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To cite this article: Alaa. M. F. AL. Jumaili *et al* 2019 *J. Phys.: Conf. Ser.* **1234** 012101

Other new types of Mappings with Strongly Closed Graphs in Topological spaces via e-θ and δ-β-θ-open sets

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Abstract: The purpose of this paper is to introduce and investigate a new types of mappings with Strongly Closed Graphs in Topological spaces namely strongly e-θ-(resp. δ-βθ)-Closed Graphs by using the concepts of e-θ-(resp. δ-βθ)-open sets and e-θ-(resp. δ-βθ)-closure operator which were introduced by Murad ÄOzkoc and GÄulhan Aslim [1] (resp. Alaa. M. F, and Xiao-Song Yang [2]), several characterizations and fundamental properties concerning of such these mappings have been studied. Furthermore, some characterizations of strongly e-θ-(δ-βθ)-Closed Graphs and nearly Φ-compact spaces have been obtained.
Keywords: e-θ-open sets, δ-β_θ-open sets, e-θ-closed graphs, δ-β_θ-closed graphs, e-θ-T_2 spaces, δ-β_θ-T_2 spaces, Φ-compact spaces.
Mathematics Subject Classification: 54B05, 54C08, 54C10.

1. Introduction

Generalized open sets are now well-known important concepts in Topology and its applications. Many topologists are focusing their research on these topics and this amounted to many important and useful results. In this respect, the variously modified forms of continuity, separation axioms etc . by utilizing generalized open sets, play a significant role in General Topology and Real analysis. Over the years quite a number of generalizations of the class of open sets in a topological space have been considered and widely investigated.

In 1970, N. Levine [3] introduced the first step of generalizing closed set. As a generalization of closed sets, e-closed sets, δ-β-closed sets and the related sets were introduced and studied by E. Ekici ([4] [5], [6], [7]) and E. Hatir and T. Noiri [8].

Of the most well-known concepts the notions of e-open sets and δ-β-open sets were introduced by E. Ekici [4] and E. Hatir & T. Noiri [8] respectively, In 2010 (resp. 2015), Murad ÄOzkoc and GÄulhan Aslim [1], (resp. Alaa. M. F, and Xiao-Song Yang [2]), used the notion of e-(resp. δ-β)-open sets and the e-(resp. δ-β)-closure [4] and [8] respectively, of a set to introduce the concepts of e-θ(resp. δ-βθ)-open sets and e-θ(resp. δ-βθ)-closed sets which provide a formulation of e-θ(resp. δ-βθ)-closure of a set in a topological spaces.

IOP Conf. Series: Journal of Physics: Conf. Series **1234** (2019) 012101 doi:10.1088/1742-6596/1234/1/012101

In this paper we introduce and investigate a new classes of mappings with strongly closed graphs by utilizing the concepts of e-θ-(resp. δ-βθ)-open sets and e-θ-(resp. δ-βθ)-closure operator in Topological spaces. Some characterizations and interesting properties for such these mappings are obtained .

2. Preliminaries

Throughout the present paper, (X, T) and (Y, T^*) (or simply X and Y) mean topological spaces. For any subset A of X, The closure and interior of A are denoted by Cl(A) and Int(A), respectively . We recall the following definitions, which will be used often throughout this paper.

Definition 2.1: A subset A of a space (X, T) is called:

a) δ-open [9] if for each $x \in A$ there exists a regular open set V such that $x \in V \subset A$. The δ-interior of A is the union of all regular open sets contained in A and is denoted by $Int_\delta(A)$. The subset A is called δ-open [9]if A = $Int_\delta(A)$. A point $x \in X$ is called a δ-cluster points of A [9] if A ∩ Int(Cl(V)) ≠ Ø for each open set V containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$. If A = $Cl_\delta(A)$), then A is said to be δ-closed [9]. The complement of δ-closed set is said to be δ-open set.
b) e-open [4] if $A \subset Cl(Int_\delta(A)) \cup Int(Cl_\delta(A))$. The complement of an e-open set is called e-closed. The intersection of all e-closed sets containing A is called the e-closure of A [4] and is denoted by e-Cl(A). The union of all e-open sets of X contained in A is called the e-interior [4] of A and is denoted by e-Int(A).
c) δ-β-open [8] or e^*-open [6], if $A \subset Cl(Int(\delta\text{-}Cl(A)))$, the complement of a δ-β-open set is called δ-β-closed. The intersection of all δ-β-closed sets containing A is called the δ-β-closure of A [8] and is denoted by δ-β-Cl(A). The union of all δ-β-open sets of X contained in A is called the δ-β-interior [8] of A and is denoted by δ-β-Int(A).
d) e-regular [1] if it is e-open and e-closed.
e) δ-β-regular [2] if it is δ-β-open and δ-β-closed.

Remark 2.1: The family of all e-open (resp. e-closed, e-regular, δ-β-open , δ-β-closed, δ-β-regular) subsets of X containing a point $x \in X$ is denoted by EΣ(X, x) (resp. EC(X, x), ER(X, x), δ-βΣ(X, x) , δ-βC(X, x), δ-βR(X, x)), the family of all e-open (resp. e-closed, e-regular, δ-β-open , δ-β-closed, δ-β-regular) sets in a space X are denoted by EΣ(X, T) (resp. EC(X, T), ER(X, T), δ-βΣ(X, T) , δ-βC(X, T), δ-βR(X, T)).

Remark 2.2: We will use the concept of δ-β-open sets instead of e^*-open sets because the concepts of δ-β-open sets and e^*-open sets are same.

Definition 2.2: A point $x \in X$ is said to be an e-θ-cluster [1] (resp. δ-β_θ-cluster [2]) point of A if e-Cl(U) ∩ A ≠ Ø (resp. δ-β-Cl(U) ∩ A ≠ Ø), for every $U \in$ EΣ(X, x) (resp. $U \in$ δ-βΣ(X, x)). The set of all e-θ-cluster [1] (resp. δ-β_θ-cluster[2]) points of A is called e-θ-closure (resp. δ-β_θ-closure) of A and is denoted by e-Cl_θ(A) (resp. δ-β-Cl_θ(A)). A subset A is said to be e-θ-closed [1] (resp. δ-β_θ-closed [2]) if A = e-Cl_θ(A) (resp. A = δ-β-Cl_θ(A)). The complement of an e-θ-closed (resp. δ-β_θ-closed) set is said to be e-θ-open (resp. δ-β_θ-open).

Remark 2.3: The family of all e-θ-open (resp. e-θ-closed, δ-β_θ-open, δ-β_θ-closed) subsets of X containing a point $x \in X$ is denoted by EθΣ(X, x) (resp. EθC(X, x), δ-β-θΣ(X, x), δ-β-θC(X, x)). The family of all e-θ-open (resp. e-θ-closed δ-β_θ-open, δ-β_θ-closed) sets in X denoted by EθΣ(X, T) (resp. EθC(X, T), δ-β-θΣ(X, T), δ-β-θC(X, T)) .

The following interesting results which are obtained by Murad ÄOzkoc and GÄulhan Aslim [1] and, Alaa. M. F, and Xiao-Song Yang [2], will play an important role in the sequel.

Lemma 2.1: For a subset A of a topological space (X, T), the following properties hold:

a) $A \in E\Sigma(X, T)$ (resp. $A \in \delta\text{-}\beta\Sigma(X, T)$) if and only if $e\text{-}Cl(A) \in ER(X, T)$ (resp. $\delta\text{-}\beta\text{-}Cl(A) \in \delta\text{-}\beta R(X, T)$).

b) $A \in ER(X, T)$ (resp. $A \in \delta\text{-}\beta R(X, T)$) if and only if A is e-θ(resp. δ-β_θ)- open and e-θ(resp. δ-β_θ)-closed.

Lemma 2.2: The following properties hold for the e-θ(resp. δ-β_θ)-closure of a subsets A and B of a topological space (X, T):

a) If $A \in E\Sigma(X, T)$ (resp. $A \in \delta\text{-}\beta\Sigma(X, T)$), Then $e\text{-}Cl(A) = e\text{-}Cl_\theta(A)$ (resp. $\delta\text{-}\beta\text{-}Cl(A) = \delta\text{-}\beta\text{-}Cl_\theta(A)$.

b) If $A \subset B$, then $e\text{-}Cl_\theta(A) \subset e\text{-}Cl_\theta(B)$(resp. $\delta\text{-}\beta\text{-}Cl_\theta(A) \subset \delta\text{-}\beta\text{-}Cl_\theta(B)$).

c) $e\text{-}Cl_\theta(e\text{-}Cl_\theta(A)) = e\text{-}Cl_\theta(A)$(resp. $\delta\text{-}\beta\text{-}Cl_\theta(\delta\text{-}\beta\text{-}Cl_\theta(A)) = \delta\text{-}\beta\text{-}Cl_\theta(A)$).

d) $e\text{-}Cl_\theta(A)$ (resp. $\delta\text{-}\beta\text{-}Cl_\theta(A)$) is e-θ(resp. δ-$\beta_\theta$)-closed and $e\text{-}Int_\theta(A)$(resp. $\delta\text{-}\beta\text{-}Int_\theta(A)$) is e-θ(resp. δ-$\beta_\theta$)-open.

e) If A_λ is e-θ-(resp. δ-β_θ)-closed in X for each $\lambda \in \Delta$. then $\bigcap_{\lambda\in\Delta} A_\lambda$ is e-θ-(resp. δ-β_θ)-closed in X.

Lemma 2.3: for any subsets A and A_λ ($\lambda \in \Delta$) of a topological space X, the following properties hold:

a) A is e-θ-(resp. δ-β_θ)-open in X if and only if for each $x \in A$ there exists $U \in ER(X, x)$ (resp. $U \in \delta\text{-}\beta R(X, x)$) such that $x \in U \subset A$.

b) If A_λ is e-θ(resp. δ-β_θ)-open in X for each $\lambda \in \Delta$, then $\bigcup_{\lambda\in\Delta} A_\lambda$ is e-θ(resp. δ-β_θ)-open in X.

Remark 2.4: The union of two e-θ-(resp. δ-β_θ)-closed sets is not necessarily e-θ (resp. δ-β_θ)-closed as shown in examples (3.6) [1] and (3.1) [2] respectively.

Definition 2.3: [10] Let *f*: $(X, T) \to (Y, T^*)$ be any mapping. Then the subset $G(f) = \{ (x, f(x)) : x \in X\}$ of the product space $(X\times Y, T\times T^*)$ is called the graph of *f*.

3. Characterizations of strongly e-θ-(δ-βθ)-Closed Graphs and their fundamental Properties

We begin this section with the following definitions which will play an important role in the sequel.

Definition 3.1: A topological space (X, T) is said to be:

a) e-θ-T_2 (resp. δ-β_θ-T_2) if for each pair of distinct points x and y in X, there exist two e-θ-open (resp. δ-β_θ-open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

b) e-T_2 [5] (resp. δ-β-T_2 [11, 5]) if for each pair of distinct points x and y in X, there exist $U \in E\Sigma(X, x)$ (resp. $U \in \delta\text{-}\beta\Sigma(X, x)$) and $V \in E\Sigma(X, y)$ (resp. $V \in \delta\text{-}\beta\Sigma(X, y)$) such that $U \cap V = \emptyset$.

Theorem 3.1: The following properties are equivalent for a topological space (X, T):

a) For every pair of distinct points $x, y \in X$, there exist $U \in E\theta\Sigma(X, x)$ (resp. $U \in \delta\text{-}\beta\text{-}\theta\Sigma(X, x)$) and $V \in E\theta\Sigma(X, y)$ (resp. $V \in \delta\text{-}\beta\text{-}\theta\Sigma(X, y)$) such that $e\text{-}Cl_\theta(U)$ (resp. $\delta\text{-}\beta\text{-}Cl_\theta(U)$) $\cap$ $e\text{-}Cl_\theta(V)$ (resp. $\delta\text{-}\beta\text{-}Cl_\theta(V)$) = Ø.

b) (X, T) is e-θ-T_2 (resp. δ-β_θ-T_2);

c) (X, T) is e-T_2(resp. δ-β-T_2);

d) For every pair of distinct points $x, y \in X$, there exist $U,V \in E\Sigma(X, T)$ (resp. $U,V \in \delta\text{-}\beta\Sigma(X, T)$) such that $x \in U$, $y \in V$ and $e\text{-}Cl(U)$ (resp. $\delta\text{-}\beta\text{-}Cl(U)$) $\cap$ $e\text{-}Cl(V)$ (resp. $\delta\text{-}\beta\text{-}Cl(V)$) = Ø;

e) For every pair of distinct points $x, y \in X$, there exist $U, V \in ER(X, T)$ (resp. $U, V \in \delta\text{-}\beta R(X, T)$) such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Proof: (a) ⇒ (b). This proof is clear thus omitted.

(b) ⇒ (c). Since EθΣ(X, T) (resp. δ-β-θΣ(X, T)⊂EΣ(X, T) (resp. δ-βΣ(X, T), thus this proof is obvious.

(c) ⇒ (d). This proof follows directly from lemma (5.1) and (6.1) in [1] and [2] respectively.

(d) ⇒ (e). By using lemma (2.1), for every U∈EΣ(X, T) (resp. U∈δ-βΣ(X, T)) we have e-Cl(U) (resp. δ-β-Cl(U))∈ ER(X, T) (resp. δ-βR(X, T)) therefore the proof follows immediately.

(e) ⇒ (a). By using lemma (2.1), since every e-regular (resp. δ-β-regular) set is an e-θ (resp. δ-β_θ)-open set and e-θ(resp. δ-β_θ)-closed set. Thus this proof is obvious.

Definition 3.2: A mapping *f*: (X, T) → (Y, T^*) is said to be:

a) Pre-e-θ-open if *f* (A)∈EθΣ(Y, T^*) for all A∈EθΣ(X, T).
b) Pre-δ-β_θ-open if *f* (A)∈δ-β-θΣ(Y, T^*) for all A∈δ-β-θΣ(X, T).

Lemma 3.1: if a mapping *f*: (X, T) → (Y, T^*) is a bijective pre e-θ-(resp. pre δ-β_θ)-open. Then *f* (B)∈ EθC(Y, T^*) (resp. *f* (B)∈δ-β-θC(Y, T^*)) for any B∈EθC(X, T) (resp. B∈δ-β-θC(X, T)).

Remark 3.1: The following theorem shows that e-θ-T_2-(resp. δ-β_θ-T_2) spaces remain invariant under bijective pre e-θ-(resp. pre δ-β_θ)-open mappings.

Theorem 3.2: let *f*: (X, T) → (Y, T^*) be a bijective pre e-θ-(resp. pre δ-β_θ)-open mapping and X be an e-θ-T_2 (resp. δ-β_θ-T_2) then Y be an e-θ-T_2 (resp. δ-β_θ-T_2).

Proof: Let y_1 and y_2 be any two distinct points of Y. Then since *f* is bijective, we have $f^{-1}(y_1)$, $f^{-1}(y_2)$ ∈X and $f^{-1}(y_1) \neq f^{-1}(y_2)$. Since X is an e-θ-$T_2$ (resp. δ-β_θ-T_2), there exists a sets U∈EθΣ(X, $f^{-1}(y_1)$) (resp. U∈δ-β-θΣ(X, $f^{-1}(y_1)$)) and V∈EθΣ(X, $f^{-1}(y_2)$) (resp. V∈δ-β-θΣ(X, $f^{-1}(y_2)$)) such that U⋂V = Ø. therefore pre e-θ(resp. pre δ-β_θ)-openness of *f* gives the existence of two sets *f* (U)∈EθΣ(Y, y_1) (resp. *f* (U)∈δ-β-θΣ(Y,y_1)) and *f* (V)∈EθΣ(Y, y_2) (resp. *f* (V)∈δ-β-θΣ(Y,y_2)) such that *f* (U) ⋂ *f* (V) = Ø. This shows that Y is e-θ-T_2-(resp. δ-β_θ-T_2).

Definition 3.3: The graph G(*f*) of a mapping *f*: (X, T) → (Y, T^*) is said to be:

a) Strongly e-θ (resp. δ-β_θ)-closed if for each (x, y)∈(X × Y)\G(*f*), there exist U∈EθΣ(X, x) (resp. U∈δ-β-θΣ(X, x)) and V∈EθΣ(Y, y) (resp. V∈δ-β-θΣ(Y, y)) such that *f* (U)⋂e-Cl_θ(V)(resp. δ-β-Cl_θ(V)) = Ø .
b) Strongly e-closed[1] if for each (x, y)∈(X×Y)\G(*f*), there exist U∈EΣ(X, x) and an open set V in Y containing y such that (e-Cl(U)×V)⋂G (*f*) = Ø.
c) Strongly δ-β-closed [2] if for each (x, y)∈(X×Y)\G(*f*), there exist U ∈ δ-βΣ(X, x) and an open set V in Y containing y such that (δ-β-Cl(U)×V)⋂ G(*f*) = Ø.

Theorem 3.3: Let G(*f*) be the graph of a mapping *f*: (X, T) → (Y, T^*). Then following statements are equivalent:

a) A graph G(*f*) of *f* is strongly e-θ-(resp. δ-β_θ)-closed;
b) For each (x, y)∈(X×Y)\G(*f*), there exist U∈EθΣ(X, x) (resp. U∈δ-β-θΣ(X, x)) and V∈EθΣ(Y, y) (resp. V∈δ-β-θΣ(Y, y)) such that *f* (U)⋂V = Ø;
c) For each (x, y)∈(X × Y)\G(*f*), there exist U∈EΣ(X, x) (resp. U∈δ-βΣ(X, x)) and V∈EΣ(Y, y) (resp. V∈δ-βΣ(Y, y)) such that *f* (e-Cl(U) (resp. δ-β-Cl(U))) ⋂ V = Ø;
d) For each (x, y)∈(X × Y)\G(*f*), there exist U∈ER(X, x) (resp. U∈δ-βR(X, x)) and V∈ ER(Y, y) (resp. V∈δ-βR(Y, y)) such that *f* (U)⋂V = Ø;
e) A graph G(*f*) of *f* is strongly e(resp. δ-β)-closed{in the sense of [1] and [2] respectively}.

IOP Conf. Series: Journal of Physics: Conf. Series **1234** (2019) 012101 doi:10.1088/1742-6596/1234/1/012101

Proof: (a) ⇒ (b). This proof is obvious thus omitted.

(b) ⇒ (c). For U ∈ EθΣ(X, x) (resp. U ∈ δ-β-θΣ(X, x)), there exists U_o ∈ EΣ(X, x) (resp. U_o ∈ δ-βΣ(X, x)) such that x ∈ U_o ⊂ e-Cl(U_o) (resp. δ-β-Cl(U_o)) ⊂ U.

(c) ⇒ (d). For each (x, y) ∈ (X × Y)\G(*f*), by using part (3) there exist U ∈ EΣ(X, x) (resp. U ∈ δ-βΣ(X, x)) and V ∈ EθΣ(Y, y) (resp. V ∈ δ-β-θΣ(Y, y)) such that *f* (e-Cl(U) (resp. δ-β-Cl(U))) ⋂ V = Ø. now by using lemma (2.1), we have e-Cl(U) (resp. δ-β-Cl(U)) ∈ ER(X, T) (resp. δ-βR(X, T)) and there exist V_o ∈ ER(Y, T^*) (resp. V_o ∈ δ-βR(Y, T^*)) such that y ∈ V_o ⊂ V; thus *f* (e-Cl(U) (resp. δ-β-Cl(U))) ⋂ V_o = Ø.

(d) ⇒ (e). This proof is clear thus omitted.

(e) ⇒ (a). For each (x, y) ∉ G(*f*), there exist U ∈ EΣ(X, x) (resp. U ∈ δ-βΣ(X, x)) and V ∈ EΣ(Y, y) (resp. V ∈ δ-βΣ(Y, y)) such that *f* (e-Cl(U) (resp. δ-β-Cl(U))) ⋂ e-Cl(V) (resp. δ-β-Cl(V)) = Ø. Since, x ∈ U ⊂ e-Cl(U) (resp. δ-β-Cl(U)) ∈ ER(X, x) (resp. δ-βR(X, x)) ⊂ EθΣ(X, x) (resp. δ-β-θΣ(X, x)) and, y ∈ V ⊂ e-Cl(V) (resp. δ-β-Cl(V)) ∈ ER(Y, y) (resp. δ-βR(Y, y)) ⊂ EθΣ(Y, y) (resp. δ-β-θΣ(Y, y)) ⋂ EθC(Y, y) (resp. δ-β-θC(Y, y)), *f* (e-Cl(U) (resp. δ-β-Cl(U))) ⋂ e-$Cl_θ$(e-Cl(V)) (resp. δ-β-$Cl_θ$(δ-β-Cl(V)) = *f* (e-Cl(U) (resp. δ-β-Cl(U))) ⋂ e-Cl(V)) (resp. δ-β-Cl(V)) = Ø.
Now we recall that, a mapping *f* is said to be weakly e-irresolute (resp. weakly δ-β-irresolute) [12] if for each point x ∈ X and each V ∈ EΣ(Y, *f* (x)) (resp. V ∈ δ-βΣ(Y, *f* (x))), there exists U ∈ EΣ(X, x) (resp. U ∈ δ-βΣ(X, x)) such that, *f* (U) ⊂ e-Cl(V)(resp. *f* (U) ⊂ δ-β-Cl(V)).

Lemma 3.2: The following properties are equivalent for a mapping *f*: (X, T) → (Y, T^*):

a) *f* is said to be weakly e-irresolute (resp. weakly δ-β-irresolute);
b) f^{-1}(V) ∈ ER(X, T) (resp. δ-βR(X, T)) ∀ V ∈ ER(Y, T^*) (resp. δ-βR(Y, T^*));
c) f^{-1}(V) ∈ EθΣ(X, T) (resp. δ-β-θΣ(X, T) ∀ V ∈ EθΣ(Y, T^*) (resp. δ-β-θΣ(Y, T^*);
d) f^{-1}(V) ∈ EθC(X, T) (resp. δ-β-θC(X, T)) ∀ V ∈ EθC(Y, T^*) (resp. δ-β-θC(Y, T^*))

Proof: It follows from [12] (Theorems 3.3 and 3.5).

Theorem 3.4: let *f*: (X, T) → (Y, T^*) be weakly e-irresolute (resp. weakly δ-β-irresolute) mappings and Y be an e-θ-T_2 (resp. δ-$β_θ$-T_2) then G(*f*) is Strongly e-θ (resp. δ-$β_θ$)-closed.

Proof: let (x, y) ∈ (X × Y)\G(*f*). The e-θ-T_2 (resp. δ-$β_θ$-T_2) of Y gives the existence of a set V ∈ EθΣ(Y, y) (resp. V ∈ δ-β-θΣ(Y, y)) such that *f* (x) ∉ e-$Cl_θ$(V)(resp. δ-β-$Cl_θ$(V)). Now Y \ e-$Cl_θ$(V) ∈ EθΣ(Y, *f* (x)) (resp. Y \ δ-β-$Cl_θ$(V) ∈ δ-β-θΣ(Y, *f* (x)) Therefore, by the weakly e-irresoluteness (δ-β-irresoluteness) of *f* there exists U ∈ EθΣ(X, x) (resp. U ∈ δ-β-θΣ(X, x)) such that *f* (U) ⊂ Y \ e-$Cl_θ$(V) (resp. *f* (U) ⊂ Y \ δ-β-$Cl_θ$(V). Consequently, *f* (U) ⋂ e-$Cl_θ$(V) = Ø (resp. *f* (U) ⋂ δ-β-$Cl_θ$(V) = Ø) and therefore G(*f*) is Strongly e-θ (resp. δ-$β_θ$)-closed.

Theorem 3.5: If *f*: (X, T) → (Y, T^*) is surjective mapping and has a strongly e-θ-(resp. δ-$β_θ$)-closed graph G(*f*), then Y is e-θ-T_2 (resp. δ-$β_θ$-T_2).

Proof: let y_1 & y_2 ∈ Y be distinct points. The surjectivity of *f* gives an x_1 ∈ X such that *f* (x_1) = y_1. Now (x_1, y_2) ∈ (X × Y)\G(*f*) the strong e-θ-closedness (resp. δ-$β_θ$-closedness) of G(*f*) provides U ∈ EθΣ(X, x_1) (resp. U ∈ δ-β-θΣ(X, x_1)) and V ∈ EθΣ(Y, y_2) (resp. V ∈ δ-β-θΣ(Y, y_2)) (s. t) *f* (U) ⋂ e-$Cl_θ$(V) = Ø (resp. *f* (U) ⋂ δ-β-$Cl_θ$(V) = Ø), whence one infers that y_1 ∉ e-$Cl_θ$(V) (resp. y_1 ∉ δ-β-$Cl_θ$(V)). This means that there exists H ∈ EΣ(Y, y_1) (resp. H ∈ δ-β-Σ(Y, y_1)) such that e-Cl(H) ⋂ V= Ø (resp. δ-β--Cl(H) ⋂ V= Ø). So, Y is e-θ-T_2 (resp. δ-$β_θ$-T_2).

Theorem 3.6: A topological space (X, T) is e-θ-T_2 (resp. δ-β_θ-T_2) iff the identity mapping *if*: (X, T) → (X, T) has a strongly e-θ-(resp. δ-β_θ)-closed graph G(*if*).

Proof (Necessity): let (X, T) be e-θ-T_2 (resp. δ-β_θ-T_2). Since the identity mapping *if*: (X, T) → (X, T) is weakly e-irresolute (resp. weakly δ-β-irresolute) then by Theorem (3.4), G(*if*) strongly e-θ-(resp. δ-β_θ)-closed.
(Sufficiency): let G(*if*) be a strongly e-θ-(resp. δ-β_θ)-closed graph. Then the surjectivity of (*if*) and strong e-θ-(resp. δ-β_θ)-closedness of G(*if*) together imply, by Theorem-(3.5), that A topological space (X, T) is e-θ-T_2 (resp. δ-β_θ-T_2).

Theorem 3.7: let *f*: (X, T) → (Y, T^*) be a injection mapping and has a strongly e-θ-(resp. δ-β_θ)-closed graph G(*f*), then X is e-θ-T_2 (resp. δ-β_θ-T_2) .

Proof: since *f* injective, for any pair of distinct points x_1 & $x_2 \in$ X, $f(x_1) \neq f(x_2)$. Then (x_1, $f(x_2)$) ∈ (X × Y) \ G(*f*). Since G(*f*) is strongly e-θ-(resp. δ-β_θ)-closed, then via Theorem (3.3) there exist U∈ EθΣ(X, x_1) (resp. U∈δ-β-θΣ(X, x_1)) and V∈EθΣ(Y, $f(x_2)$) (resp. V∈δ-β-θΣ(X, $f(x_2)$)) such that *f* (U) ⋂ V = Ø. Therefore $x_2 \notin$ U. via Lemma (2.3) there exists, K∈ER(X, x) (resp. K ∈ δ-βR(X, x)) such that $x_1 \in K \subset U$. Therefore, $x_2 \in X \setminus U \subset X \setminus K \in$ ER(X, T) (resp. $x_2 \in X \setminus U \subset X \setminus K \in$ δ-βR(X, T)) also via Theorem (3.1), X is e-θ-T_2 (resp. δ-β_θ-T_2).

Theorem 3.8: let *f*: (X, T) → (Y, T^*) be a bijective mapping with a strongly e-θ-(resp. δ-β_θ)-closed graph G(*f*), then both X and Y are e-θ-T_2 (resp. δ-β_θ-T_2).

Proof: The proof is consequence immediately of Theorems (3.5) and (3.7).

Theorem 3.9: if a bijection mapping *f*: (X, T) → (Y, T^*) is pre-e-θ-(resp. pre-δ-β_θ)-open and X is e-θ-T_2 (resp. δ-β_θ-T_2), then G(*f*) is strongly e-θ-(resp. δ-β_θ)-closed.

Proof: let (x, y)∈(X × Y) \ G(*f*). Then y ≠ *f* (x), since *f* bijective, $f^{-1}(y) \neq x$. Since X is e-θ-T_2 (resp. δ-β_θ-T_2), there exists U_x & U_y ∈ EθΣ(X, T) (resp. U_x & U_y ∈δ-β-θΣ(X, T)) such that x ∈ U_x & $f^{-1}(y) \in U_y$ and $U_x \cap U_y$ = Ø. Furthermore, since *f* pre-e-θ-(resp. pre-δ-β_θ)-open and bijective, thus $f(x) \in f(U_x)$ ∈ EθΣ(Y, T^*)(resp. $f(x) \in f(U_x)$ ∈ δ-β-θΣ(Y, T^*)) & y ∈ $f(U_y)$ ∈ EθΣ(Y, T^*)(resp. y ∈ $f(U_y)$ ∈ δ-β-θΣ(Y, T^*)) and $f(U_x) \cap f(U_y)$ = Ø. This shows that G(*f*) is strongly e-θ-(resp. δ-β_θ)-closed.

Theorem 3.10: let *f*: (X, T) → (Y, T^*) be a mapping with a strongly e-θ-(resp. δ-β_θ)-closed graph, then, *f* (x) = ∩ { e-Cl_θ(*f* (U)) (resp. δ-β-Cl_θ(*f* (U)) : U∈ EθΣ(X, x) (resp. U∈ δ-β-θΣ (X, x))} for each x ∈ X.

Proof: Suppose that there exists a point y ∈ Y, such that y ≠ *f* (x) and
y ∈ ∩ { e-Cl_θ(*f* (U)) (resp. δ-β-Cl_θ(*f* (U)) : U∈ EθΣ(X, x) (resp. U∈ δ-β-θΣ (X, x))}. This implies that y ∈ e-Cl_θ(*f* (U)) (resp. y ∈ δ-β-Cl_θ(*f* (U)) for every U∈ EθΣ(X, x) (resp. U∈ δ-β-θΣ (X, x)). Thus, e-Cl(V) ⋂ *f* (U) ≠ Ø (resp. δ-β-Cl(V) ⋂ *f* (U ≠ Ø for each V∈ EΣ(Y, y) (resp. V∈ δ-β-Σ(Y, y)). This indicates that
e-Cl(V) ⋂ *f* (U) ≠ Ø ⊂ e-Cl_θ(V) ⋂ *f* (U) (resp. δ-β-Cl(V) ⋂ *f* (U) ≠ Ø ⊂ δ-β-Cl_θ(V) ⋂ *f* (U)). Which contradicts the hypothesis that *f* is a mapping with a strongly e-θ-(resp. δ-β_θ)-closed graph. Hence the theorem holds.

4. Characterizations of strongly e-θ-(δ-βθ)-Closed Graphs and nearly Φ-compact spaces

In this section first of all, as a consequence of Remark (2.4) we introduce the following definition:

Definition 4.1: A topological space (X, T) is said to be an e-θ-(resp. δ-β_θ)-space if the union of any two e-θ-(resp. δ-β_θ)-closed sets is an e-θ-(resp. δ-β_θ)-closed set .

Theorem 4.1: let *f*, *g*: (X, T) → (Y, T^*) be mappings, Y an e-θ-T_2 (resp. δ-β_θ-T_2) and let X be an e-θ-(resp. δ-β_θ)-space. If *f* & *g* weakly e-irresolute (resp. weakly δ-β-irresolute), then Ą= {x ∈ X: *f(x)* = *g(x)*} is e-θ-(resp. δ-β_θ)-closed in X.

Proof: if x ∈ X\Ą, then *f* (x) ≠ *g* (x). Since Y is e-θ-T_2 (resp. δ-β_θ-T_2), there exist V_1∈ EθΣ(Y, *f* (x)) (resp. V_1∈ δ-β-θΣ(Y, *f* (x)) and V_2∈ EθΣ(Y, *g* (x)) (resp. V_2∈ δ-β-θΣ(Y, *g*(x)) such that $V_1 \cap V_2$ = Ø. By the fact that *f* & *g* are weakly e-irresolute (resp. weakly δ-β-irresolute), $f^{-1}(V_1)$ and $g^{-1}(V_2)$ are e-θ-(resp. δ-β_θ)-open. Since X is an e-θ-(resp. δ-β_θ)-space, so x ∈ $f^{-1}(V_1) \cap g^{-1}(V_2)$ ∈ EθΣ(X, T) (resp. x ∈ $f^{-1}(V_1) \cap g^{-1}(V_2)$ ∈ δ-β-θΣ (X, T)). Put U = $f^{-1}(V_1) \cap g^{-1}(V_2)$, therefore U ∈ EθΣ(X, T) (resp. U ∈ δ-β-θΣ (X, T)) and U ⋂ Ą = Ø. This explains that Ą is e-θ-(resp. δ-β_θ)-closed in X.

Definition 4.2: A subset Ą of a Topological space (X, T) is said to be nearly Φ-compact relative to X if for every cover {V_λ: λ∈ Δ} of Ą via e-θ-(resp. δ-β_θ)-open sets of X, there exists a finite subset Δ_0 of Δ such that Ą ⊂ ⋃ {e-Cl(V_λ) (resp. δ-β- Cl(V_λ)) : λ∈ Δ_0 }. If Ą = X, the space X is said to be nearly Φ-compact.

Theorem 4.2: For a Topological space (X, T), the following properties are equivalent:

a) X is nearly Φ-compact;
b) Every cover of X via e-(resp. δ-β)-Regular sets has a finite sub-cover;
c) For every family {V_λ: λ∈ Δ} of e-(resp. δ-β)-Regular sets of (X, T) such that ∩ {V_λ: λ∈ Δ } = Ø, there exists a finite subset Δ_0 of Δ such that ∩ {V_λ: λ∈ Δ_0} = Ø.

Proof: (a) ⇒ (b) ⇒ (c): This proof is obvious thus omitted.

(c) ⇒ (a): Let {V_λ: λ∈ Δ} be a cover of X via e-θ-(resp. δ-β_θ)-open sets of X. Via Lemma (2.1), {e-$Cl_\theta(V_\lambda)$ (resp. δ-β-$Cl_\theta(V_\lambda)$) : λ∈ Δ} is e-(resp. δ-β)-Regular cover of *X*. Therefore {X \ e-$Cl_\theta(V_\lambda)$ (resp. X \ δ-β-$Cl_\theta(V_\lambda)$) : λ∈ Δ} is a family of e-(resp. δ-β)-Regular sets of *X* having the empty intersection. Via (c), there exists a finite subset Δ_0 of Δ such that ∩ {X \ e-$Cl_\theta(V_\lambda)$ (resp. X \ δ-β-$Cl_\theta(V_\lambda)$) : λ∈ Δ_0 } = Ø; thus X = ⋃ {e-$Cl_\theta(V_\lambda)$ (resp. δ-β-$Cl_\theta(V_\lambda)$) : λ∈ Δ_0 }. This shows that *X* is nearly Φ-compact.

Lemma 4.1: Let Q be an e-θ-(resp. δ-β_θ)-closed sets of a nearly Φ-compact space (X, T), then Q is nearly Φ-compact relative to X.
Proof: Let {V_λ: λ∈ Δ} be a cover of Q via e-(resp. δ-β)-Regular sets of X. For each x ∈ X \ Q, there exists U_x ∈ EΣ(X, x) (resp. U_x ∈ δ-β-Σ(X, x)) such that e-Cl(U_x) ⊂ X\ Q (resp. δ-β-Cl(U_x) ⊂ X\ Q). Via Lemma (2.1), e-Cl(U_x) ∈ ER(X, x) (resp. δ-β-Cl(U_x) ∈ δ-βR(X, x)) for each x∈ X and the family{e-Cl(U_x) (resp. δ-β-Cl(U_x)): x ∈ X \ Q}⋃ {V_λ: λ∈ Δ} is a cover of X via e-(resp. δ-β)-Regular sets of X. Since X is nearly Φ-compact, there exists a finite subset Δ_0 of Δ such that Q ⊂ ⋃ {V_λ: λ∈ Δ_0}. Hence, by Theorem (4.2) Q is nearly Φ-compact relative to X.

Remark 4.1: Noiri [13] showed that if G(*f*) is strongly closed then *f* has the following property:

Proposition 4.1: [13] for every set *B* quasi *H*-closed relative to (Y, T^*), f^{-1} (*B*) is a closed set of (X, T).
Analogously, we have the following result.

Theorem 4.3: let *f*: (X, T) → (Y, T^*) be a mapping has a strongly e-θ-(resp. δ-β_θ)-closed graph and let (X, T) be an e-θ-(resp. δ-β_θ)-space, then it has the following property:

(P*)- For every set *M* which is nearly Φ-compact relative to (Y, T^*), f^{-1} (*M*) is e-θ-(resp. δ-β_θ)-closed subset of (X, T).
***Proof*:** let f^{-1} (*M*) be not e-θ-(resp. δ-β_θ)-closed in X. Then there exists x ∈ e-Cl_θ(f^{-1} (*M*)) \ f^{-1} (*M*) (resp. x ∈ δ-β-Cl_θ(f^{-1} (*M*)) \ f^{-1} (*M*)). Let y ∈ *M*. then (x, y) ∈ (X × Y) \ G(*f*). Strong e-θ-(resp. δ-β_θ)-closedness of G(*f*) gives the existence of $U_y(x)$ ∈ EθΣ(X, x) (resp. $U_y(x)$ ∈ δ-β-θΣ (X, x)) and V_y ∈ EθΣ(Y, y) (resp. V_y ∈ δ-β-θΣ (Y,y)) (s. t) *f* ($U_y(x)$) ∩ e-$Cl_\theta(V_y)$ (resp. *f* ($U_y(x)$) ∩ δ-β-$Cl_\theta(V_y)$) = Ø .
Clearly { V_y : y ∈ *M* } is a cover of *M* by e-θ-(resp. δ-β_θ)-open sets in Y. The near Φ-compactness of *M* relative to Y guarantees the existence e-θ-(resp. δ-β_θ)-open sets V_{y1}, V_{y2}, V_{y3}, ….., V_{yn} in Y (s. t)

$$\boldsymbol{M} \subset \bigcup_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{e-Cl_\theta}\left(\mathbf{V_{y_i}}\right) (\mathbf{resp.}\, \boldsymbol{M} \subset \bigcup_{\mathbf{i=1}}^{\mathbf{n}} \boldsymbol{\delta} - \boldsymbol{\beta} - \mathbf{Cl_\theta}\left(\mathbf{V_{y_i}}\right))$$

$$\mathbf{Let}\ \boldsymbol{U} = \bigcap_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{U_{y_i}}\,(\boldsymbol{x})$$

Where U_{yi} (x) are the e-θ-(resp. δ-β_θ)-open sets in X satisfying (a), moreover U∈ EθΣ(X, x) (resp. U∈ δ-β-θΣ (X, x)). Now we have

$$f(U) \cap M \subset f\left[\bigcap_{i=1}^{n} U_{y_i}(x)\right] \cap \left(\bigcup_{i=1}^{n} e - Cl_\theta\left(V_{y_i}\right)\right) \subset \bigcup_{i=1}^{n}\left(f\left[U_{y_i}(x)\right] \cap e - Cl_\theta\left(V_{y_i}\right)\right) = \varphi.$$

Respectively:

$$f(U) \cap M \subset f\left[\cap_{i=1}^{n} U_{y_i}(x)\right] \cap \left(\cup_{i=1}^{n} \delta - \beta - Cl_\theta\left(V_{y_i}\right)\right) \subset \cup_{i=1}^{n}\left(f\left[U_{y_i}(x)\right] \cap \delta - \beta - Cl_\theta\left(V_{y_i}\right)\right) = \varphi.$$

But *x* ∈ e-Cl_θ(f^{-1} (*M*)) (resp. x ∈ δ-β-Cl_θ(f^{-1} (*M*)), therefore U ⋂ f^{-1} (*M*) ≠ Ø. which contradicts to the above deduction. Thus the result holds.

Theorem 4.4: let (X, T) be an e-θ-(resp. δ-β_θ)-space and let (Y, T^*) be nearly Φ-compact, then a mapping *f*: (X, T) → (Y, T^*) with a strongly e-θ-(resp. δ-β_θ)-closed graph is weakly e-irresolute (resp. weakly δ-β-irresolute).
Proof: let Q be any e-θ-(resp. δ-β_θ)-closed set of Y. Since Y is nearly Φ-compact, by Lemma (4.1) Q is nearly Φ-compact relative to Y and it follows from Theorem (4.3) that f^{-1} (Q) is e-θ-(resp. δ-β_θ)-closed set in X. then via Lemma (3.2) We get *f* is weakly e-irresolute (resp. weakly δ-β-irresolute) mapping .

Corollary 4.1: let (X, T) be an e-θ-(resp. δ-β_θ)-space and let (Y, T^*) be an e-irresolute semi-T_2 space (resp. δ-β-irresolute semi-T_2 space) then for mapping *f*: (X, T) → (Y, T^*) the following properties are equivalent:

a) G(*f*) is strongly e-θ-(resp. δ-β_θ)-closed graph;
b) f^{-1} (Q) is e-θ-(resp. δ-β_θ)-closed set in X for every subset Q which is nearly Φ-compact relative to Y;
c) *f* is weakly e-irresolute (resp. weakly δ-β-irresolute) mapping.

Proof (a) ⇒ (b): This proof is an immediate consequence of Theorem (4.3).

Proof (b) ⇒ (c): This proof follows from Lemmas (4.1) and (3.2).

Proof (c) ⇒ (a): This proof follows from Theorems (3.4) and (3.3).

5. Conclusion

Topology as a field of mathematics is concerned with all questions directly or indirectly related to continuity. It is well known that various types of functions play a significant role in the theory of classical point set topology and a great number of papers dealing with such functions have appeared [14, 15, 16]. In this article we introduced and investigated new classes of mappings with Strongly Closed Graphs in Topological spaces namely mappings with strongly e-θ-(resp. δ-β_θ)-Closed Graphs which is may have possible applications in quantum physics, high energy physics, superstring theory, computer science, digital topology, computational topology for geometric and molecular design. Several characterizations and fundamental properties concerning of such these mappings are obtained. On the other hand, the mathematical theory of fuzzy sets is highly developed and used extensively in many practical and engineering problems. Fractal geometry is, by its very nature, fuzzy, and that is how K3 which is used in string theory for other purposes could be given, a fuzzy outlook. So the fuzzy Topological version of the notions and results introduced in this paper are very important .

Acknowledgments

I would like to express my sincere gratitude to the referees for their valuable suggestions and comments which improved the paper and I am thankful to professor. Erdal Ekici (Turkey) for sending many of his papers as soon as I had requested and for help .

References

[1] M. ÄOzkoc, G. Aslim, On Strongly θ-e-Continuous Functions. Bull. Korean Math. Soc. Vol. 47(5), 1025-1036, (2010).
[2] A. M. F. AL. Jumaili and X. S. Yang, New Type of Strongly Continuous Functions In topological Spaces Via δ-β-Open Sets, European Journal of Pure and Applied Mathematics, 8 (2), 185-200 (2015).
[3] N. Levine, Generalized closed sets in topology, Rend. Circ. Math. Palermo, 19, 89-96, (1970).
[4] E. Ekici, On e-open sets, DP^*-sets and DPE^*-sets and decompositions of continuity, The Arabian J. for Sci. Eng, 33 (2A), 269-282, (2008).
[5] E. Ekici, New forms of contra-continuity. Carpathian J. Math. 24(1), 37-45, (2008).
[6] E. Ekici, On e^*-Open Sets and $(D, S)^*$-Sets, Math. Moravica, 13 (1), 29-36, (2009).
[7] E. Ekici, A note on *a*-open sets and e^*-sets, Filomat, 22 (1), 87-94, (2008).
[8] E. Hatir, and T. Noiri. Decompositions of continuity and complete continuity, Acta, Math. Hungar. 113 (4), 281-287, (2006).
[9] N. V. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl. Ser. 278, 103-118, (1968).
[10] T. Husain. Topology and Maps. Plenum press, New York, (1977).
[11] E. Hatir and T. Noiri, On δ-β-continuous functions, Chaos, Solitons and Fractals 42, 205–211, (2009).
[12] Alaa. M. F. AL. Jumaili, On Some Weak and Strong Forms of Irresolute Mappings in Topological Spaces via E-Open and δ-β-Open Sets, International Conference on Mathematics, Engineering and Industrial Applications 2018 (ICoMEIA 2018), AIP Conf. Proc. 2013.
[13] T. Noiri, on functions with strongly closed graphs. Acta Math. Acad. Sci. Hungar. 32, 1- 4, (1978).
[14] Caldas, M, S. Jafari, T. Noiri and M. Simoes, A new generalization of contra-continuity via Levine's g-closed sets. Chaos. Solitons & Fractals., 32, 1597–1603, (2007).

[15] Alaa. M. F. AL. Jumaili, New types of Strongly Functions and Quasi Functions in Topological Spaces Via e-Open Set, Computational and Applied Mathematics Journal. 1(5), 386-392, (2015).
[16] Alaa. M. F. Al. Jumaili, Alaa A. Auad and Majid M. Abed, A New Type of Strongly Faint Continuous Mappings and Their Applications in Topological Spaces, Journal of Engineering and Applied Sciences 14 (3): 905-912, (2019).