho\begin{pmatrix}a_1, & c_1, & d_1\\ a_2, & c_2, & d_2\end{pmatrix}:\mathcal{F}\to\mathcal{F},$$

defined by

$$\mho\begin{pmatrix}a_1, & c_1, & d_1\\ a_2, & c_2, & d_2\end{pmatrix}\mathcal{g}=\ \mathcal{g}*(\psi_1+\overline{\psi_2})=h*\psi_1+\overline{\phi*\psi_2}$$

for any function $\mathcal{g}=h+\bar{\phi}$, in $\mathcal{F}$. Letting

$$\mho\begin{pmatrix}a_1, & c_1, & d_1\\ a_2, & c_2, & d_2\end{pmatrix}\mathrm{E}(w)=H(w)+\overline{\Phi(w)}\,,$$

we have

$$H(w)=w+\sum_{\iota=2}^{\infty}\frac{(a_1)_{\iota-1}(c_1)_{\iota-1}}{(d_1)_{\iota-1}(1)_{\iota-1}}A_\iota\,w^\iota\ ,\ \ \Phi(w)=\sum_{\iota=1}^{\infty}\frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}}C_\iota\,w^\iota,\quad (w\in\mathcal{L})\tag{5}$$

We observe that

$$\mho\begin{pmatrix}a_1, & 1, & a_1\\ a_2, & 1, & a_2\end{pmatrix}\mathcal{g}(w)=\mathcal{g}(w)=\ \mathcal{g}(w)*\left(\frac{w}{1-w}+\overline{\frac{w}{1-w}}\right),$$

it's the identity mapping method.

In[4], the author described and studied this convolution operator $\mho$. Denote by $S_{\mathcal{F}}$ the subset of $\mathcal{F}$ in $\mathcal{L}$ that is uniform and sense-preserving.

Note that $\frac{g-\overline{C_1 g}}{1-|C_1|^1} \in S_{\mathcal{F}}$ whenever $g \in S_{\mathcal{F}}$. We also let the subclass $S_{\mathcal{F}}^0$ of $S_{\mathcal{F}}$

$$S_{\mathcal{F}}^0 = \{g = h + \bar{\phi} \in S_{\mathcal{F}} : \phi'(0) = C_1 = 0\}.$$

In [7], the classes $S_{\mathcal{F}}^0$ and $S_{\mathcal{F}}$ were first studied. We also let $K_{\mathcal{F}}^0, S_{\mathcal{F}}^{*,0}$ and $C_{\mathcal{F}}^0$ denote the subclasses of $S_{\mathcal{F}}^0$ of harmonic functions that are convex, starlike and close-to-convex in $\mathcal{L}$, respectively. One can refer to ([5,7,10]) or [9] for definitions and properties of these classes.

For $0 \leq \tau < 1, \mu \in \mathbb{N}$ and $m \in \mathbb{N}_0 = \{0,1,2, \ldots\}$, let

$$N_{\mathcal{F}}(\tau) = \left\{g \in \mathcal{F} : \mathcal{R}e \frac{g'(w)}{w'} \geq \tau, w = re^{i\gamma} \in \mathcal{L}\right\},$$

$$\Phi_{\mathcal{F}}(\tau) = \left\{g \in \mathcal{F} : \mathcal{R}e \left\{\left(1 + \rho e^{i\theta}\right) \frac{\mathrm{M}^{\mu} g(w)}{\mathrm{M}^{m} g(w)} - \rho e^{i\theta}\right\} \geq \tau, \ \ \theta \in \mathbb{R}, \ \ w \in \mathcal{L}\right\},$$

where $w' = \frac{\partial}{\partial \gamma}(w = re^{i\gamma}), g'(w) = \frac{\partial}{\partial \gamma} g(re^{i\gamma})$.

Define $DN_{\mathcal{F}}(\tau) = \mathrm{N}_{\mathcal{F}}(\tau) \cap D$ and $D\Phi_{\mathcal{F}}(\tau) = \Phi_{\mathcal{F}}(\tau) \cap D$, where D consists of the functions $g = h + \bar{\phi}$ in $S_{\mathcal{F}}$ so that h and ϕ are of the form

$$h(w) = w - \sum_{\iota=2}^{\infty} |A_{\iota}| w^{\iota}, \quad \phi(w) = \sum_{\iota=1}^{\infty} |C_{\iota}| w^{\iota}. \tag{6}$$

In [1, 3,11], classes $\mathrm{N}_{\mathcal{F}}(\tau)$ and $\Phi_{\mathcal{F}}(\tau)$were initially introduced and studied. A function in $\Phi_{\mathcal{F}}(\tau)$ is called the harmonic univalent function in $\mathcal{L}$ of the Goodman-Salagean-type.

We will also use the notations in this paper

$$\mho(g) = \mho\begin{pmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \end{pmatrix} g, \ \ M_{\iota-1} = \frac{(|a_1|)_{\iota-1}(|c_1|)_{\iota-1}}{(|d_1|)_{\iota-1}(1)_{\iota-1}}, \ \ \mathcal{F}_{\iota-1} = \frac{(|a_2|)_{\iota-1}(|c_2|)_{\iota-1}}{(|d_2|)_{\iota-1}(1)_{\iota-1}}$$

and a well-known formula

$$\mathrm{E}(a, c; d;\ 1) = \frac{\Gamma(d-a-c)\Gamma(d)}{\Gamma(d-a)\Gamma(d-c)}, \mathcal{R}e(d-a-c) > 0.$$

The main objective of this paper is to create some significant relations between the classes $K_{\mathcal{F}}^0$, $S_{\mathcal{F}}^{*,0}$, $C_{\mathcal{F}}^0$, $\mathrm{N}_{\mathcal{F}}(\tau)$ and $\Phi_{\mathcal{F}}(\tau)$ by applying the convolution operator.

MAIN RESULTS

The following results are required for Lemma 2.1.[7], Lemma 2.2.[1] and Lemma 2.4.[4] to create ties between harmonic convex functions.

Lemma 2.1. If $g = h + \bar{\phi} \in K_{\mathcal{F}}^0$ where h and ϕ are given by Eqn. (4) with $C_1 = 0$, then

$$|A_\iota| \leq \frac{\iota + 1}{2} \quad , \quad |C_\iota| \leq \frac{\iota - 1}{2}.$$

Lemma 2.2. Let $g = h + \bar{\phi}$ be given by Eqn. (4). If

$$\sum_{\iota=2}^{\infty}[(1+q)\iota^{\mu} - \iota^{m}(\tau + q)]|a_\iota| + \sum_{\iota=1}^{\infty}[(1+q)\iota^{\mu} - (-1)^{\mu-m}\iota^{m}(\tau+q)]\,|c_\iota| \leq 1 - \tau, \qquad (7)$$

Then g is sense-preserving, harmonic univalent functions of Goodman-Salagean-type in $\mathcal{L}$ and $g \in \Phi_{\mathcal{F}}(\tau)$.

Remark 2.3. In [1,14], $g = h + \bar{\phi}$ given by Eqn. (6) is also shown belongs to the $D\Phi_{\mathcal{F}}(\tau)$ family, if and only if the coefficient condition (7) holds. Moreover, if $g \in D\Phi_{\mathcal{F}}(\tau)$, then

$$|A_\iota| \leq \frac{1-\tau}{(1+q)\iota^{\mu} - \iota^{m}(\tau+q)}\,, (\iota \geq 2), \qquad |C_\iota| \leq \frac{1-\tau}{(1+q)\iota^{\mu} - (-1)^{\mu-m}\iota^{m}(\tau+q)}\,, (\iota \geq 1).$$

Lemma 2.4. If $a, c, d > 0$, then

(i) $\mathrm{E}(a+m, c+m; d+m; 1) = \frac{(c)_m}{(d-a-c-m)_m}\mathrm{E}(a, c; d;\ 1)$, for $m = 0,1,2,3, \ldots$, if $d > a + c + m$.

(ii) $\sum_{\iota=2}^{\infty}(\iota-1)\frac{(a)_{\iota-1}(c)_{\iota-1}}{(d)_{\iota-1}(1)_{\iota-1}} = \frac{ac}{d-a-c-1}\mathrm{E}(a, c; d;\ 1)$, if $d > a + c + 1$.

(iii) $\sum_{\iota=2}^{\infty}(\iota-1)^2\frac{(a)_{\iota-1}(c)_{\iota-1}}{(d)_{\iota-1}(1)_{\iota-1}} = \left[\frac{(a)_2(c)_2}{(d-a-c-2)_2} + \frac{ac}{d-a-c-1}\right]\mathrm{E}(a, c; d;\ 1)$, if $d > a + c + 2$.

(iv) $\sum_{\iota=2}^{\infty}(\iota-1)^3\frac{(a)_{\iota-1}(c)_{\iota-1}}{(d)_{\iota-1}(1)_{\iota-1}} = \left[\frac{(a)_3(c)_3}{(d-a-c-3)_3} + \frac{(a)_2(c)_2}{(d-a-c-2)_2} + \frac{ac}{d-a-c-1}\right]\mathrm{E}(a, c; d;\ 1)$, if $d > a + c + 3$.

(v) $\sum_{\iota=2}^{\infty}(\iota-1)^4\frac{(a)_{\iota-1}(c)_{\iota-1}}{(d)_{\iota-1}(1)_{\iota-1}} = \left[\frac{(a)_4(c)_4}{(d-a-c-4)_4} + \frac{(a)_3(c)_3}{(d-a-c-3)_3} + \frac{(a)_2(c)_2}{(d-a-c-2)_2} + \frac{ac}{d-a-c-1}\right]\mathrm{E}(a, c; d;\ 1)$, if $d > a + c + 4$.

Theorem 2.5. Let $a_i, c_i \in \mathbb{C} \setminus \{0\}, d_i \in \mathbb{R}$ and $d_i > |a_i| + |c_i| + 3$ for $i = 1,2$. If for some $q (0 \leq q \leq 1)$ and $\tau (0 \leq \tau < 1)$, when $\mu = m + 1, m = 1$ the inequality

$$\Upsilon_1 \mathrm{E}(|a_1|, |c_1|; d_1;\ 1) + \Omega_1 \mathrm{E}(|a_2|, |c_2|; d_2; 1) \leq 4(1-\tau),$$

is satisfied, then $\mho(K_{\mathcal{F}}^0) \subset \Phi_{\mathcal{F}}(\tau)$, where

$$\Upsilon_1 = (1+q)\frac{(|a_1|)_3(|c_1|)_3}{(d_1-|a_1|-|c_1|-3)_3} + (4+3q-\tau)\frac{(|a_1|)_2(|c_1|)_2}{(d_1-|a_1|-|c_1|-2)_2} + (5+2q-3\tau)\frac{|a_1c_1|}{(d_1-|a_1|-|c_1|-1)} - 2(\tau-1)\ ,$$

$$\Omega_1 = (1+q)\frac{(|a_2|)_3(|c_2|)_3}{(d_2-|a_2|-|c_2|-3)_3} + (2+3q+\tau)\frac{(|a_2|)_2(|c_2|)_2}{(d_2-|a_2|-|c_2|-2)_2} + (1+2q+\tau)\frac{|a_2c_2|}{(d_2-|a_2|-|c_2|-1)}.$$

Proof. Let $g = h + \bar{\phi} \in K_{\mathcal{F}}^0$ where h and ϕ are of the form Eqn.(4) with $C_1 = 0$. We need to demonstrate that $\mho(g) = \mathrm{H} + \bar{\Phi} \in \Phi_{\mathcal{F}}(\tau)$, where analytic functions in $\mathcal{L}$ are H and Φ defined by Eqn.(5). In view of Lemma 2.2., we have to show the $\Lambda_1 \leq 1-\tau$, where

$$\Lambda_1 = \sum_{\iota=2}^{\infty}[(1+q)\iota^{\mu} - \iota^{m}(\tau+q)]\left|\frac{(a_1)_{\iota-1}(c_1)_{\iota-1}}{(d_1)_{\iota-1}(1)_{\iota-1}}A_\iota\right| + \sum_{\iota=1}^{\infty}[(1+q)\iota^{\mu} - (-1)^{\mu-m}\iota^{m}(\tau+q)]\left|\frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}}C_\iota\right|$$

Taking Lemma 2.1. and Lemma 2.4. into account, it follows that

$$\begin{aligned}
\Lambda_1 &\leq \frac{1}{2}\sum_{\iota=2}^{\infty}(\iota+1)[(1+q)\iota^{\mu} - \iota^{m}(\tau+q)]M_{\iota-1} + \frac{1}{2}\sum_{\iota=1}^{\infty}(\iota-1)\,[(1+q)\iota^{\mu} - (-1)^{\mu-m}\iota^{m}(\tau+q)]\mathcal{F}_{\iota-1} \\
&= \frac{1}{2}\sum_{\iota=2}^{\infty}[(1+q)(\iota-1)^3 + (4+3q-\tau)(\iota-1)^2 + (5+2q-3\tau)(\iota-1) - 2(\tau-1)]M_{\iota-1} \\
&+ \frac{1}{2}\sum_{\iota=1}^{\infty}[(1+q)(\iota-1)^3 + (2+3q+\tau)(\iota-1)^2 + (1+2q+\tau)(\iota-1)]\,\mathcal{F}_{\iota-1} \\
&= \frac{1}{2}\Upsilon_1\mathrm{E}(|a_1|,|c_1|;d_1;\ 1) + \frac{1}{2}\Omega_1\mathrm{E}(|a_2|,|c_2|;d_2;1) - (1-\tau).
\end{aligned}$$

Hence $\Lambda_1 \leq 1-\tau$ that follows from the condition given.

Lemma 2.6.[3] and Lemma 2.8.[2] need the following results in order to evaluate the relation between $DN_{\mathcal{F}}(\delta)$ and $\Phi_{\mathcal{F}}(\tau)$.

Lemma 2.6. If $g = h + \bar{\phi}$ where h and ϕ are given by Eqn.(6) with $C_1 = 0$, and suppose that $0 \leq \delta < 1$. Then

$$g \in DN_{\mathcal{F}}(\delta) \Leftrightarrow \sum_{\iota=2}^{\infty}\iota|A_\iota| + \sum_{\iota=1}^{\infty}\iota\,|C_\iota| \leq 1-\delta$$

Remark 2.7. If $g \in DN_{\mathcal{F}}(\delta)$, then

$$|A_\iota| \leq \frac{1-\delta}{\iota}\ \ ,\ \iota \geq 2\quad ,\qquad |C_\iota| \leq \frac{\iota-\delta}{\iota}\quad ,\quad \iota \geq 1.$$

Lemma 2.8. Let $a, c \in \mathbb{C}\setminus\{0\}, a \neq 1, c \neq 1, d \in (0,1)\cup(1,\infty)$ and $d > max\{0, |a|+|c|-1\}$. Then

$$\sum_{\iota=1}^{\infty}\frac{1}{\iota}\frac{(|a|)_{\iota-1}\,(|c|)_{\iota-1}}{(d)_{\iota-1}\,(1)_{\iota-1}}=\frac{(d-|a|-|c|)}{(|a|-1)(|c|-1)}\mathrm{E}(|a|,|c|;d;\ 1)-\frac{(d-1)}{(|a|-1)(|c|-1)}.$$

Theorem 2.9. Let $a_i, c_i \in \mathbb{C}\setminus\{0\}, a_i \neq 1, c_i \neq 1, d_i \in \mathbb{R}$ and $d_i > max\{0, |a_i| + |c_i| - 1\}$ for $i = 1,2$. If for some $\delta(0 \leq \delta \leq 1)$ and $\tau(0 \leq \tau < 1)$, when $\mu = m+1, m = 0$, $\mu = m+1, m = 1$ and $\mu = m+1, m = 2$ the inequality

$$\Upsilon_2\mathrm{E}(|a_1|,|c_1|;d_1;\ 1)+\Omega_2\mathrm{E}(|a_2|,|c_2|;d_2;1)$$

$$\leq\frac{2(1-\tau)-\delta(1-\tau)}{(1-\delta)}-[\tau+q]\left[\frac{(d_1-1)}{(|a_1|-1)(|c_1|-1)}-\frac{(d_2-1)}{(|a_2|-1)(|c_2|-1)}\right],$$

is satisfied, then $\mho\big(DN_{\mathcal{F}}(\delta)\big) \subset \Phi_{\mathcal{F}}(\tau)$, where

$$\Upsilon_2=(1+q)-(\tau+q)\frac{(d_1-|a_1|-|c_1|)}{(|a_1|-1)(|c_1|-1)}\,,\qquad \Omega_2=(1+q)+(\tau+q)\frac{(d_2-|a_2|-|c_2|)}{(|a_2|-1)(|c_2|-1)}.$$

Proof. Let $g = h + \bar{\phi} \in DN_{\mathcal{F}}(\delta)$ where h and ϕ are of the form Eqn.(6). With Lemma 2.2. in mind, it is necessary to demonstrate that $\Lambda_2 \leq 1-\tau$ and

$$\Lambda_2=\sum_{\iota=2}^{\infty}[(1+q)\iota^{\mu}-\iota^{m}(\tau+q)]\left|\frac{(a_1)_{\iota-1}(c_1)_{\iota-1}}{(d_1)_{\iota-1}(1)_{\iota-1}}A_\iota\right|+\sum_{\iota=1}^{\infty}[(1+q)\iota^{\mu}-(-1)^{\mu-m}\iota^{m}(\tau+q)]\left|\frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}}C_\iota\right|$$

Using Remark 2.7. and Lemma 2.8. if $\mu = m+1, m = 0$. Then

$$\Lambda_2\leq(1-\delta)\left(\sum_{\iota=2}^{\infty}\left[(1+q)-\frac{(\tau+q)}{\iota}\right]M_{\iota-1}+\sum_{\iota=1}^{\infty}\left[(1+q)+\frac{(\tau+q)}{\iota}\right]\mathcal{F}_{\iota-1}\right)$$

$$=(1-\delta)\big(\Upsilon_2\mathrm{E}(|a_1|,|c_1|;d_1;\ 1)+\Omega_2\mathrm{E}(|a_2|,|c_2|;d_2;1)\big)-(1-\delta)(1-\tau)$$

$$+(\tau+q)(1-\delta)\left[\frac{(d_1-1)}{(|a_1|-1)(|c_1|-1)}-\frac{(d_2-1)}{(|a_2|-1)(|c_2|-1)}\right]\leq(1-\tau)\,,$$

by the hypothesis given.

Now, if $\mu = m+1, m = 1$, then

$$\Lambda_2\leq(1-\delta)\left(\sum_{\iota=2}^{\infty}[(1+q)\iota-(\tau+q)]M_{\iota-1}+\sum_{\iota=1}^{\infty}[(1+q)\iota+(\tau+q)]\,\mathcal{F}_{\iota-1}\right)$$

$$=(1-\delta)\left(\sum_{\iota=2}^{\infty}[(1+q)(\iota-1)-(\tau-1)]M_{\iota-1}+\sum_{\iota=1}^{\infty}[(1+q)(\iota-1)+(\tau+2q+1)]\,\mathcal{F}_{\iota-1}\right)$$

$$=(1-\delta)\big(\Upsilon_2\mathrm{E}(|a_1|,|c_1|;d_1;\ 1)+\Omega_2\mathrm{E}(|a_2|,|c_2|;d_2;1)\big)-(1-\delta)(1-\tau)\leq(1-\tau)$$

and

$$\Upsilon_2=(1+q)\frac{|a_1c_1|}{(d_1-|a_1|-|c_1|-1)}-(\tau-1),\ \Omega_2=(1+q)\frac{|a_2c_2|}{(d_2-|a_2|-|c_2|-1)}+(1+2q+\tau).$$

Finally, if $=m+1, m=2$, then

$$\Lambda_2\le(1-\delta)\left(\sum_{\iota=2}^{\infty}[(1+q)\iota^2-\iota(\tau+q)]M_{\iota-1}+\sum_{\iota=1}^{\infty}[(1+q)\iota^2+\iota(\tau+q)]\mathcal{F}_{\iota-1}\right)$$

$$=(1-\delta)\left(\sum_{\iota=2}^{\infty}[(1+q)(\iota-1)^2+(2+q-\tau)(\iota-1)-(\tau-1)]M_{\iota-1}\right.$$

$$\left.+\sum_{\iota=1}^{\infty}[(1+q)(\iota-1)^2+(2+3q+\tau)(\iota-1)+(1+2q+\tau)]\mathcal{F}_{\iota-1}\right)$$

$$=(1-\delta)\big(\Upsilon_2\mathrm{E}(|a_1|,|c_1|;d_1;\ 1)+\Omega_2\mathrm{E}(|a_2|,|c_2|;d_2;1)\big)-(1-\delta)(1-\tau)\le(1-\tau)$$

and

$$\Upsilon_2=(1+q)\frac{(|a_1|)_2(|c_1|)_2}{(d_1-|a_1|-|c_1|-2)_2}+(2+q-\tau)\frac{|a_1c_1|}{(d_1-|a_1|-|c_1|-1)}-(\tau-1),$$

$$\Omega_2=(1+q)\frac{(|a_2|)_2(|c_2|)_2}{(d_2-|a_2|-|c_2|-2)_2}+(2+3q+\tau)\frac{|a_2c_2|}{(d_2-|a_2|-|c_2|-1)}+(1+2q+\tau).$$

Next, we find similarities with $\Phi_{\mathcal{F}}(\tau)$ of classes $C_{\mathcal{F}}^0, S_{\mathcal{F}}^{*,0}$ and $D_{\mathcal{F}}^0$. We require, however, the following result first, which can be found in [5, 7,13] or [20] .

Lemma 2.10. If $g=h+\bar{\phi}\in C_{\mathcal{F}}^0\left(S_{\mathcal{F}}^{*,0},D_{\mathcal{F}}^0\right)$ where h and ϕ are given by Eqn.(4) with $C_1=0$, then

$$|A_\iota|\le\frac{(2\iota+1)(\iota+1)}{6}\quad,\qquad|C_\iota|\le\frac{(2\iota-1)(\iota-1)}{6}\ .$$

Theorem 2.11. Let $a_i,c_i\in\mathbb{C}\setminus\{0\}, d_i\in\mathbb{R}$ and $d_i>|a_i|+|c_i|+4$ for $i=1,2$. If for some $q(0\le q\le 1)$ and $\tau(0\le\tau<1)$, when $\mu=m+1, m=1$ the inequality

$$\Upsilon_3\mathrm{E}(|a_1|,|c_1|;d_1;\ 1)+\Omega_3\mathrm{E}(|a_2|,|c_2|;d_2;1)\le 12(1-\tau),$$

is satisfied, then $\mho(C_{\mathcal{F}}^0)\subset\Phi_{\mathcal{F}}(\tau)$, $\mho\big(S_{\mathcal{F}}^{*,0}\big)\subset\Phi_{\mathcal{F}}(\tau)$, $\mho(D_{\mathcal{F}}^0)\subset\Phi_{\mathcal{F}}(\tau)$, where

$$\Upsilon_3=2(1+q)\frac{(|a_1|)_4(|c_1|)_4}{(d_1-|a_1|-|c_1|-4)_4}+(8+7q-2\tau)\frac{(|a_1|)_3(|c_1|)_3}{(d_1-|a_1|-|c_1|-3)_3}$$
$$+(14+7q-7\tau)\frac{(|a_1|)_2(|c_1|)_2}{(d_1-|a_1|-|c_1|-2)_2}+(9+2q-7\tau)\frac{|a_1c_1|}{(d_1-|a_1|-|c_1|-1)}-2(\tau-1)\ ,$$

$$\Omega_3 = 2(1+q)\frac{(|a_2|)_4(|c_2|)_4}{(d_2-|a_2|-|c_2|-4)_4} + (5+7q+2\tau)\frac{(|a_2|)_3(|c_2|)_3}{(d_2-|a_2|-|c_2|-3)_3} + (4+7q+3\tau)\frac{(|a_2|)_2(|c_2|)_2}{(d_2-|a_2|-|c_2|-2)_2} + (1+2q+\tau)\frac{|a_2c_2|}{(d_2-|a_2|-|c_2|-1)}.$$

Proof. Let $\mathcal{g} = h + \bar{\phi} \in C_{\mathcal{F}}^0\left(S_{\mathcal{F}}^{*,0}, D_{\mathcal{F}}^0\right)$ where h and ϕ are of the form Eqn.(4) with $C_1 = 0$. We need to demonstrate that $\mho(\mathcal{g}) = \mathrm{H} + \bar{\Phi} \in \Phi_{\mathcal{F}}(\tau)$, where analytic functions in $\mathcal{L}$ are H and Φ defined by Eqn.(5). In view of Lemma 2.2., we have to show the $\Lambda_3 \leq 1-\tau$, where

$$\Lambda_3 = \sum_{\iota=2}^{\infty}[(1+q)\iota^{\mu} - \iota^{m}(\tau+q)]\left|\frac{(a_1)_{\iota-1}(c_1)_{\iota-1}}{(d_1)_{\iota-1}(1)_{\iota-1}}A_\iota\right| + \sum_{\iota=1}^{\infty}[(1+q)\iota^{\mu} - (-1)^{\mu-m}\iota^{m}(\tau+q)]\left|\frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}}C_\iota\right|$$

Taking Lemma 2.4. and Lemma 2.10. into account, it follows that

$$\begin{aligned}
\Lambda_3 &\leq \frac{1}{6}\sum_{\iota=2}^{\infty}(2\iota+1)(\iota+1)[(1+q)\iota^2 - \iota(\tau+q)]M_{\iota-1} + \frac{1}{6}\sum_{\iota=1}^{\infty}(2\iota-1)(\iota-1)\,[(1+q)\iota^2 + \iota(\tau+q)]\mathcal{F}_{\iota-1} \\
&= \frac{1}{6}\sum_{\iota=2}^{\infty}\begin{bmatrix}2(1+q)(\iota-1)^4 + (8+7q-2\tau)(\iota-1)^3 + (14+7q-7\tau)(\iota-1)^2 \\ +(9+2q-7\tau)(\iota-1) - 2(\tau-1)\end{bmatrix}M_{\iota-1} \\
&+ \frac{1}{6}\sum_{\iota=1}^{\infty}\begin{bmatrix}2(1+q)(\iota-1)^4 + (5+7q+2\tau)(\iota-1)^3 + (4+7q+3\tau)(\iota-1)^2 \\ +(1+2q+\tau)(\iota-1)\end{bmatrix}\mathcal{F}_{\iota-1} \\
&= \frac{1}{6}\Upsilon_3\mathrm{E}(|a_1|,|c_1|;d_1;\ 1) + \frac{1}{6}\Omega_3\mathrm{E}(|a_2|,|c_2|;d_2;1) - (1-\tau).
\end{aligned}$$

Hence $\Lambda_3 \leq 1-\tau$ that follows from the condition given.

We create ties between $D\Phi_{\mathcal{F}}(\tau)$ and $\Phi_{\mathcal{F}}(\tau)$ in the next Theorem.

Theorem 2.12. Let $a_i, c_i \in \mathbb{C}\setminus\{0\}, d_i \in \mathbb{R}$ and $d_i > |a_i| + |c_i|$ for $i = 1,2$. If for some $\tau(0 \leq \tau < 1)$, when $\mu = m+1, m = 1$ the inequality

$$\mathrm{E}(|a_1|,|c_1|;d_1;\ 1) + \mathrm{E}(|a_2|,|c_2|;d_2;1) \leq 2,$$

is satisfied, then $\mho\left(\mathrm{D}\Phi_{\mathcal{F}}(\tau)\right) \subset \Phi_{\mathcal{F}}(\tau)$.

Proof. By using Lemma 2.2. and the definition of Λ_2 in Theorem 2.9., we need to prove that $\Lambda_2 \leq 1-\tau$.

By Remark 2.3., it follows that

$$\Lambda_2 = \sum_{\iota=2}^{\infty}[(1+q)\iota^{\mu} - \iota^{m}(\tau+q)]\left|\frac{(a_1)_{\iota-1}(c_1)_{\iota-1}}{(d_1)_{\iota-1}(1)_{\iota-1}}A_\iota\right| + \sum_{\iota=1}^{\infty}[(1+q)\iota^{\mu} - (-1)^{\mu-m}\iota^{m}(\tau+q)]\left|\frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}}C_\iota\right|$$

$$\leq (1-\tau)\left(\sum_{\iota=2}^{\infty}M_{\iota-1} + \sum_{\iota=1}^{\infty}\mathcal{F}_{\iota-1}\right) = (1-\tau)\left(\mathrm{E}(|a_1|,|c_1|;d_1;\ 1) + \mathrm{E}(|a_2|,|c_2|;d_2;1)\right) - (1-\tau) \leq (1-\tau).$$

The proof is satisfied by the specified condition.

In the next results, we create ties between $D\Phi_{\mathcal{F}}(\tau)$ and $\Phi_{\mathcal{F}}(\tau)$. By diluting the drawbacks of Theorem 2.12. on the complex coefficients.

Theorem 2.13. Let $a_1, c_1 < 0, a_1, c_1 > -1,\ d_1 > max\{0, a_1 + c_1\}, a_2, c_2 \in \mathbb{C} \setminus \{0\}$ and $d_2 > |a_2| + |c_2|,$ then a sufficient condition for $\mho\big(\mathrm{D}\Phi_{\mathcal{F}}(\tau)\big) \subset \Phi_{\mathcal{F}}(\tau)$ is that

$$\mathrm{E}(|a_1|, |c_1|; d_1;\ 1) - \mathrm{E}(|a_2|, |c_2|; d_2; 1) \geq 0,$$

for some $q(0 \leq q \leq 1)$ and $\tau(0 \leq \tau < 1)$, when $\mu = m + 1, m = 1$.

Proof. Let $g = h + \bar{\phi} \in D\Phi_{\mathcal{F}}(\tau)$ where h and ϕ are of the form Eqn.(6). Then

$$\mho(g) = w - \sum_{\iota=2}^{\infty} \frac{(a_1)_{\iota-1}(c_1)_{\iota-1}}{(d_1)_{\iota-1}(1)_{\iota-1}}\ |A_\iota| w^\iota + \overline{\sum_{\iota=1}^{\infty} \frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}} |C_\iota|\, w^\iota}.$$

It is possible to rewrite this function as

$$\mho(g) = w + \frac{|a_1 c_1|}{d_1} \sum_{\iota=2}^{\infty} \frac{(a_1+1)_{\iota-2}(c_1+1)_{\iota-2}}{(d_1+1)_{\iota-2}(1)_{\iota-1}}\ |A_\iota| w^\iota + \overline{\sum_{\iota=1}^{\infty} \frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}} |C_\iota|\, w^\iota}.$$

In view of Lemma 2.2., we have to demonstrate that $\Lambda_4 \leq 1$ is where

$$\Lambda_4 = \frac{|a_1 c_1|}{d_1} \sum_{\iota=2}^{\infty} \left[\frac{(1+q)\iota^2 - \iota(\tau+q)}{(1-\tau)}\right] \frac{(a_1+1)_{\iota-2}(c_1+1)_{\iota-2}}{(d_1+1)_{\iota-2}(1)_{\iota-1}} |A_\iota| + \sum_{\iota=1}^{\infty} \left[\frac{(1+q)\iota^2 + \iota(\tau+q)}{(1-\tau)}\right] \frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}} |C_\iota|$$

$$\leq \frac{|a_1 c_1|}{d_1} \sum_{\iota=2}^{\infty} \frac{(a_1+1)_{\iota-2}(c_1+1)_{\iota-2}}{(d_1+1)_{\iota-2}(1)_{\iota-1}} |A_\iota| + \sum_{\iota=1}^{\infty} \mathcal{F}_{\iota-1} = -\mathrm{E}(|a_1|, |c_1|; d_1;\ 1) + \mathrm{E}(|a_2|, |c_2|; d_2; 1) + 1 \leq 1,$$

by the given condition.

We present a condition on the parameters $a_1, a_2, c_1, c_2 d_1, d_2$ in the next theorem and obtain an operator characterization that maps $D\Phi_{\mathcal{F}}(\tau)$ on itself.

Theorem 2.14. Let $a_i, c_i < 0, d_i > a_i + c_i\ (i = 1{,}2), q(0 \leq q \leq 1)$ and $\tau(0 \leq \tau < 1)$, when $\mu = m + 1, m = 1$, then $\mho\big(\mathrm{D}\Phi_{\mathcal{F}}(\tau)\big) \subset \mathrm{D}\Phi_{\mathcal{F}}(q, \tau)$ if and only if

$$\mathrm{E}(|a_1|, |c_1|; d_1;\ 1) + \mathrm{E}(|a_2|, |c_2|; d_2; 1) \leq 2.$$

Proof. Let $g = h + \bar{\phi} \in D\Phi_{\mathcal{F}}(q, \tau)$ where h and ϕ are of the form Eqn.(6). We need to demonstrate that $\mho(g) = \mathrm{H} + \bar{\Phi} \in \mathrm{D}\Phi_{\mathcal{F}}(q, \tau)$, where analytic functions in $\mathcal{L}$ are H and Φ defined by Eqn.(5) $\Lambda_4 \leq 1$, where

$$\Lambda_4 = \sum_{\iota=2}^{\infty} \left[\frac{(1+q)\iota^2 - \iota(\tau+q)}{(1-\tau)}\right] \left|\frac{(a_1)_{\iota-1}(c_1)_{\iota-1}}{(d_1)_{\iota-1}(1)_{\iota-1}} A_\iota\right| + \sum_{\iota=1}^{\infty} \left[\frac{(1+q)\iota^2 + \iota(\tau+q)}{(1-\tau)}\right] \left|\frac{(a_2)_{\iota-1}(c_2)_{\iota-1}}{(d_2)_{\iota-1}(1)_{\iota-1}} C_\iota\right|.$$

By using Remark 2.3., we get

$$\Lambda_4 \le \sum_{\iota=1}^{\infty} M_\iota + \sum_{\iota=0}^{\infty} \mathcal{F}_\iota \le \mathrm{E}(|a_1|, |c_1|; d_1;\ 1) + \mathrm{E}(|a_2|, |c_2|; d_2; 1) - 1.$$

Hence $\Lambda_4 \le 1$ that follows from the condition given.

REFERENCES

1. R. Aghalary, Goodman-Salagean-Type Harmonic Univalent Functions with Varying Arguments, International Journal of Mathematical Analysis, 22(1), 1051-1057 (2007).
2. O.P. Ahuja, Connections between various subclasses of planar harmonic mappings Involving hypergeometric functions , Applied Mathematics and Computation, 198(1), 305-316 (2008).
3. O.P. Ahuja, Jahangiri J.M., Noshiro-type harmonic univalent functions, Scientiae Mathematicae Japonicae, 6(2), 253-259 (2002).
4. O.P. Ahuja, Planar harmonic convolution operators generated by hypergeometric functions, Integral Transforms and Special Functions, 18(3), 165-177 (2007).
5. O.P. Ahuja, Planar harmonic univalent and related mappings, Journal of Inequalities in Pure and Applied Mathematics, 6(4) Art. 122, 1-18 (2005).
6. B.C. Carleson and D.B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM Journal on Mathematical Analysis, 15, 737-745 (1984).
7. J. Clunie and T. Sheil-Small, Harmonic univalent functions, Annales Academie Scientiarum Fennice, Series A. I. Mathematica 9, 3-25 (1984).
8. L. De Branges, A proof of the Bieberbach conjecture, Acta Mathematica, 154, 137-152 (1985).
9. P. Duren, Harmonic Mappings in the plane, Cambridge Tracts in Mathematics, Vol. 156, Cambridge University Press, Cambridge, ISBN 0-521064121-7 (2004).
10. M.I. Hameed, C. Ozel, A. Al-Fayadh and A.R.S. Juma, Study of certain subclasses of analytic functions involving convolution operator, AIP Conference Proceedings, Vol. 2096. No. 1. AIP Publishing LLC (2019).
11. M. Hameed and I. Ibrahim, Some Applications on Subclasses of Analytic Functions Involving Linear Operator, 2019 International Conference on Computing and Information Science and Technology and Their Applications (ICCISTA). IEEE (2019).
12. Y.E. Hohlov, Convolution operators preserving univalent functions, Ukrainian Mathematical Journal, 37, 220-226 (1985).
13. A.R.S. Juma, R.A. Hameed and M.I. Hameed, Certain subclass of univalent functions involving fractional q-calculus operator, Journal of Advance in Mathematics 13.4 (2017).
14. A.R.S. Juma, R.A. Hameed and M.I. Hameed, SOME RESULTS OF SECOND ORDER DIFFERENTIAL SUBORDINATION INVOLVING GENERALIZED LINEAR OPERATOR., Acta Universitatis Apulensis No. (53), pp. 19-39 (2018).
15. S. Miller and P.T. Mocanu, Univalence of Gaussian and confluent hypergeometric Functions, Proceedings of American Mathematical Society, 110(2), 333-342 (1990).
16. S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canadian Journal of Mathematics, 39, 1057-1077 (1987).
17. S. Ruscheweyh and V. Singh, On the order of starlikeness of hypergeometric functions, Journal of Mathematical Analysis and Applications, 113, 1-11 (1986).

18. H.M. Srivastava and H.L. Manocha, A Treatise on Generating Functions, Ellis Horwood Limited and John Wiley & Sons, New York, Chichester, Toronto (1984).
19. A. Swaminathan, Certain Sufficiency conditions on Gaussian hypergeometric functions, Journal of Inequalities in Pure and Applied Mathematics, 5(4), Article 83, 1-10 (2004).
20. X.T. Wang, X.Q. Liang and Y.L. Zhang, Precise coefficient estimates for close-toconvex harmonic univalent mappings, Journal of Mathematical Analysis and Applications, 263(2), 501-509 (2001).