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To cite this article: Abdul Rahman S. Juma and Mohammed H. Saloomi 2018 *J. Phys.: Conf. Ser.* **1003** 012046

Generalized Differential Operator on Bistarlike and Biconvex Functions Associated By Quasi-Subordination

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Abstract. In this paper, the generalized differential operator is applied to derive some subclasses of function class σ of bi-univalent functions defined in unit disk 𝔘. We estimate the bounds of the coefficients a_2 and a_3 for all functions which belong to the derived subclasses of σ.

Keywords: Holomorphic function, Quasi-subordination, Bistarlike function, Biconvex function, Bi-univalent function.

1. Introduction

Let $\mathcal{A}$ be the class of all functions f of the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,, \tag{1.1}$$

which are holomorphic in unit disk 𝔘

$\mathfrak{U} = \{z : |z| < 1\}$

and normalized by the conditions $f(0)=0$ and $f'(0)=1$. Let S denote the class of all univalent and holomorphic functions.

Let φ(z) be holomorphic function in 𝔘 and $|\varphi(z)| \leq 1$, such that

$$\varphi(z) = A_\circ + A_1 z + A_2 z^2 + A_3 z^3 + \cdots \,, \tag{1.2}$$

Where $A_\circ, A_1, A_2, A_3, \ldots$ are real. Let ϕ (z) be a holomorphic and univalent function with positive real part in𝔘, ϕ(0)=1, ϕ′(0)>0 and ϕ(𝔘) is a region starlike with the respect to 1and symmetric with the respect to the real axis. Further, let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \,, \tag{1.3}$$

where $B_1, B_2, B_3, \ldots$ are real and $B_1 > 0$.

Here, we suppose that the functions φ and ϕ are hold the above conditions one or otherwise stated.

The Koebe one–quarter Theorem [1] states that the image of 𝔘 under every function f in S contains a disk of radius $\frac{1}{4}$. So each univalent function has an inverse $g = f^{-1}$ satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathfrak{U}),$$

$$f\,(f^{-1}(\omega)) = \omega,\ (|\omega| < r_\circ(f)\ ,\ r_\circ(f)\geq \frac{1}{4}\ ), \quad (1.4)$$

A holomorphic function is called bi-univalent in $\mathfrak{U}$ if both f and f^{-1} are univalent in $\mathfrak{U}$. By σ, we denote the class of all bi-univalent functions defined in (1.1). Since f in σ has the form (1.1), the computation proves that the invers $g = f^{-1}$ has the following expansion

$$g\,(\omega) = f^{-1}(\omega) = \omega - a_2\,\omega^2 + (2a_2^2 - a_3)\omega^3 - \cdots . \quad (1.5)$$

A holomorphic function f is *subordinate* to holomorphic function g, written by

$$f \prec g \ \text{ or } f(z) \prec g(z)\ \ (z \in \mathfrak{U}), \quad (1.6)$$

provided there is holomorphic function k defined on $\mathfrak{U}$,with k (0)=0 and$|k(z)|$ <1 such that $f(z) = g(k(z))$. Moreover if g is univalent in$\mathfrak{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathfrak{U}) \subset g(\mathfrak{U})$. For more details on the notion of subordination.(see [1]).

In [2], the concept of quasi-subordinate introduced by Robertson. For holomorphic functions f and g in $\mathfrak{U}$, the function f is *quasi-subordination* to , written as follows:

$$f(z) \prec_q g(z)\ (z \in \mathfrak{U}) \quad (1.7)$$

if there exist holomorphic functions φ and k with |φ(z) |≤1, k (0) = 0 and $|k$ (z)|<1 such that $f(z) = \varphi(z) g(k(z))$. (z ∈ $\mathfrak{U}$).

Note that when $\varphi(z) = 1$, then $f(z) = g(k(z))$ so that $f(z) \prec g(z)$ in $\mathfrak{U}$. Also notice that, if k (z)= z, then $f(z) = \varphi(z)\, g(z)$ and it is said that f is *majorized* by g , in $\mathfrak{U}$. From previous statement it is clear that quasi-subordination is generalization of subordination as well as majorization. (For more details related to quasi-subordination see [2].)

Ma and Minda in [3] indicated to the unified classes by $S^*(\phi)$ and $K(\phi)$ and defined as following

$$S^*(\phi) := \{ f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec \phi(z);\ z \in \mathfrak{U} \}, \quad (1.8)$$

$$K(\phi) := \{ f \in \mathcal{A}: 1 + \frac{zf'''(z)}{f'(z)} \prec \phi(z);\ z \in \mathfrak{U} \}.$$

The classes $S^*(\phi)$ and $K(\phi)$ are amplification of a classical set of starlike and convex functions. (See [3]).

El-Ashwah and Kanas in [4] studied the classes

$$S_q^*(\gamma,\phi) := \{ f \in \mathcal{A}: \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec_q \phi(z) - 1; z \in \mathfrak{U}, 0 \neq \gamma \in \mathbb{C}\},$$

$$K_q(\gamma,\phi) := \{ f \in \mathcal{A}: \frac{1}{\gamma}\left(\frac{zf''(z)}{f(z)}\right) \prec_q \phi(z) - 1; z \in \mathfrak{U}, 0 \neq \gamma \in \mathbb{C}\}.$$

If $\varphi(z) \equiv 1$, the classes $S_q^*(\gamma,\phi)$ and $K_q(\gamma,\phi)$ convert respectively, to $S^*(\gamma,\phi)$ and $K^*(\gamma,\phi)$ of Ma.Minda starlike and convex functions of order γ ($\gamma \in \mathbb{C} \setminus \{0\}$), in unit disk $\mathfrak{U}$ ([5]). The classes $S_q^*(\gamma,\phi)$ and $K_q(\gamma,\phi)$ minimize, to $S^*(\phi)$ and $K(\phi)$, respectively. When γ=1, that are similar to Ma.Minda starlike and convex functions, determined by Mohd and Darus [6].

Let φ(b,d; z) be function defined by

$$\phi(b, d; z) = \sum_{n=0}^{\infty} \frac{(b)_n}{(d)_n} z^{n+1},\ (d \neq 0, -1, -2, \ldots\ z \in \mathfrak{U}),$$

where $(\alpha)_n$ denote the Pochhammer symbol defined by

$$(\alpha)_k = \begin{cases} 1 & \text{for } k = 0,\ \alpha \in \mathbb{C} \setminus \{0\}, \\ \alpha(\alpha+1)(\alpha+2)\ldots\ldots\ldots(\alpha+k-1) & \text{for } k \in \mathbb{N} = \{1,2,3\ldots.\},\ \alpha \in \mathbb{C}. \end{cases}$$

In [7] corresponding to the function $\phi(b,d;z)$,the generalized deferential operator defined as

$D_\lambda^m(b,d)f: \mathfrak{U} \longrightarrow \mathfrak{U}$,

$D_\lambda^0(b,d)f(z) = f(z) * \phi(b,d;z)$

$D_\lambda^1(b,d)f(z) = (1-\lambda)(f(z) * \phi(b,d;z)) + \lambda z(f(z) * \phi(b,d;z))'$

$$D_\lambda^m(b,d)f(z) = D_\lambda^1\left(D_\lambda^{m-1}(b,d)f(z)\right)$$

Let $f \in \mathcal{A}$, then from last two relations, we may deduce that

$$D_\lambda^m(b,d)f(z) = z + \sum_{n=2}^{\infty}[1 + (n-1)\lambda]^m \frac{(b)_{n-1}}{(d)_{n-1}} a_n z^n,$$

where $m \in N_\circ = \{0,1,2\ldots\ldots\}$, and λ≥0.

From the last relation, we have

$$z(D_\lambda^m(b,d)f(z))' = bD_\lambda^m(b+1,d)f(z) - (b-1)\,D_\lambda^m(b,d)f(z),$$

and
$\lambda z(D_\lambda^m(b,d)f(z))' = D_\lambda^{m+1}(b,d)f(z) - (1-\lambda)\, D_\lambda^m(b,d)f(z)$.
Lately, Srivastava et al. [8] found estimates for coefficients of the first two factors $|a_2|$ and $|a_3|$. For the functions of these classes, we use this motivation in this research to define a unified subclass of bi- univalent function class σ as follows.
A function $f \in \sigma$ defined in (1.1) is said to be in the class $\mathcal{M}_{q,\sigma}^{\alpha,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$ if the quasi-subordination conditions are satisfied:

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(\mathcal{D}_\lambda^{m+1}(b,d)f(z))'}{\mathcal{D}_\lambda^{m+1}(b,d)f(z)} + \alpha\{1+\frac{z(\mathcal{D}_\lambda^{m+1}(b,d)f(z))''}{(\mathcal{D}_\lambda^{m+1}(b,d)f(z))'}\}-1\right] \prec_q \phi(z)-1,$$

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(\mathcal{G}_\lambda^{m+1}(b,d)g(w))'}{\mathcal{G}_\lambda^{m+1}(b,d)g(w)} + \alpha\{1+\frac{w(\mathcal{G}_\lambda^{m+1}(b,d)g(w))''}{(\mathcal{G}_\lambda^{m+1}(b,d)g(w))'}\}-1\right] \prec_q \phi(w)-1,$$

where $g(w) = f^{-1}(w)$ given by (1.5), and
$\mathcal{D}_\lambda^{m+1}(b,d)f(z) = (1-\lambda)\mathcal{D}_\lambda^m(b,d)f(z) + \lambda z(\mathcal{D}_\lambda^m(b,d)f(z))'$
$\mathcal{G}_\lambda^{m+1}(b,d)g(w) = (1-\lambda)\mathcal{G}_\lambda^m(b,d)g(w) + \lambda w(\mathcal{G}_\lambda^m(b,d)g(w))'$, $0 \le \lambda \le 1$. (see [7])
For special values of α,λ,γ,b, d, m,ϕ and φ(z), the class $\mathcal{M}_{q,\sigma}^{\alpha,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$ unify known and new classes.
Remark (1.1): *Putting λ=0 in the class* $\mathcal{M}_{q,\sigma}^{\alpha,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$*, we get*

$$\mathcal{M}_{q,\sigma}^{\alpha,0}\mathcal{D}^m(b,d,\gamma,\phi) \coloneqq \mathcal{M}_{q,\sigma}^{\alpha,}\mathcal{D}^m(b,d,\gamma,\phi).$$

For b=d and m =0, we have the class $\mathcal{M}_{q,\sigma}^{\alpha,\lambda}(\gamma,\phi)$ was introduced and studied by N.Magesh,V.K.Balaji and J.Yamini [9].

In particular for b=d, m =0, and γ=1, we have the class

$$\mathcal{M}_{q,\sigma}^{\alpha,}\mathcal{D}^0(b,b,1,\phi) := \mathcal{M}_{q,\sigma}^{\alpha,}(\phi),$$

which introduced by Goyal and Kumar [10]. In this case for $\varphi(z) \equiv 1$, we get

$$\mathcal{M}_{q,\sigma}^{\alpha,}(\phi) \coloneqq \mathcal{M}_{\sigma}^{\alpha,}(\phi),$$

this class was studied by Ali et al. [11,12].

Remark(1.2): *Putting λ=α=0 in the class* $\mathcal{M}_{q,\sigma}^{\alpha,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$*,we get*

$$\mathcal{M}_{q,\sigma}^{0,0}\mathcal{D}^m(b,d,\gamma,\phi) \coloneqq \mathcal{M}_{q,\sigma}\mathcal{D}^m(b,d,\gamma,\phi).$$

In special case for γ=1,b=d and m =0, we have the class $\mathcal{M}_{q,\sigma}\mathcal{D}^0(b,b,1,\phi) = S_{q,\sigma}^*(\phi)$, was introduced by Goyal and Kumar [10].We observe that, for $\varphi(z) \equiv 1$, we get the class

$$S_{q,\sigma}^*(\gamma,\phi) := S_\sigma^*(\gamma,\phi),$$

was studied by Deniz [12].

Remark(1.3): *For λ=0 and α=1 in this class* $\mathcal{M}_{q,\sigma}^{\alpha,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$*, we obtain*

$$\mathcal{M}_{q,\sigma}^{1,0}\mathcal{D}^m(b,d,\gamma,\phi) \coloneqq \mathcal{M}_{q,\sigma}^{1}\mathcal{D}^m(b,d,\gamma,\phi).$$

In particular, put b=d, m =0,and γ=1,we obtain $\mathcal{M}_{q,\sigma}^{1}\mathcal{D}^0(b,b,1,\phi) \coloneqq K_{q\sigma,}(\phi)$ this is special case of the class $\mathcal{M}_{q,\sigma}^{\alpha,}(\phi)$ when α=1,this class was studied by Goyal and Kumar [10].

Remark (1.4): *Put α=0, we get the class* $\mathcal{M}_{q,\sigma}^{0,\lambda}\mathcal{D}^m(b,d,\gamma,\phi) \equiv \mathcal{M}_{q,\sigma}\mathcal{D}^m(b,d,\lambda,\gamma,\phi)$ *which defined as follow.*

A function $f\in\sigma$ is in the class $\mathcal{M}_{q,\sigma}\mathcal{D}^m(b,d,\lambda,\gamma,\phi)$, $\gamma\in\mathbb{C}\setminus\{0\}$, $0\leq\lambda\leq1$ if the quasi-subordination conditions are satisfied:

$$\frac{1}{\gamma}\left[\frac{z(\mathcal{D}_\lambda^m(b,d)f(z))'+\lambda z^2(\mathcal{D}_\lambda^m(b,d)f(z))''}{(1-\lambda)\mathcal{D}_\lambda^m(b,d)f(z)+\lambda z(\mathcal{D}_\lambda^m(b,d)f(z))'}-1\right]\prec_q \phi(z)-1,$$

and

$$\frac{1}{\gamma}\left[\frac{w(\mathcal{G}_\lambda^m(b,d)g(w))'+\lambda w^2(\mathcal{G}_\lambda^m(b,d)g(w))''}{(1-\lambda)\mathcal{G}_\lambda^m(b,d)g(w)+\lambda w(\mathcal{G}_\lambda^m(b,d)g(w))'}-1\right]\prec_q \phi(w)-1.$$

In particular for b=d and m =0 in above class, we obtain the class

$$\mathcal{M}_{q,\sigma}\mathcal{D}^0(b,d,\lambda,\gamma,\phi):=\mathcal{P}_{q,\sigma}(\gamma,\lambda,\phi).$$

The functions of the class $\mathcal{P}_{q,\sigma}(\gamma,\lambda,\phi)$ are called bi-convex and bi-starlike functions of complex order γ of Ma-Minda type. This class was studied by Nanjundan, Vitalrao and Jagadesan [9].

Remark(1.5): *Putting α=1, we get the class $\mathcal{M}_{q,\sigma}^{1,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)\equiv\mathcal{M}_{q,\sigma,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$ which is defined as follow.*

A function $f\in\sigma$ is said to be in the class $\mathcal{M}_{q,\sigma,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$, $\gamma\in\mathbb{C}\setminus\{0\}$, $0\leq\lambda\leq1$, if the following conditions are satisfied

$$\frac{1}{\gamma}\left[\frac{z(\mathcal{D}_\lambda^m(b,d)f(z))'+(1+2\lambda)z^2(\mathcal{D}_\lambda^m(b,d)f(z))''+\lambda z^3(\mathcal{D}_\lambda^m(b,d)f(z))'''}{z(\mathcal{D}_\lambda^m(b,d)f(z))'+\lambda z^2(\mathcal{D}_\lambda^m(b,d)f(z))''}\right]-1\prec_q \phi(z)-1,$$

and

$$\frac{1}{\gamma}\left[\frac{w(\mathcal{G}_\lambda^m(b,d)g(w))'+(1+2\lambda)w^2(\mathcal{G}_\lambda^m(b,d)g(w))''+\lambda w^3(\mathcal{G}_\lambda^m(b,d)g(w))'''}{w(\mathcal{G}_\lambda^m(b,d)g(w))'+\lambda w^2(\mathcal{G}_\lambda^m(b,d)g(w))''}\right]-1\prec_q \phi(w)-1.$$

In particular for b=d and m =0 in above class, we obtain the class

$$\mathcal{M}_{q,\sigma}^{\lambda}\mathcal{D}^0(b,b,\gamma,\phi)\equiv\mathcal{K}_{q,\sigma}(\gamma,\lambda,\phi)$$

was studied by Nanjundan, Vitalrao and Jagadesan [9].

To find out our results, we needed to talk about the following lemma.

Lemma (1.6)*[13]. If $p\in\mathcal{P}$, then $|c_j|\leq 2$ for each j, where $\mathcal{P}$ is the family of all functions p, holomorphic in $\mathfrak{U}$, for which*

$$\text{Re}\{\,p(z)\}>0,\ (z\in\mathfrak{U}),$$

where

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots,\ (z\in\mathfrak{U}).$$

2. Coefficients Bounds

In this section, we find the initial Taylor coefficients $|a_2|$ and $|a_3|$ for functions in class $\mathcal{M}_{q,\sigma}^{\alpha,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$.

Theorem(2.1): *If f belonging to the class $\mathcal{M}_{q,\sigma}^{\alpha,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$, then*

$$|a_2|\leq\frac{|\gamma||A_\circ|B_1\sqrt{B_1}}{\sqrt{\left|\gamma[2(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}-(1+3\alpha)(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(d)_1}\}^2]A_\circ B_1{}^2-(1+\alpha)^2(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(d)_1}\}^2(\mathrm{B}_2-\mathrm{B}_1)\right|}} \quad (2.1)$$

$$|a_3|\leq \frac{|\gamma|^2|A_\circ|^2B_1{}^3}{\left|\gamma[2(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_1}{(d)_1}-(1+3\alpha)(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(d)_1}\}^2]A_\circ B_1{}^2-(1+\alpha)^2(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(d)_1}\}^2(\mathrm{B}_2-\mathrm{B}_1)\right|}$$
$$+\frac{|\gamma||A_1|B_1}{2(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}}+\frac{|\gamma||A_\circ|B_1}{2(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}} \quad (2.2)$$

Proof: Let $f \in\mathcal{M}_{q,\sigma}^{\alpha,\lambda}\mathcal{D}^m(b,d,\gamma,\phi)$ and $g=f^{-1}$. Then there exist holomorphic functions u, v: $\mathfrak{U}\rightarrow\mathfrak{U}$, with u(0)=v(0)=0, such that

$$\frac{1}{\gamma}[(1-\alpha)\frac{\mathrm{z}(\mathcal{D}_\lambda^{\mathrm{m+1}}(\mathrm{b,d})\mathrm{f(z)})'}{\mathcal{D}_\lambda^{\mathrm{m+1}}(\mathrm{b,d})\mathrm{f(z)}}+\alpha\{1+\frac{\mathrm{z}(\mathcal{D}_\lambda^{\mathrm{m+1}}(\mathrm{b,d})\mathrm{f(z)})''}{(\mathcal{D}_\lambda^{\mathrm{m+1}}(\mathrm{b,d})\mathrm{f(z)})'}\}\text{-}1]=\varphi(\mathrm{z})(\phi(\mathrm{z})-1), \quad (2.3)$$

$$\frac{1}{\gamma}[(1-\alpha)\frac{\mathrm{w}(\mathcal{G}_\lambda^{\mathrm{m+1}}(\mathrm{b,d})\mathrm{g(w)})'}{\mathcal{G}_\lambda^{\mathrm{m+1}}(\mathrm{b,d})\mathrm{g(w)}}+\alpha\{1+\frac{\mathrm{w}(\mathcal{G}_\lambda^{\mathrm{m+1}}(\mathrm{b,d})\mathrm{g(w)})''}{(\mathcal{G}_\lambda^{\mathrm{m+1}}(\mathrm{b,d})\mathrm{g(w)})'}\}\text{-}1]=\varphi(\mathrm{w})(\phi(w)-1). \quad (2.4)$$

Now, we define the functions p and q by

$$\mathrm{p(z)}=\frac{1+u(\mathrm{z})}{1-u(\mathrm{z})}=1+c_1\mathrm{z}+c_2\mathrm{z}^2+c_3\mathrm{z}^3+\cdots$$

and

$$\mathrm{q(z)}=\frac{1+v(\mathrm{z})}{1-v(\mathrm{z})}=1+b_1\mathrm{z}+b_2\mathrm{z}^2+b_3\mathrm{z}^3+\cdots,$$

It is obvious, Re p(z) >0 and Re q(z) >0 . From last two relations, we derive

$$\mathrm{u(z)}=\frac{p(\mathrm{z})-1}{p(\mathrm{z})+1}=\frac{1}{2}[c_1\mathrm{z}+(c_2-\frac{c_1^2}{2})\mathrm{z}^2+\cdots \quad (2.5)$$

$$\mathrm{v(z)}=\frac{q\mathrm{z}-1}{q(\mathrm{z})+1}=\frac{1}{2}[b_1\mathrm{z}+(b_2-\frac{b_1^2}{2})\mathrm{z}^2+\cdots. \quad (2.6)$$

It is obvious that p and q are holomorphic functions in $\mathfrak{U}$ with $p(0)=q(0)=1$.

Using (2.5), (2.6) in (2.3) and (2.4), respectively, we obtain

$$\frac{1}{\gamma}[(1-\alpha)\frac{\mathrm{z}(\mathcal{D}_\lambda^{m+1}(b,d)f(\mathrm{z}))'}{\mathcal{D}_\lambda^{m+1}(b,d)f(\mathrm{z})}+\alpha\{1+\frac{\mathrm{z}(\mathcal{D}_\lambda^{m+1}(b,d)f(\mathrm{z}))''}{(\mathcal{D}_\lambda^{m+1}(b,d)f(\mathrm{z}))'}\}\text{-}1]=\varphi(\mathrm{z})(\phi\left(\frac{p(\mathrm{z})-1}{p(\mathrm{z})+1}\right)-1), \quad (2.7)$$

$$\frac{1}{\gamma}[(1-\alpha)\frac{w(\mathcal{G}_\lambda^{m+1}(b,d)g(w))'}{\mathcal{G}_\lambda^{m+1}(b,d)g(w)}+\alpha\{1+\frac{w(\mathcal{G}_\lambda^{m+1}(b,d)g(w))''}{(\mathcal{G}_\lambda^{m+1}(b,d)g(w))'}\}-1]=\varphi(\mathrm{w})(\phi\left(\frac{q(\mathrm{z})-1}{q(\mathrm{z})+1}\right)-1). \quad (2.8)$$

Utilize (2.5), (2.6) together with (1.3) it is evident that

$$\varphi\,(\mathrm{z})(\phi\left(\frac{p(\mathrm{z})-1}{p(\mathrm{z})+1}\right)-1)=\frac{1}{2}A_\circ B_1C_1\mathrm{z}+\{\frac{1}{2}A_1B_1C_1+\frac{1}{2}A_\circ B_1(C_2-\frac{c_1^2}{2})+\frac{1}{4}A_\circ B_2c_1^2\}\,\mathrm{z}^2+\ldots \quad (2.9)$$

$$\varphi(\mathrm{w})(\phi\left(\frac{q(\mathrm{z})-1}{q(\mathrm{z})+1}\right)-1)=\frac{1}{2}A_\circ B_1b_1w+\{\frac{1}{2}A_1B_1b_1+\frac{1}{2}A_\circ B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}A_\circ B_2b_1^2\}\,w^2+\ldots.. \quad (2.10)$$

It follows from (2.7),(2,8),(2.9) and (2.10) that

$$\frac{1}{\gamma}(1+\alpha)(1+\lambda)^{m+1}\frac{(b)_1}{(d)_1}a_2=\frac{1}{2}A_\circ B_1C_1 \quad (2.11)$$

$$\frac{1}{\gamma}[2(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}a_3-(1+3\alpha)(1+\lambda)^{2(m+1)}(\frac{(b)_1}{(d)_1})^2a_2^2]=\frac{1}{2}A_1B_1C_1+\frac{1}{2}A_\circ B_1(C_2-\frac{c_1^2}{2})+\frac{1}{4}A_\circ B_2c_1^2\} \quad (2.12)$$

$$\frac{-1}{\gamma}(1+\alpha)(1+\lambda)^{m+1}\frac{(b)_1}{(d)_1}a_2=\frac{1}{2}A_\circ B_1b_1 \quad (2.13)$$

$$\frac{1}{\gamma}[(4(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}-(1+3\alpha)(1+\lambda)^{2(m+1)}(\frac{(b)_1}{(d)_1})^2)a_2^2-2(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}a_3]=\frac{1}{2}A_1B_1b_1+\frac{1}{2}A_\circ B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}A_\circ B_2b_1^2\}. \quad (2.14)$$

From (2.11) and (2.13), we obtain

$$c_1=-b_1 \quad (2.15)$$

and

$$a_2=\frac{\gamma A_\circ B_1C_1}{2(1+\alpha)(1+\lambda)^{m+1}\frac{(b)_1}{(d)_1}}=\frac{-\gamma A_\circ B_1b_1}{2(1+\alpha)(1+\lambda)^{m+1}\frac{(b)_1}{(d)_1}} \quad (2.16)$$

$$8(1+\alpha)^2(1+\lambda)^{2(m+1)}a_2^2=\gamma^2A_\circ{}^2B_1{}^2(\frac{(b)_1}{(d)_1})^2(b_1{}^2+C_1{}^2), \quad (2.17)$$

adding (2.12) and (2.14) it follows that

$$\frac{a_2^2}{\gamma}[(4(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}-(1+3\alpha)(1+\lambda)^{2(m+1)}(\frac{(b)_1}{(d)_1})^2)]=\frac{1}{2}A_\circ B_1(c_2+b_2)+\frac{A_\circ(B_2-B_1)}{4}(c_1^2+b_1^2). \quad (2.18)$$

Substituting (2.15) and (2.16) into (2.18), we have

$$a_2^2=\frac{\gamma^2A_\circ{}^2B_1{}^3(c_2+b_2)}{4\gamma[(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}-(1+3\alpha)(1+\lambda)^{2(m+1)}(\frac{(b)_1}{(d)_1})^2]A_\circ B_1{}^2-4(B_2-B_1)(1+\alpha)^2(1+\lambda)^{2(m+1)}(\frac{(b)_1}{(d)_1})^2}. \quad (2.19)$$

Applying Lemma (1.6), we get the desired inequality (2.1).

Subtracting (2.12) from (2.14) and computation using (2.15), we obtain

$$a_3=a_2^2+\frac{\gamma A_1B_1C_1}{4(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}}+\frac{\gamma A_\circ B_1(c_2-b_2)}{8(1+2\alpha)(1+2\lambda)^{m+1}\frac{(b)_2}{(d)_2}}.$$

By applying for Lemma (1.6) again, we had the estimate (2.2).

Taking special values for α and λ in previous theorem, we obtain the following results

Corollary (2.2): *Let* $\gamma\in\mathbb{C}\setminus\{0\}$ *and* $\alpha\geq 0$. *If* $f\in\mathcal{M}_{q,\sigma}^{\alpha,}\mathcal{D}^m(b,d,\gamma,\phi)$, *then*

$$|a_2|\leq\frac{|\gamma||A_\circ|B_1\sqrt{B_1}}{\sqrt{\left|\gamma[2(1+2\alpha)\frac{(b)_2}{(d)_2}-(1+3\alpha)\{\frac{(b)_1}{(d)_1}\}^2]A_\circ B_1{}^2-(1+\alpha)^2\{\frac{(b)_1}{(d)_1}\}^2(B_2-B_1)\right|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2|A_\circ|^2{B_1}^3}{\left|\gamma[2(1+2\alpha)\frac{(b)_1}{(d)_1}-(1+3\alpha)\{\frac{(b)_1}{(d)_1}\}^2]A_\circ{B_1}^2-(1+\alpha)^2\{\frac{(b)_1}{(d)_1}\}^2(B_2-B_1)\right|}+\frac{|\gamma||A_1|B_1}{2(1+2\alpha)\frac{(b)_2}{(d)_2}}+\frac{|\gamma||A_\circ|B_1}{2(1+2\alpha)\frac{(b)_2}{(d)_2}}$$

Remark (2.3): *For b=d and m =0, the above corollary reduces [9, Corollary 12, p.5].*

Corollary (2.4): *Let f be in the class $\mathcal{M}_{q,\sigma}\mathcal{D}^m(b,d,\gamma,\phi)$, and $\gamma\in\mathbb{C}-\{0\}$,α≥0. Then*

$$|a_2|\leq\frac{|\gamma||A_\circ|B_1\sqrt{B_1}}{\sqrt{\left|\gamma[2\frac{(b)_2}{(d)_2}-\{\frac{(b)_1}{(d)_1}\}^2]A_\circ{B_1}^2-\{\frac{(b)_1}{(d)_1}\}^2(B_2-B_1)\right|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2|A_\circ|^2{B_1}^3}{\left|\gamma[2\frac{(b)_1}{(d)_1}-\{\frac{(b)_1}{(d)_1}\}^2]A_\circ{B_1}^2-\{\frac{(b)_1}{(d)_1}\}^2(B_2-B_1)\right|}+\frac{|\gamma||A_1|B_1}{2\frac{(b)_2}{(d)_2}}+\frac{|\gamma||A_\circ|B_1}{2\frac{(b)_2}{(d)_2}}$$

Remark (2.5): *For $b=d$ and $m=0$, the Corollary (2.4) reduces to [9, Corollary 9, p.5].*

Corollary (2.6): *Let f be in the class $\mathcal{M}^{1,}_{q,\sigma}\mathcal{D}^m(b,d,\gamma,\phi)$, and $\gamma\in\mathbb{C}-\{0\}$,.Then*

$$|a_2|\leq\frac{|\gamma||A_\circ|B_1\sqrt{B_1}}{\sqrt{\left|\gamma[6\frac{(b)_2}{(c)_2}-4\{\frac{(b)_1}{(c)_1}\}^2]A_\circ{B_1}^2-4\{\frac{(b)_1}{(c)_1}\}^2(B_2-B_1)\right|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2|A_\circ|^2{B_1}^3}{\left|\gamma[6\frac{(b)_1}{(c)_1}-4\{\frac{(b)_1}{(c)_1}\}^2]A_\circ{B_1}^2-4\{\frac{(b)_1}{(c)_1}\}^2(B_2-B_1)\right|}+\frac{|\gamma||A_1|B_1}{6\frac{(b)_2}{(c)_2}}+\frac{|\gamma||A_\circ|B_1}{6\frac{(b)_2}{(c)_2}}$$

Remark (2.7): *For b=d and m =0, the Corollary (2.6) reduces to[9, Corollary 11,p.5].*

Corollary (2.8): *Let f be in the class $\mathcal{M}_{q,\sigma}\mathcal{D}^m(b,d,\lambda,\gamma,\phi)$, and $\gamma\in\mathbb{C}-\{0\}$ and 0≤λ≤1. Then*

$$|a_2|\leq\frac{|\gamma||A_\circ|B_1\sqrt{B_1}}{\sqrt{\left|\gamma[2(1+2\lambda)^{m+1}\frac{(b)_2}{(c)_2}-(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(c)_1}\}^2]A_\circ{B_1}^2-(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(c)_1}\}^2(B_2-B_1)\right|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2|A_\circ|^2{B_1}^3}{\left|\gamma[2(1+2\lambda)^{m+1}\frac{(b)_1}{(c)_1}-(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(c)_1}\}^2]A_\circ{B_1}^2-(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(c)_1}\}^2(B_2-B_1)\right|}+\frac{|\gamma||A_1|B_1}{2(1+2\lambda)^{m+1}\frac{(b)_2}{(c)_2}}+\frac{|\gamma||A_\circ|B_1}{2(1+2\lambda)^{m+1}\frac{(b)_2}{(c)_2}}$$

Remark (2.9):*By taking b=d and m =0,the above corollary gives to the result obtained in Corollary (13) in [9].*

Corollary (2.10): *If f be in the class $\mathcal{M}^{\lambda}_{q,\sigma}\mathcal{D}^m(b,d,\lambda,\gamma,\phi)$, $\gamma\in\mathbb{C}-\{0\}$ and 0≤λ≤1, then*

$$|a_2|\leq\frac{|\gamma||A_\circ|B_1\sqrt{B_1}}{\sqrt{\left|\gamma[6(1+2\lambda)^{m+1}\frac{(b)_2}{(c)_2}-4(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(c)_1}\}^2]A_\circ{B_1}^2-4(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(c)_1}\}^2(B_2-B_1)\right|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2|A_\circ|^2{B_1}^3}{\left|\gamma[6(1+2\lambda)^{m+1}\frac{(b)_1}{(c)_1}-4(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(c)_1}\}^2]A_\circ{B_1}^2-4(1+\lambda)^{2(m+1)}\{\frac{(b)_1}{(c)_1}\}^2(B_2-B_1)\right|}+\frac{|\gamma||A_1|B_1}{6(1+2\lambda)^{m+1}\frac{(b)_2}{(c)_2}}+\frac{|\gamma||A_\circ|B_1}{6(1+2\lambda)^{m+1}\frac{(b)_2}{(c)_2}}$$

Remark (2.11): *If we set b=d and m =0, the above corollary leads to get coefficient estimates $|a_2|$ and $|a_3|$ in the class $K_{q\sigma,}(\gamma,\lambda,\phi)$.[9, Corollary 14, p.5].*

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