Amply Supplemented Module over Commutative Ring

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Abstract: No doubt, a notion of the amply supplemented module can constitute a very important situation in the module theory, because a generalization of supplemented module depend on some types of modules as amply supplemented module. Here we introduce a characterization of the relations between amply supplemented and some concepts as, semi-perfect, perfect rings, Rad-supplemented and lifting modules. We prove that if M be a nonzero R-module is linearly compact then M is amply supplemented. Also we prove that if R be a semi-local ring with every simple R-module has a flat cover then M is amply supplemented. Let R be a Bass ring and let M be Rad-supplemented R-module then M is amply supplemented. Any lifting R-module M is amply supplemented module.

Keywords: Amply supplemented module, perfect ring, semi-local ring, Rad-supplemented module, lifting module, Noetherian ring

1. Definition and Notions

All the rings in this paper are considered to be associative with identity and all modules are unitary left *R*-module. A sub-module N of M is called small in M ($N \ll M$) if for every sub-module of M, $N+L=M$ then $L=M$. A sub-module N of M is called a supplement of K in M if $N+K=M$ and N is minimal with respect to this property [11]. A module M is called supplemented if any sub-module N of M has a supplement in M. Let R be a ring and let N be sub-module of an R-module M such that N is supplement and lies above a direct summand of M then R is perfect ring. In [6], Abed and Ahmad defined amply supplemented modules and related it with Rad-supplemented. Here we use local property to obtain amply supplemented module. Moreover If M is projective and local also it is amply supplemented. Also if M is (D_1) module then it is amply supplemented because every N supplement sub-module of M lies above a direct summand of M this implies R is perfect ring and so supplemented and finally M is amply supplemented module. Also if M is projective and semi-perfect then it is supplemented module. If any module M has no maximal sub-module this means $M = Rad(M)$ such that $Rad(M)$ is the intersection of all maximal sub-modules of M. A ring R is called left Bass if $Rad(M)$ is small in M such that $M \neq 0$ [4]. A module M is called hollow if every proper sub-module N of M is small in M. A module M is said to have the summand intersection property (briefly (SIP)) if the intersection of any two direct summands is again a direct summand. An R-module M is called semi-simple if any exact sequence of R-modules $0 \to N \to M \to N_1 \to 0$ splits. For integral domain R, an R-module M is called torsion free if $Ann(a)=0$, for each $0 \neq a \in M$. This article divided into 5 sections. In section 2 we study some properties of amply supplemented module and the relation between linearly compact property and amply supplemented module. In section 3 we introduce some results about the relation between amply supplemented module and semi-perfect and perfect ring. In section 4 we study the relation between Rad-supplemented module and amply supplemented. In section 5 we conclude some results which it explain the relation between lifting module and amply supplemented module.

2. Some Properties of Amply Supplemented Module

A module M is called amply supplemented if for any two sub-modules H and K with $H+K, K$ contains a supplement of H in M. Also, we say that a sub-module N of the R-module M has ample supplements in M if, for every $K \subset M$ with $N+K=M$, there is a supplement L of N with $L \subset K$. Therefore if every (finitely generated) sub-module of M has ample supplements in M, then we call M amply (finitely) supplemented. A sub-module N is a supplement of K in M if and only if $N+K=M$ and $K \bigcap N \subset N$, because if N be a supplement of K in M such that $N+K=M$. Suppose $(K \bigcap N)+A=N$ for some $A \subseteq N$, then $N+K=M=K+(K \bigcap N)+A=K+A$. Since N has minimal property then $A=N$. Therefore $(K \bigcap N)$ is small in N. If $N+K=M$ and $(K \bigcap N)$ is small in N. If $M=K+B$ for some $(B \bigcap N)$, then by (Modular Law) we can say $N=M \bigcap N=(K+B) \bigcap N=(K \bigcap N)+B$, therefore $B=N$ because $K \bigcap N \subset N$ and this means N is a supplement of K in M. A module M is called local if it has a largest proper sub-module, equivalently, a module is local if and only if it is cyclic nonzero and has a unique maximal proper sub-module. If an R-module M is local, then M is hollow module.

Lemma 2.1. Every local R-module M is supplemented module.

Proof: Since M is local module then for every proper sub-module A of M, $A \subseteq Rad(M) \ll M$. Therefore $A \ll M$. Then M is hollow module, therefore M be an R-module and A be a sub-module of M. Then $A+M=M$. By definition of hollow module we have $A \bigcap M = A \ll M$. Thus M is supplemented module.

Remark 2.2. Let M be an amply supplemented R-module, then:

(1) every factor module of amply supplemented module is also amply supplemented.

(2) A module M is amply supplemented if and only if every maximal sub-module has ample supplements in M.

Some modules are not amply supplemented especially when M is not satisfy the conditions supplemented module. For example, Z-module such that Z every nonzero proper sub-module has no supplements. But we can give some modules are amply supplemented, for example an Artinian module is amply supplemented and semi-simple module is amply supplemented therefore every injective R-module has ((SIP)) property is semi-simple modules and so amply supplemented.

Note that if M is amply supplemented R-module then it is supplemented, therefore we can easily introduce the following lemma:

Lemma 2.3. Let M be an R-module. Then M is a supplemented module if and only if it is amply supplemented module.

A module M is called linearly compact if for every family of co-sets $\{x_i + M_i\}_\Delta ; x_i \in M$ and sub-modules $M_i \subset M$ (with ($\frac{M}{M_i}$) finitely cogenerated) such that the intersection of any finitely many of these cosets is not empty, the intersection is also not empty.

Theorem 2.5. Let N be a linearly compact sub-module of an R-module M. Then N has ample supplements in M.

Proof: Suppose N, $K \subseteq M \ni N$ is linearly compact and let $M = N + K$. We take η ={ $K_1 \subseteq K \mid N + K_1 = M$ }. Since $K \in \eta$ then $\eta \neq 0$. Let $\{K_\lambda\}$ a chain in η. We can consider it is an inverse family of sub-modules K_λ since $\{K_\lambda\}$ is a chain. Now $\bigcap K_\lambda$ is a lower r bound for $\{K_\lambda\}$ and by definition of linearly compact module we get $N + (\bigcap K_\lambda) \bigcap (N + K_\lambda) = M$.

Thus $\bigcap K_\lambda \in \eta$ By Zorn's Lemma we have L is minimal element in η such that $M = N + L$ and this means L is a supplement of N and $L \subseteq K$. Therefore N has ample supplements in M.

Theorem 2.6. Every direct sum of two submodules linearly compact of M has ample supplements in M.

Proof: Let $N = N_1 + N_2$ be linearly compact and let $K = K_1 + K_2 \subset M \ni (N_1 + N_2) + (K_1 + K_2) = M$. If inverses of sub-modules K_i of $(K_1 + k_2)$ such that $(K_1 + k_2) + \mathrm{K}_i = M$, then we get $(N_1 + N_2) + \bigcap \mathrm{K}_i = \bigcap (N_1 + N_2) + K_i = M$. Then we have $\{(L_1 + L_2) \subset (\mathrm{K}_1 + K_2)\ (N_1 + N_2) + (L_1 + L_2) = M\}$ is inductive and by (Zorn`s Lemma) this set has a minimal element. Hence $N_1 + N_2$ has ample supplements in M.

Remark 2.7. We can easily generalization last theorem for any linearly sub-modules N_i of M such that $N_i; i = 1, \ldots n$ (N_i has ample supplement).
Now we can use the linearly compact property for any module in order to obtain amply supplemented module and also we make a generalization of linearly compact property of the module M :

Theorem 2.8. Let M be a nonzero R-module. Let M to be linearly compact then M is amply supplemented.
Proof: Let M be linearly compact then every sub-module N of M is linearly compact and then N has ample supplements in M. Then M is amply supplemented [Remark 2.2 (2)].

Theorem 2.9. If $M = M_1 + M_2$ be linearly compact. Then M is amply supplemented module.
Proof: Let $M = M_1 + M_2$ be linearly compact and let ($N_1 + N_2$) be sub-module of M. By [Theorem 2.5]. since every sub-module ($N_1 + N_2$) of M is linearly compact, then ($N_1 + N_2$) has ample supplements in M. Then M is amply supplemented module.

Theorem 2.10. For an R-module M. If M is supplemented and π-projective then M is amply supplemented.
Proof: Let $M = N + K$ and A be a supplement of N in M. Let g belong to endomorphism of M (End (M)) such that $\operatorname{Im} g$ (g) subset of K and $\operatorname{Im} g$ (1− g) subset of N since we have g (N) subset of N, $M = N + g(B)$ and g ($N \cap B$)= $N \cap g$ (B) (since $n = g$ (b) then we can say $b - n$ =(1− g)(b) $\in N$ and $b \in N$). Since $N \cap B$ □ A, $N \cap g(A)$□ $g(A)$, and this means $g(A)$ is a supplement of N such that $g(A)$ subset of K .Then M is amply supplemented module.

Corollary 2.11. Let M be an R-module. If M is projective and local then M is amply supplemented.
Proof: Let M be projective module. Let f be epimorphism. Here we must show that f is split. Let 1_M be identity mapping. Since M is projective module then there exists a mapping g from M into $(U \oplus V)$ such that $f \circ g = 1_M$ and this means f is split and hence M is π-projective. Now we have M is local module then if M be an R-module. Therefore for every proper sub-module N of M, $N \subseteq Rad(M)$□ M. Hence N □ M and M is hollow and if K be a sub-module of M. Then

$K+M=M$ and since, $K \cap M = K$ □ M. Therefore M is supplemented. Now M is π-projective and supplemented then by [Theorem 2.10] M is amply supplemented module.

Proposition 2.12. Let M be an R-module. If M is a π-projective δ-supplemented, then M is amply δ-supplemented module.

Proof: Let N, K sub-module of M such that $M=N+K$. We have M is π-projective module, then there exists a mapping β from M into M such that β(M) sub-module of N and (1−β)(M) sub-module of K. We know that (1−β)(N) sub-module of N. Let L to be a δ-supplement of N in M. Then $M=\beta(M)$+(1−β)(M)=$\beta(M)$+(1−β)($N+L$) sub-module of N+(1−β)(L) sub-module of M, also $M=N$+(1−β) (L). Note that (1−β)(L) sub-module of K. Suppose that n belong to $N \cap$ (1−β)(L). Then n belong to N and n=(1−β) (s)=$s-\beta(s)$ for some s belong to L. From now we consider $s=n+\beta(s)$ belong to N, so that n belong to (1−β) ($N \cap L$). But $N \cap L$ □ δ(L)implies that $N \cap$(1−β) (L)=(1−β) ($N \cap L$)δ(1−β) (L). Therefore (1−β) (L) is a δ-supplement of N in M and hence M is an amply δ-supplemented module.

Definition 2.13. Let M be an R-module and $K \leq N \leq M$. If $N=K$ □ $M=K$ then we say N lies above K. Therefore, N lies above a sub-module K of M if and only if K □ N and for every sub-module L of M such that $N+L=M$, then $K+L=M$.

Lemma 2.14. Every sub-module of M lies above a supplement sub-module of M if and only if for any sub-module N of M, there exists sub-modules K and K_1 of N with K is supplemented, $N=K+K_1$ and K_1 □ M **.**

Proof: Let N be a sub-module in M. Since N lies above a supplement sub-module A in M. Suppose that A is a supplement of B in M then $M=A+B=N+B$. Again by hypothesis B lies above a supplement sub-module K in M. Hence $M=A+B=A+K$. Since A is a supplement of B, $A \cap K$ □ M. Furthermore since K is a supplement in M, $A \cap K$ □ K and K is a supplement of for any $K_1 \leq Y$ with $N+K_1=M$, since N lies above A and K is a supplement of A, $A+K_1=M$ and $K_1=K$. Thus K is a supplement of N. Finally, M is supplemented therefore by [Lemma 2.3] M is amply supplemented module.

Corollary2 .15. Let M be an R-module. Then every sub-module of M lies above a supplement sub-module of M, if and only if M is amply supplemented.

Corollary 2.16. For an R-module M .If every cyclic sub-module lies above a direct summand; then M is amply finitely supplemented and every supplement is a direct summand.

3. Amply Supplemented Property over Perfect and Semi-local Rings

In this section we study amply supplemented module over some rings which is commutative as semi-local, semi-perfect and perfect ring. A ring R is called semi-perfect if every finitely generated R-module has a projective cover. A module M is called semi-perfect if every factor module of M has a projective cover. A ring R is called perfect if every R-module M has projective cover. Also in this section we study amply property over ring R is called semi-local.

Lemma 3.1. [2, Theorem 2.4]. For every ring R ,if every finitely generated free R-module is ⊕-supplemented. Then R is semi-perfect.

Lemma 3.2. [13, Corollary 3.15].The following statements are equivalent for a ring R.
(1) R is a left perfect.
(2) Every R-module is supplemented.
(3) Every projective R-module is supplemented.

Theorem 3.3. Let R be a Noetherian ring. If R is semi-perfect ring, then every finitely generated left R-module is amply supplemented module.

Corollary 3.4. Let R be a Noetherian ring. If every finitely generated free R-module M is ⊕-supplemented then M is amply supplemented.

Theorem 3.5. Let R be a ring and let M be an R-module such that it is strongly ⊕-supplemented. Then M is amply supplemented.
Proof: Since M is strongly-⊕-supplemented module then R is perfect ring and so M is supplemented and hence M is amply supplemented.
Let A and B be sub-modules of an amply supplemented module M such that $M = A + B$. Then there exist sub-modules A_1 and B_1 of $M \ni A_1 \subseteq A$, $B_1 \subseteq B$ and A_1, B_1 are supplements of each other in M, for example:
Let $R = \begin{bmatrix} G & G \\ 0 & G \end{bmatrix}$ be the ring of all upper triangular $n \times n$ matrices with entries in G, where G is a field. It is clear that R is a right perfect ring and hence R is an amply supplemented right R-module.

Consider the right R-modules $A = \begin{bmatrix} G & G \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & G \\ 0 & G \end{bmatrix}$. Then $R_R = A + B$. On the other hand, $R_R = \begin{bmatrix} G & G \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}$. Now we can take $A_1 = A$ and $B_1 = \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} \leq B$. Clearly A_1 and B_1 are supplements of each other in R_R.

A ring R is called semi-local if $(\frac{R}{Jac(R)})$ is a semi-simple ring. Also R is semi-local if R has finitely many maximal ideals.

Theorem 3.6. Let R be a semi-local ring and every simple R-module has a flat cover then M is amply supplemented.

Proof: Let R be semi-local and consider $(\frac{R}{Jac(R)}) = E_1 \oplus \ldots \oplus E_n$ with E_i simple R-modules. Every simple R-module is isomorphic to one of the all E_i. By hypothesis every E_i has a flat cover L_i. Thus $L := L \oplus \ldots \oplus L_n$ is a flat cover of $R / Jac(R)$. Hence we obtain the following:

$$\begin{array}{ccccc} & & R & & \\ & & \downarrow & & \\ L & \longrightarrow & (\frac{R}{Jac(R)}) & \longrightarrow & 0 \end{array}$$

That can be extended by a homomorphism $g : R \to L$ Since f is a small epimorphism and gf is epimorphism, g must be epimorphism with $Ker(g) \subseteq Ker(gf) = Jac(R)$. Hence R is a projective cover of the flat module L. By [9], $L \approx R$ and hence all L_i must be projective. Thus each simple R-

module has a projective cover and so R is semiperfect. Therefore we have R semi-perfect with M is finitely generated (simple module) then M is amply supplemented module.

Theorem 3.7. Let R be a semilocal Bass ring. Then any R-module M is supplemented.

Theorem 3.8. Let R be a ring such that $(\frac{R}{Jac(R)})$ is Artinian. Then R has the lifting property of simple modules as a right or left R-module if and only if R is semi-perfect.

Corollary 3.9. For any ring R, if M is R-module and satisfy the following:
1. Every left R-module M is semi-local.
2. Every simple R-module M has a flat cover.
Then M is amply supplemented.

Corollary 3.10. Let R be a left perfect ring. Then any R-module M is amply supplemented.

Corollary 3.11. Let R be a semi-local Bass ring. Then any R-module M is amply supplemented module.

Corollary 3.12. Let R be semi-perfect ring then any finitely generated R-module (M is simple) is amply supplemented.

4. Amply Supplemented Module and Rad-Supplemented Module

Let R be a Rad-supplemented ring and has hereditary property then R is semi-perfect ring. Let M be an R-module. If $M = Rad(M)$ then M is Rad-supplemented. Any local module with Bass ring, are satisfying a condition of amply supplemented module. If M is hollow module then every proper sub-module of M is small in M and this means every sub-module of M is supplement in M, so M is supplemented and hence M is Rad-supplemented module. Therefore if M is a hollow module, then M is Rad-supplemented.

Theorem 4.1. Let R be a commutative Bass ring and let M be an injective R-module, then M is supplemented.

Proof: Since M is injective R-module then it is radical module ($Rad(M) = M$) and this implies M is Rad-supplemented. Now M is Rad-supplemented and R Bass ring. Let M be a Rad-supplemented module and N be a proper sub-module of M. There exists $K \leq M \ni M = N + K$ and $N \cap K \leq Rad(K)$. But R is left Bass ring, $Rad(K) \ll K$. Then $(N \cap K) \ll K$, and this means K is a supplement of N in M. Therefore, M is supplemented.

Corollary 4.2. Let R be a Bass ring and let M be Rad-supplemented R-module then M is amply supplemented.

Corollary 4.3. Let R be a left Bass ring and let M be be injective R-module then M is amply supplemented.

Proof: Since M is local module then for every proper sub module N of M, $N \subseteq Rad(M) \ll M$. Therefore $N \ll M$ and then M is hollow. Now M is hollow module and this implies M is Rad-supplemented and hence M is amply supplemented module.

Theorem 4.4. Let R be a semi-simple and Bass ring. If M be $\oplus$-supplemented, then M is amply supplemented.

Proof: Since R is semi-simple with M is $\oplus$-supplemented then M is injective R-module, but R is Bass ring then M supplemented and hence it is amply supplemented.

Lemma 4.5. Let R be a Dedekind domain, then every torsion free divisible R-module is Rad-supplemented.

Corollary 4.6. If M is torsion free divisible R-module and let R be a ring such that satisfying the following conditions:

1) R is Dedekind domain;

2) R is Bass ring;

then M is amply supplemented module.

Theorem 4.7. Let R be a ring. If M is projective R-module and is $Rad(M)$-semiperfect, then M is amply supplemented module.

Proof: Suppose that M is projective R-module. We must prove that R is perfect ring. Since M is $Rad(M)$-semiperfect therefore we need prove $Rad(M) \ll M$ in order to we get R perfect ring.. Let N be a submodule of M such that $M = Rad(M) + N$. Then, $M = L \oplus K$, where $L \leq N$ and $K \cap N \leq Rad(M)$. Then $N = L \oplus (K \cap N)$ and so $M = Rad(M) + L$. Since L is a summand of M, there exists a submodule N_1 of $Rad(M)$ such that $M = N_1 \oplus L$ by [9]. Then $Rad(N_1) = N_1 \cap Rad(M) = N_1$. Since N_1 is projective and $N_1 = 0$ then $M = N$. So R is perfect ring and hence by [Corollary 3.10] M is amply supplemented module.

Theorem 4.8. Let R be a left Bass ring. If every sub-module of R-module M is supplemented in M, then M is amply supplemented.

Proof: Since every supplemented module is amply supplemented then we must prove that M is supplemented. Let $Rad(M) \neq 0$ then there exists a nonzero element $n \in Rad(M)$. We have Rn is a supplement that is $Rn + N = M$ and $Rn \cap N \ll Rn$ for some $N \subseteq M$. Since $n \in Rad(M)$, $Rn \subseteq M$ and $N = M$. Thus $Rn \ll Rn$, and this impossible and then $Rad(M) = 0$. Let $N \leq M$. Since N is a supplement $N + N_1 = M$ and $N \cap N_1 \ll N$ for some N_1 subset of M. Then $N \cap N_1$ subset of $Rad(M)=0$ therefore $N \cap N_1 = 0$. So $M = N \oplus N_1$ and M is semi-simple, then every sub-module of M is supplement in M and this means M is supplemented. So M is amply supplemented.

Corollary 4.9. If R is hereditary Rad-supplemented ring, then any finitely generated R-module M is amply supplemented.

Corollary 4.10. If R be a Rad-supplemented ring and has hereditary property. Then R is semi-perfect ring and so amply supplemented module.

5. Amply Supplemented Module and Lifting Property

A module M is called lifting or satisfies (D_1), if for every sub-module N of M there exists a direct summand K of M such that K is a coessential sub-module of N in M. Let M be an R-module such that M is projective and lifting then R is perfect ring and is amply supplemented. Any R-module M having no factor module is amply supplemented because any module have this property is lifting module. A module M is called (D_{12}) if for every sub-module K of M there exist a direct summand N of M and an epimorphism $\delta : M = N \rightarrow (\frac{M}{K})$ with $Ker(\delta) \ll (\frac{M}{N})$. A module M is said to satisfying (T_1) if for every sub-module K of M, where $(\frac{M}{N})$ is isomorphic to a co-closed sub-module of M, every homomorphism $\mu : M \rightarrow (\frac{M}{N})$ lifts to a homomorphism $\beta : M \rightarrow M$.

Lemma 5.1. Any lifting R-module M is amply supplemented module.

Theorem 5.2. For an R-module M, if M is (D_1) module then M is amply supplemented and every supplement sub-module is a direct summand.

Proof: Since M is (D_1) module then every sub-module of M lies above a direct summand; then M is obviously supplemented and every sub-module $N \subset M$ is of the form $N = A + B$, with A a direct summand of M and $B \ll M$. Since A is again supplemented it follows, from that M is amply supplemented. The converse true because if M is amply supplemented module then for $A \subset M$, let K be a supplement in M and A a supplement of K in M with $A \subset N$. Then $M = A \oplus A_1$ for a suitable direct summand $A_1 \subset M$. Since $N \cap K \ll M$, this A_1 is a supplement of A+($N \cap K$) = N and hence $N \cap A_1 \ll A_1$.

Definition 5.3. A module M is called hollow-lifting if every sub-module N of M with $(\frac{M}{N})$ is hollow has a coessential sub-module that is a direct summand of M. Equivalently for an indecomposable module M, the module M is hollow-lifting if and only if M is hollow, or else M has no hollow factor modules [8].

Proposition 5.4. Let N_1 and N_2 be hollow modules. If $M = N_1 \oplus N_2$ such that M is hollow-lifting then M is lifting module and so M is amply supplemented module.

Proof: We must prove that M is lifting module. Let L sub-module of M and let the two projections the first $\alpha : M \rightarrow N_1$ and the second $\beta : M \rightarrow N_2$. Suppose α (L)≠ N_1 and β :(L)≠ N_2, therefore L is small in M. Now, let α (L) = N_1. Therefore $M = L + N_2$ and then $(\frac{M}{N})$ is hollow. Thus there exists a direct summand K of M such that K sub-module of L and $(\frac{L}{K})$ is small in $(\frac{M}{K})$. Therefore M is lifting module and then M is amply supplemented module [Lemma 5.1].

Proposition 5.5. Let M be an R-module. Then the following statements are equivalent:

1. M is D_1-module.
2. M is lifting module.
3. M is hollow-lifting module.
4. M is amply supplemented module.

Lemma 5.6. Let M be a projective module. Then the following statements are equivalent:

(i) Every factor module of M has a projective cover;

(ii) M is lifting.

We will say that K is a strong supplement of N in M if K is a supplement of N in M and $K \cap N$ is a direct summand of N.

Theorem 5.7. Let M be a finitely generated module over a commutative ring R. If M is hollow-lifting then M is lifting.

Proof: Let N be a sub-module of M such that $(\frac{M}{N})$ is cyclic. Since R is local, $(\frac{M}{N})$ is a local module. Then N has a strong supplement in M and so M is lifting.

A module M is called finitely lifting, or f -lifting for short, if every finitely generated sub-module of M lies above a direct summand. Let R be a V -ring. An R -module M is lifting if and only if it is semi-simple.

Lemma 5.8. [12, Theorem 3.3] If M is cf -lifting then M is f -lifting.

Theorem 5.9. For Noetherian R -module M , if M is f -lifting then M is amply supplemented module.

Proof: Since M is Noetherian, every sub-module of M is finitely generated so M is lifting and by [Lemma 5.1] M is amply supplemented.

Theorem 5.10. Let N_1 and N_2 be hollow modules. Then the following are equivalent for the module $M = N_1 \oplus N_2$. If M is f -hollow-lifting module then M is amply supplemented.

Proof: We must proof M is f-lifting module. Let N be a finitely generated sub-module of M . Consider the two natural projections maps $\alpha : M \to N_1$ and $\beta : M \to N_2$. If $\alpha(N) \neq N_1$ and $\beta(N) \neq N_2$. Then by our assumption $\alpha(N)$ □ N_1 and $\beta(n_2$ □ n_2 , so by [22, Lemma (1.2)], $\alpha(N)\beta(n_2)$ □ $N_1 \oplus N_2$. Now claim that N subset of $\alpha(N_1) \oplus \beta(N_2)$, to see that, let $n \in N$ then $n \in M = N_1 \oplus N_2$ and hence $n = (n_1, n_2)$, where $n_1 \in N_1$ and $n_2 \in N_2$. Now, α (n) $= \alpha((n_1, n_2)) = n_1$ and $\beta(n) = \beta((n_1, n_2)) = n_2$. This implies that $n = (\alpha(n), \beta($ n)) and we get N subset of $\alpha(N) \oplus \beta(N)$ and $N \ll M$ .Thus M is f -lifting module. Then M is amply supplemented supplemented module

Corollary 5.11. If M is cf -lifting module then M is then M is amply supplemented.

Corollary 5.12. Let R be a V -ring. An R -module M is semi-simple then M is amply supplemented supplemented module.

Corollary 5.13. Let R be a right perfect ring. Then any projective R -module M is lifting and so amply supplemented.

Theorem 5.14. If M is a strongly $\oplus$-supplemented module. Then M is amply supplemented module [2]

Proof: Let M be a strongly $\oplus$-supplemented module. Then from definition of strongly-$\oplus$-supplemented we get M is lies above a direct summand of M and this mean M is (D_1) module and by [Theorem 5.2] M is lifting module. So M is amply supplemented.

Any module M is called a weak lifting module provided, for each semi-simple sub-module N of M , there exists a direct summand K of M such that $K \leq N$ and $(\frac{N}{K})$ □ $(\frac{M}{K})$ equivalently, there exists a decomposition $M = M_1 \oplus M_2$, such that $M_1 \leq N$ and $M_2 \cap N$ □ M_2 .

Theorem 5.15. Every amply supplemented R -module M is weak lifting.

Proof: Let A be a semi-simple sub-module of M . Then there exists $B \leq M$ with $M = A + B$ and $A \cap B$ small in B . Now there exists $C \leq M$ such that $M = C + B$ and $C \cap B$ small in $C \leq$ A. Since C is semi-simple $C \cap B = 0$ and hence $M = C \oplus B$. Thus M is weak lifting.

Remark 5.16. Weak lifting module not implies amply supplemented, but the converse is true.

Example 5.17. The ring Z is weak lifting but not amply supplemented module.

At the end of the paper we can review the most important results that give a clearer view:

1. M linearly compact $\Rightarrow M$ is amply supplemented.
2. M supplemented module and π -projective $\Rightarrow M$ amply supplemented module.
3. R perfect ring $\Rightarrow$ M is amply supplemented module.

4. M D_1-module $\Rightarrow$ M is amply supplemented module and (D_{12})- module.
5. M is amply supplemented module with (D_{12})-module and $T_1 \Rightarrow M$ discrete $\Rightarrow M$ quasi-discrete.
6. M having no factor module $\Rightarrow$ M amply supplemented module.
7. M hollow-lifting$\Rightarrow M$ Lifting module$\Rightarrow M$ amply supplemented module$\Rightarrow M$ weak lifting module.

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