GENERALIZATION OF RAD-D_{11}-MODULE

MAJID MOHAMMED ABED, ABD GHAFUR AHMAD AND A. O. ABDULKAREEM

ABSTRACT. This paper gives generalization of a notion of supplemented module. Here, we utilize some algebraic properties like supplemented, amply supplemented and local modules in order to obtain the generalization. Other properties that are instrumental in this generalization are D_i, SSP and SIP. If a module M is *Rad*-D_{11}-module and has D_3 property, then M is said to be completely-*Rad*-D_{11}-module (C-*Rad*-D_{11}-module). Similarly it is for M with SSP property. We provide some conditions for a supplemented module to be C-*Rad*-D_{11}-module.

1. Introduction and Preliminaries

Throughout this paper all rings are unital and modules are considered to be right modules. A submodule N of M is small in M ($N \ll M$) if for every submodule L of M with $N+L=M$, $L=M$. In [1], we have that any module M is called hollow module if every proper submodule of M is a small in M. The direct summand plays vital role in generalization of supplemented module. A submodule N of M is called supplement of K in M if $N+K=M$ and N is minimal with respect to this property. A module M is called supplemented if any submodule N of M has a supplement in M.

In [2], Y. Talebi and A. Mahmoudi studied C-*Rad*-D_{11}-module through *Rad*-$\bigoplus$-supplemented modules and D_3 property to get C-*Rad*-D_{11}-module. Here we use other algebraic properties such as supplemented, amply supplemented and local modules to get generalization of *Rad*-D_{11}-module.

Also summand sum property (SSP) and summand intersection property (SIP) are very important in generalization of supplemented module (see Definition 3.1). From [2], if M is a *Rad*-D_{11}-module and has SSP property, then M is a C-*Rad*-D_{11}-module. If M_1 and M_2 are direct summands of M with $M=M_1+M_2$ and $M_1 \cap M_2$ is also direct summand of M and so M has D_3 property. Note that, the D_3-module with *Rad*-D_{11}-module already gives a C-*Rad*-D_{11}-module. There is another module called T_1-module which has relationship with C-*Rad*-D_{11}-module. If for every submodule K of M such that M/K is isomorphic to a co-closed submodule of M and every homomorphism $\mu : M \to M/K$ lifts to a homomorphism β:$M \to M$ in this case M is called T_1-module [4]. If M is a local module then it is a C-*Rad*-D_{11}-module. On the other hand, M is C-*Rad*-D_{11}-module if it is projective, supplemented and has D_3 properties. There have been different notions

2000 *Mathematics Subject Classification.* 54C05, 54C08, 54C10
Corresponding author majid mohammed abed .
Key words and phrases. *Rad*-supplemented, D_3-module, UC-module, (SSP) property, SIP property, indecomposable module.

of generalization of supplemented module conducted by many researchers. These generalizations are motivated by different properties of supplemented module. However, in this study we try to give another notion of generalization for supplemented module. It is interesting to note that several properties of supplemented module have been harnessed to give important properties of the generalization considered.

In Section 2, we give some properties of C-Rad-D_{11}-module. We proved that if M is a projective and local module with D_2 property then it is a C-Rad-D_{11}-module. Necessary condition for a supplemented module to have a generalization of rank three is also given. In section 3, we study three properties (injective, SSP and SIP) of supplemented module over Rad-D_{11}-module. An easy to follow proof of the consequence of each property is provided. Using unique closure (UC) and extending properties of a module, we give necessary and sufficient condition for Rad-D_{11}-module to be C-Rad-D_{11}-module

2. ALGEBRAIC PROPERTIES AND C-RAD-D_{11} MODULE

In this section, we utilize some algebraic properties in order to obtain rank three generalization of supplemented module. Let M be an R-module. From [5], a module is said to be a D_i-module) if it satisfies D_i (i=1, 2, 3) condition. A module M is called D_1 if for every submodule A of a module M, there is a decomposition $M = M_1 \bigoplus M_2$ such that $M_1 \in A$ and $A \cap M_2 \ll M_2$.
Equivalently, any module M is called lifting if for all N submodule of M there is a decomposition

$$M = H \bigoplus G \ni H \in N \text{ and } N \cap H \leq M$$

.
A module M is called D_2 if $A \leq M$ such that M/K is isomorphic to a summand of M and this implies that A is a summand of M.
For N and L are submodules of M, L is a radical supplement (Rad-supplement) of N in M if

$$N + L = M \text{ and } N \cap L \ll Rad(L).$$

On the other hand from [6], M is called D_{11}-module if every submodule of M has supplement which is a direct summand of M. Therefore, any module M is called a Rad-D_{11}-module if every submodule of M has a Rad-supplement that is a direct summand of M. Also, a module M is called semiperfect if every factor module of M has a projective cover.

Definition 2.1. A module M is called C-Rad-D_{11}-module if every direct summand of M is Rad-D_{11}-module.

In [7], N.O. Ertas, gives the direct sum of additive Abelian groups $A \bigoplus B$, A and B are called direct summands. The map

$$\alpha_1 : A \to A \bigoplus B$$

defined by the rule $\alpha(a) = a \bigoplus 0$ is called the injection of the first summand and the map

$$p_1 : A \bigoplus B \to A$$

defined by $p_1(a \bigoplus b) = a$ is called the projection onto the first summand. Similar maps α_2, p_2 are defined for the second summand B. Equivalently, the direct sum of objects A_i with $\alpha \in I$ is denoted by $A = \bigoplus A_\alpha$ and each A_α is called direct summand of A.

Remark 2.2. Let A and B be two direct summands of an Abelian group S such that $A + B = S$. Then the intersection of A and B is not a direct summand of S. An example is given by

$$S = K_2 \bigoplus K_8, \ A = \langle(1,1)\rangle \text{ and } B = \langle(0,1)\rangle.$$

Then S is the direct sum of A and ⟨(1,4)⟩ and of B and ⟨(1,0)⟩. The intersection is

$$\langle(0,2)\rangle \cong K_4,$$

which is not a direct summand of S because S is not isomorphic to $K_4 \bigoplus K_4$ or to $K_2 \bigoplus K_2 \bigoplus K_4$. The group generated by A and B is S because it contains $(0,1)$ and $(1,0) = (1,1) - (0,1)$.

From the definitions of D_2 and D_3 properties, it is obvious that D_2 implies D_3. Other properties like supplemented, amply supplemented and local modules are inherited by summands property. Also Rad-D_{11}-module take the same inherited property. It is a fact that there is an equivalence between supplemented and amply supplemented module. Finally, we say that:

(*) If M is a *Rad*-D_{11}-module with D_2 property then it is a C-*Rad*-D_{11}-module.

Lemma 2.3. *Let M be an R-module. If M has largest submodule then it is a supplemented module.*

The next theorem whose proof shall be given at the end of this section is one of the main results of this study.

Theorem 2.4. *Let M be an R-module, if M satisfies the following conditions;*
1. M is a projective module,
2. M is local module,
3. M is D_2-module,
then M is a C-Rad-D_{11}-module.

Lemma 2.5. *Let M be an R-module. If M is a projective and supplemented then it is a Rad-D_{11}-module.*

Proof. Since M is a projective module with $M = N + K$ then M is a π-projective. But M is a supplemented module. Therefore M is amply supplemented. Since a projective module with amply supplemented property is a semiperfect module, we infer that M is a *Rad*-D_{11}-module. □

Theorem 2.6. *Let M be a projective and local module. If M is D_2-module then it is a C-Rad-D_{11}-module.*

Proof. Let M be a projective and local module. Then from Lemma 2.5 we get M is a *Rad*-D_{11}-module. Now we must show that B is a *Rad*-supplemented. In other word, B has a *Rad*-supplement in A that is a direct summand of A. Let A be a direct summand of M and B, submodule of A. Since M is a *Rad*-D_{11}-module, then there exists a direct summand C of $M \ni M = B + C$ and $B \cap C \leq Rad(C)$.

So,

$$A = B + (A \cap C).$$

But from definition of D_2-module, if $A \leq M$ such that $M = A$ is isomorphic to a summand of M, then A is a summand of M. Thus, M has D_3 property and therefore $A \cap C$ is a direct summand of M. Hence $M_1 \cap M_2$ is also direct summand of M.
Thus,

$$B \cap (A \cap C) = B \cap C,\ B \cap C \leq Rad(M) \text{ and } B \cap C \leq A \cap C.$$

So

$$B \cap C \leq (A \cap C \cap Rad(M)) = Rad(A \cap C).$$

Consequently, by Definition 2.1, M is a C-Rad-D_{11}-module. □

Theorem 2.7. *([8]) Let R be a ring. Then the following statements are equivalent.*
1. R is a left perfect.
2. Every R-module M is a supplemented.
3. Every projective R-module is amply supplemented

The following theorem gives necessary condition for a supplemented module to possess a ran three generalization.

Theorem 2.8. *Let M be projective supplemented module with D_2 property. If every supplement submodule of M is a direct summand, then M has generalization of rank three.*

Proof. Since every supplement submodule of M is a direct summand, M is a D_{11}-module. But M is a supplemented module then it is a strongly-D_{11}-module and thus, R is a perfect ring. From Theorem 2.7 we get M is amply supplemented. But M is projective module. Hence, M is Rad-D_{11}-module. As a consequent of (*), M has a generalization of rank three (C-Rad-D_{11}-module). □

Theorem 2.9. *Let M be a Rad-D_{11}-module with D_1 property. If $M = M_1 \bigoplus M_2$ is a direct sum of submodules M_1 and M_2, then M_1 and M_2 are relatively projective and so M is a C-Rad-D_{11}-module.*

Proof. By [9] and Lemma 2.5. □

Theorem 2.10. *Let M be an R-module. If M satisfies the following conditions:*
1. M is a projective module;
2. M is a semiperfect module;
3. M is D_3-module;
then M is a C-Rad-D_{11}-module.

Proof. Let $A \leq M$. Then by assumption, there exists a projective cover $\varphi : P \to M/K$ and there is an epimorphism $\varphi : M \to M/K$.
Since M is projective then there exists a homomorphism

$$\mu : M \to P \ni \varphi \circ \mu = \beta.$$

Also since φ is small and is an epimorphism, μ splits (P is projective). We have a homomorphism

$$g : P \to M \ni \mu \circ g = 1_P \text{ and } \varphi = \varphi \circ \mu \circ g = \delta \circ g.$$

Since

$$M = Ker(\mu) \bigoplus g(P) \text{ and } Ker(\mu) \leq A; \text{then} M = A + g(p).$$

Let α be the restriction of φ to $g(p)$. Then φ=α and so α is an epimorphism. Also since φ is small, α is also small. That is

$$Ker(\alpha) = A \cap g(p) \ll g(p)$$

then $g(p)$ is a supplement of A. Thus M is *Rad*-D_{11}-module. But M has D_3 property. Then if M_1 and M_2 are direct summands of M with

$$M = M_1 + M_2 \text{ and} M_1 \cap M_2.$$

is also direct summand of M. Thus M is a *C*-*Rad*-D_{11}-module. □

The following is an example of matrix over *Rad*-D_{11}-module which gives *C*-*Rad*-D_{11}-module.

Example 2.11. *Let $M_{4\times4}$ be a matrix over field F such that it satisfies D_2 property.*

$$M = \begin{bmatrix} a & 0 & 0 & 0 \\ y & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & x & 0 \end{bmatrix}$$

such that a, b, x, y in F.

Corollary 2.12. *Let A and B submodules of projective R-module M. If B is a minimal with respect to the property $A + B = M$ then M is C-Rad-D_{11}-module.*

Proof. Let A and B be submodules of M and let A be supplement of B in M. Then M is a supplemented module. We need to show that the homomorphism β from M into M is a split homomorphism. Let 1_M be identity mapping. Let M be projective module then there exists a mapping from M into $A \bigoplus B$ such that $\beta \circ \gamma = 1_M$. This means β is split (M is a π-projective). Let g belongs to endomorphism of M such that $Im(g) \subset A$ and $Im(1 - g) \subset B$ with $M = A + B$. Since $g(A) \subseteq A$, $M = A + g(B)$ and $g(A \cap B) = A \cap g(B)$ such that $n = g(b)$ then $b - n = (1 - g)(b)$ and $b \in A$. Since $A \cap B \ll A$ and $A \cap g(A) \ll g(A)$ where $g(A)$ is a supplement of A and $g(A) \subset B$, M is *C*-*Rad*-D_{11}-module. □

Corollary 2.13. *Let M be a Rad-D_{11}-module. If M is a T_1-module then it is a C-Rad-D_{11}-module.*

Remark 2.14. We are now ready to prove the main result of this section, Theorem 2.4. This theorem further emphasizes the role of D_i-modules in our rank three generalization of supplemented module.

Proof of Theorem 2.4. Let M be a projective module with $M = N + K$. Then M is π-projective. Since M has largest submodule then M is a local module. M contains all proper submodules such that $A \subseteq Rad(M) \ll M$ and so $A \ll M$. Hence M is a hollow module (A submodule of M then $A + M = M$). Again by definition of hollow module we get $A \cap M = A$. We have $A \ll M$ therefore A supplement in M and so M is a supplemented module. Hence M is amply supplemented, with projective property, implies that M is semiperfect and so is a *Rad*-D_{11}-module. Since M has D_2 property, then it is D_3-module. Thus by Lemma 2.5 M is a *C*-*Rad*-D_{11}-module. □

3. INJECTIVE, SSP AND SIP PROPERTIES OVER RAD-D_{11}-MODULE

In this section,our attention is drawn to three properties of supplemented modules; injective, SSP and SIP. Here we investigate these properties over Rad-D_{11}-module for the purpose of our notion of generalization of supplemented module.

Definition 3.1. A module M is said to have the summand sum property SSP if the sum of any pair of direct summands of M is a direct summands of M, i.e., if N and K are direct summands of M then $N+K$ is also a direct summand of M.

Lemma 3.2. *([2]) Let M be a Rad-D_{11}-module. If M has SSP property then M is a C-Rad-D_{11}-module.*

Let N be a submodule of left R-module M. Hence there exists submodule L of M where M is the internal direct sum of N and L. In other words, $N+L=M$ and $N \cap L = 0$. This implies that M is an injective module. Also, a module P is called projective if and only if for every surjective module homomorphism $f : M \to P$ there exists a module homomorphism $h : P \to M$ such that $fh{=}id_P$.

Theorem 3.3. *Let M be an R-module. If A and B any two direct summands of M such that $A \cap B$ is injective R-module then M is a C-Rad-D_{11}-module.*

Proof. Let A and B be direct summands of M. By hypothesis M is injective module because $M \cap M = M$. Therefore any direct summand of M is injective and so A and B are also injective. Again by hypothesis $M = \bigoplus K$ for some $K \leq M$. Hence,

$$A = A \cap B \bigoplus A \cap K.$$

Also,

$$B = (A \cap B) \bigoplus (B \cap K).$$

Thus, $A \cap B$, $A \cap K$ and $B \cap K$ are injective.

We have

$$A + B = (A \cap B) \bigoplus (A \cap K \bigoplus B \cap K).$$

Then, it follows that $A+B$ is injective and so it is a direct summand of M, M has SSP property. Thus, from Lemma 3.2, M is a C-Rad-D_{11}-module. □

Corollary 3.4. *Let M be a projective and supplemented R-module. If $M = A \bigoplus B$ (direct summand of M) and $(A \cap B)$ is an injective R-module, then M is a C-Rad-D_{11}-module.*

Lemma 3.5. *([9]) Let R be a ring. If R is a semisimple then every R-module M has SSP property.*

Definition 3.6. Any module M has C_3 property if M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$ then $M_1 \bigoplus M_2$ is a summand of M.

Recall that an R-module M has the summand intersection property SIP if the intersection of two summands is again a summand. Let M be a projective R-module then M has the SIP property if and only if for any direct summands A and B of M, $A+B$ is a projective R-module.

Theorem 3.7. *Let M be C_3-module. If M has the SIP then M is a C-Rad-D_{11}-module.*

Proof. Let M be C_3-module and has the (SIP) property. Let A and B be a direct summands of M. We must show that $A+B$ is direct summand of M. Since M has the SIP then there exists

$$D \leq M \ni A \cap B \bigoplus D = M.$$

By modularity law, we obtain

$$A = A \cap B \bigoplus D \cap A \text{ and}$$

$$B = A \cap B \bigoplus D \cap A.$$

Then we have

$$A + B = A \cap B + [D \cap A \bigoplus D \cap A].$$

Next we prove that

$$(A \cap B) \cap [D \cap A \bigoplus D \cap B] = 0.$$

For if

$$x \in (A \cap B) \cap [D \cap A \bigoplus D \cap A],$$

then

$$x = n_1 + n_2 \text{ where } n_1 \in (D \cap A) \text{ and } n_2 \in D \cap B.$$

We have

$$n_2 = x - n_1 \in [A \cap B + D \cap A] \cap D \cap B \bigoplus A \cap D \cap B = 0.$$

Hence,

$$n_2 = 0 \text{ and } x = n_1.$$

Now

$$x = n_1 \in A \cap B \cap D \cap A = A \cap B \cap D = 0.$$

Thus,

$$A + B = A \cap B \bigoplus D \cap A \bigoplus D \cap B = B \bigoplus D \cap A.$$

Since M has the SIP property and D and A are direct summands, $D \cap A$ is a direct summand. From C_3 property it follows that $(A + B) = B \bigoplus D \cap A$ is a direct summand of M. Thus M has SSP property and from Lemma 3.2 M is a C-*Rad*-D_{11}-module. □

Corollary 3.8. *Any projective module M with C_3 property over right hereditary ring R is C-Rad-D_{11}-module.*

Proof. Suppose that R is right hereditary and M is any projective R-module. Since every submodule of a projective R-module over right hereditary is projective. Hence M has the SIP. Thus from Theorem 3.7 M is C-*Rad*-D_{11}-module. □

Theorem 3.9. *Let R be left hereditary ring. If M is an injective and Rad-D_{11}-module then it is a C-Rad-D_{11}-supplemented module.*

Proof. Let R be a left hereditary ring. We must prove that M has SIP property. Factor module of every injective R-module is injective. Let M be an injective module which has a decomposition $M = L \bigoplus N$. Let f be a homomorphism from L to N. Then L is injective.
By assumption,

$$Im(h) \approx (L/Ker(h)) \text{ is injective.}$$

Hence $Im(h)$ is direct summand of N. From [10] M has the SIP and so has SSP property. But M is *Rad*-D_{11}-module, thus, by Lemma 2.4 M is a C-*Rad*-D_{11}-module. □

Corollary 3.10. *Let M be Rad-D_{11}-module. If every injective R-module has the SIP property then M is a C-Rad-D_{11}-module.*

Let $\Lambda(M)=\{k\in K{:}kM\subseteq M\}$. Note that $\Lambda(M)$ is a subring of K containing R. For example, if M is the R-module R, then $\Lambda(M)=R$. On the other hand, if S is any subring of K containing R and M is the R-module S then $\Lambda(M)=S$. In particular, $\Lambda(R_K)=K$. For integral domain R, an R-module M is called torsion free if $Ann(a)=0$, for each $0\neq a\in M$ and an R-module M is called uniform if every non-zero submodule of M is essential in M. According to [6], every finitely generated torsion-free uniform R-module is a C-*Rad*-D_{11}-module. Recall that a submodule A of M is called a fully invariant submodule if $g(A)\subseteq A$, for every $g\in Hom(M, M)$[10]. Moreover; in [11], a module M is called a duo-module if every submodule of M is fully invariant.

Theorem 3.11. *Let M be Rad-D_{11}-module. If a commutative domain R is an integrally closed then every finitely generated torsion-free uniform R-module is a C-Rad-D_{11}-module.*

Proof. Suppose that R is integrally closed. Let T be any finitely generated torsion-free uniform R-module. Let k be an element in $\Lambda(T)$. Since $kT\subseteq T$ and k is integral over R, then $k\in R$ and so, $\Lambda(T)=R$. By [11], T is a duo module with *Rad*-D_{11}-module lead to M is a C-*Rad*-D_{11}-module. □

Lemma 3.12. *([2, Lemma 3.4]). Let M be a duo module. Then M has the SIP property.*

In [2], Y. Talebi and A. Mahmoudi calls a module M a UC-module if every submodule of M has a unique closure in M. A module M is called extending if every closed submodule of M is a direct summand of M. Therefore any UC-extending module has D_3 property.

Theorem 3.13. *Let M be UC-extending module. Then M is a Rad-D_{11}-module if and only if M is a C-Rad-D_{11}-module.*

Proof. Sufficiency is clear. Conversely, assume that M is M-supplemented module. From [1], M has D_3 property. Hence M is a completely-*Rad*-D_{11}-module. □

Lemma 3.14. *([12]) Let M be an R-module with Rad(M)=0. If M is a closed weak supplemented module then M is extending.*

Theorem 3.15. *For any ring R the following are equivalent:*

1. *Every left R-module is a lifting.*
2. *Every left R-module is extending.*

Theorem 3.16. *Let M be an Rad-D_{11}-module with the following conditions:*

1. *M is UC-module;*

2. *Rad(M)=0;*

3. *Every nonsingular right* D_{11}*-module is projective;*

then M *is a* C*-Rad-*D_{11}*-module.*

Proof. Let M be a nonsingular module and N a closed submodule of M. Then (M/K) is nonsingular. Since M is a projective then N is a direct summand of M. From [12], M is closed weak supplemented with *Rad*(M)=0 implies that M is extending module. Now from condition (1) with *Rad*-D_{11}-module we obtain M is a *C*-*Rad*-D_{11}-module. (see Theorem 3.13). □

Theorem 3.17. *Let* M *be an* R*-module. If* M *satisfies the following conditions:*

1. M *is* UC*-module;*

2. M *is Rad-*D_{11}*-module;*

3. M *is lifting module;*

then M *is a* C*-Rad-*D_{11}*-module.*

Corollary 3.18. *Let* M *be an* R*-module. If* M *is a local then* M *is a* C*-Rad-*D_{11}*-module.*

Corollary 3.19. *Let* M *be an* R*-module such that every direct summand of* M *is a finite direct sum of hollow modules. If* M *has* D_3 *property then* M *is a* C*-Rad-*D_{11}*-module.*

Corollary 3.20. *Any duo module* M *has* $C3$ *property is* C*-Rad-*D_{11}*-module*

Remark 3.21*.* We are now ready to prove the main result of this section, Theorem 3.7. The classification of *Rad*-D_{11}-module is very important in the process of generalization of supplemented module:

Let S=$\{s_1, \ldots, s_n\}\subset M$ be a set of generators for M over $End_R(M)$. Since a direct sum of semisimple modules is also a sum of simple modules then it is semisimple. So every direct sum of semisimple modules is again semisimple. Hence M^S is a semisimple module. Moreover; we have R-homomorphism $R \to M^S$ and $r \mapsto (rs_1, \ldots, rs_n)$ is injective: Suppose that rs_i=0 for all generators of R as an End_R M-module.

Therefore we can write every

$$x \in M \text{ as } \beta_1(s_1) + \ldots + \beta_n(s_n) \text{ for } \beta_i \in End_R M.$$

So we have,

$$rx = r(\beta_1(s_1) + \cdots + \beta_n(s_n)) = \beta_1(rs_1) + \cdots + \beta_n(rs_n) = 0.$$

Also we see that $rx = 0 \; \forall \; x \in M$. By the faithfulness of M, we conclude that $r = 0$. This shows that R is (as an R-module) isomorphic to a submodule of M^S. Hence R is a semisimple and by Lemma 3.5 we obtain M has SSP property. But M is *Rad*-D_{11}-module. Thus M is a *C*-*Rad*-D_{11}-module. Theorem 3.7 is thus proved.

4. Conclusion

The supplemented module is very important in module theory specially when we study the generalization of supplemented module. In addition we obtained the third generalization of this module by use many concepts as injective module; semiperfect module and D_i-modules. Also we found if M is a Rad-D_{11}-module with D_2 property gives C-Rad-D_{11}-module. Moreover; if M is a C_3-module having the SIP and C_3 properties lead to M is a C-Rad-D_{11}-module.

5. Acknowledgement

The authors would like to acknowledge the financial support received from Universiti Kebangsaan Malaysia under the research Grant UKM-DLP-2013-020. The authors also wish to gratefully acknowledge all those who have generously given of their time to referee our paper.

REFERENCES

[1] N. Orhan, D.T. Tutuncu and R. Tribak, On hollow-lifting modules, Taiwanese J. Math., 11, no. 2, (2007) 545-568.

[2] Y. Talebi and A. Mahmoudi, On Rad-$\bigoplus$-Supplemented Modules, Thai Journal of Mathematics, 9, no. 2, (2011) 373-381.

[3] A. Harmanci, D. Keskin and P.F. Smith, On $\bigoplus$-supplemented modules, Acta Math. Hungar., 83, (1991) 161-169.

[4] N. Orhan and D.T. Tutuncu, Generalization of Weak Lifting Modules, Soochow. J of Math., 32, no. 1, (2006) 71-76.

[5] M. Alkan and A. Harmanci, On Summand Sum and Summand Intersection Property of Modules, Turk J Math., 26, (2002) 131- 147.

[6] B. Talaee, Generalization of D_{11} and $D^+{}_{11}$ Modules. Tarbiat Moallem University, 20^{th} Seminar on Algebra, Ordibehesht. 1388, no. 2-3, (2009) 213-216.

[7] N.O. Ertas, (*)-generalized projective module and lifting modules, International Mathematical Forum, 5, no. 2, (2010) 5968.

[8] Y. Wang and N. Ding, Generalized supplemented modules. Taiwanese. Journal Mathematics, 20, no. 6, (2006) 15891601.
[9] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, Philadelphia, (1991).

[10] M. Alkan and A. Harmanci, On Summand Sum and Summand Intersection Property of Modules, Turk. J. Math., 26, (2002) 131- 147.

[11] A.C. Ozcan and A. Harmanci, Duo Modules, Glasgow Math. J., 48, (2006) 533-545.

[12] Z. Qing-yi and S. Mei-hua, On closed weak supplemented modules, Journal of Zhejiang University Science, 7, no. 2, (2006) 210-215.

References

School of mathematical Sciences, Faculty of Science and Technology,Universiti Kebangsaan Malaysia,43600 UKM, Selangor Darul Ehsan, Malaysia.

E-mail address: m_m_ukm@ymail.com.(Corresponding author)

E-mail address: ghafur@ukm.my

E-mail address: afeezokareem@gmail.com