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Certain subclass of p-valent meromorphic Bazilević functions defined by fractional q-calculus operators

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Abstract

The aim of the present paper is to introduce and investigate a new subclass of Bazilević functions in the punctured unit disk $\mathcal{U}^*$ which have been described through using of the well–known fractional q–calculus operators, Hadamard product and a linear operator. In addition, we obtain some sufficient conditions for the functions belonging to this class and for some of its subclasses.

Keywords: Meromorphic p–valent functions, Hadamard product, Bazilević function, Fractional q–calculus operators.
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1. Introduction

The fractional calculus operators have been widely used in describing and solving different issues in applied sciences and also in the geometric function theory of complex analysis. The theory of q–calculus operators has been applied lately in areas such as ordinary fractional calculus, solutions of the q–difference, optimal control problems, q–differential and q–integral equations, q–transform analysis and many more. Recently, there have been a significant increase of activity in the area of q–calculus because of its applications in arithmetic, statistics and physics. For more details, one may refer to the books [11, 13] and the recent papers [2, 3, 9] on the subject. Purohit and Raina [19] used fractional q–calculus operators in investigating certain classes of functions which are analytic in the open unit disk. Recently, several authors have presented new subclasses of analytic functions using

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q–calculus operators for some recent investigations on the subclasses of analytic functions introduced by using q–calculus operators and related topics, we refer the reader to [4, 18, 20] and the references cited therein.

Let $\Sigma_p(n)$ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_{k-p+1} z^{k-p+1} \quad (p, n \in \mathbb{N} := \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p–valent in the punctured unit disk $\mathcal{U}^* = \mathcal{U}\backslash\{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For functions $f \in \Sigma_p(n)$ given by (1.1) and $g \in \Sigma_p(n)$ is given by

$$g(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} b_{k-p+1} z^{k-p+1} \quad (z \in \mathcal{U}^*),$$

we define $f * g$ as

$$(f * g)(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_{k-p+1} b_{k-p+1} z^{k-p+1} = (g * f)(z) \quad (z \in \mathcal{U}^*),$$

where $*$ denotes the usual Hadamard product (convolution) of analytic functions.

For δ $(0 \leq \delta < p)$, a function f of the form (1.1) is called meromorphic p–valent starlike function of order δ, denote by $\Sigma s^*(p, \delta)$, if it satisfies

$$-\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \delta \quad (z \in \mathcal{U}^*).$$

We denote the former class of functions as $\Sigma_s^*(\delta)$, see [10].

Also, a function f of the form (1.1) belongs to class $\Sigma_k(p, \delta)$ of meromorphic close–to–convex functions, if there exist a function $g \in \Sigma s^*(p, \delta)$ such that

$$-\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > \delta \quad (z \in \mathcal{U}^*).$$

This class of functions was introduced and studied by Libera and Robertson [16].

Furthermore $\Sigma s^*(1, \delta) = \Sigma s^*(\delta)$ and $\Sigma_k(1, \delta) = \Sigma_k(\delta)$, where $\Sigma s^*(\delta)$ and $\Sigma_k(\delta)$ are subclasses of $\Sigma_1(n)$ consisting of all meromorphic univalent functions which are respectively, starlike and close–to–convex of order δ $(0 \leq \delta < 1)$.

Using the operator $D_{q,z}^{\xi}$ which introduced in [1], we define a q–differintegral operator

$$\Phi_{q,z}^{\xi} : \Sigma_p(n) \longrightarrow \Sigma_p(n)$$

as follows:

$$\begin{aligned} \Phi_{q,z}^{\xi} f(z) &= \frac{\Gamma_q(1-p-\xi)}{\Gamma_q(1-p)} z^{\xi} D_{q,z}^{\xi} f(z) \\ &= \frac{1}{z^p} + \sum_{k=n}^{\infty} \frac{\Gamma_q(k-p+2)\Gamma_q(1-p-\xi)}{\Gamma_q(1-p)\Gamma_q(k-p+2-\xi)} a_{k-p+1} z^{k-p+1} \end{aligned} \tag{1.2}$$

where $\xi < -2, n \in \mathbb{N}, 0 < q < 1, z \in \mathcal{U}^*$ and $D^{\xi}_{q,z}$ in (1.2) represents, fractional q–derivative operator of a function $f(z)$ of order ξ defined by

$$D^{\xi}_{q,z} f(z) \equiv D_{q,z} I^{1-\xi}_{q,z} f(z) = \frac{1}{\Gamma_q(1-\xi)} D_{q,z} \int_0^z (z-tq)_{-\xi} f(t) d_q t \quad (0 \leq \xi < 1),$$

such that $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\xi}$ is given by the following equation

$$(z-tq)_{\xi-1} = z^{\xi-1} \quad {}_1\Omega_0[q^{-\xi+1}; -; q, tq^{\xi}/z].$$

Following Gasper and Rahman [13], the series ${}_1\Omega_0[\xi; -; q, z]$ is single valued when $\mid \arg(z) \mid < \pi$ and $\mid z \mid < 1$. Therefore, the function $(z-tq)_{\xi-1}$ is single valued when $\mid \arg(-tq^{\xi}/z) \mid < \pi, \mid tq^{\xi}/z \mid < 1$ and $\mid \arg(z) \mid < \pi$.

Now, we define a linear multiplier fractional q–differintegral operator $\Omega^{\xi,m}_{q,\lambda}$ as follows:

$$\begin{aligned}
\Omega^{\xi,0}_{q,\lambda} f(z) &= f(z) \\
\Omega^{\xi,1}_{q,\lambda} f(z) &= (1-\lambda)\Phi^{\xi}_{q,z} f(z) + \frac{\lambda z}{p} D_q(\Phi^{\xi}_{q,z} f(z)) + \frac{2\lambda}{z^p} \\
\Omega^{\xi,2}_{q,\lambda} f(z) &= \Omega^{\xi,1}_{q,\lambda}(\Omega^{\xi,1}_{q,\lambda} f(z)) \\
&\vdots \\
\Omega^{\xi,m}_{q,\lambda} f(z) &= \Omega^{\xi,1}_{q,\lambda}(\Omega^{\xi,m-1}_{q,\lambda} f(z)) \quad (m \in \mathbb{N}).
\end{aligned} \tag{1.3}$$

If $f(z)$ is given by (1.1), then by (1.3), we get

$$\Omega^{\xi,m}_{q,\lambda} f(z) = \frac{1}{z^p} + \left(\sum_{k=n}^{\infty} \frac{\Gamma_q(k-p+2)\Gamma_q(1-p-\xi)}{\Gamma_q(1-p)\Gamma_q(k-p+2-\xi)} [1 - \lambda + \frac{[k-p+1]_q \lambda}{p}] \right)^m a_{k-p+1} z^{k-p+1}.$$

Now, we introduce a new subclass $\mathcal{Q}^{m,\lambda}_{q,\xi}(\alpha, \delta)$ of analytic functions involving the linear multiplier q–fractional differintegral operator $\Omega^{\xi,m}_{q,\lambda}$ defined by (1.3).

Definition 1.1. A function $f(z) \in \Sigma_p(n)$ is called meromorphic p–valent Bazilević of type α and order δ, if there exists a function g belonging to the class Σ^*_s such that

$$-\operatorname{Re} \left(\frac{z^p (\Omega^{\xi,m}_{q,\lambda} f(z))'}{(\Omega^{\xi,m}_{q,\lambda} f(z))^{1-\alpha} (\Omega^{\xi,m}_{q,\lambda} g(z))^{\alpha}} \right) > \delta \quad (z \in \mathcal{U}^*), \tag{1.4}$$

for some $\alpha \geq 0$ and $0 \leq \delta < p$. We denote the former class of functions as $\mathcal{Q}^{m,\lambda}_{q,\xi}(\alpha, \delta)$.

Remark 1.2. It should be remarked that the class $\mathcal{Q}^{m,\lambda}_{q,\xi}(\alpha, \delta)$ is a generalization of many classes considered earlier. By giving specific values to the parameters p, m and α in the class $\mathcal{Q}^{m,\lambda}_{q,\xi}(\alpha, \delta)$, we obtain the following subclasses studied by earlier authors. Let us see some of the examples:

(i) If $p = 1, m = 0$ and $\alpha = 0$ in the class $\mathcal{Q}^{m,\lambda}_{q,\xi}(\alpha, \delta)$, then we have

$$-\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > \delta,$$

it reduces to the class $\Sigma s^*(p, \delta)$, introduced in [10].

(ii) If $p=1, m=0$ and $\alpha=1$ in the class $\mathcal{Q}_{q,\xi}^{m,\lambda}(\alpha,\delta)$, then we have

$$-\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right)>\delta,$$

it reduces to the case from the class $\Sigma_k(p,\delta)$, introduced in [16].

We refer to many authors ([5],[6],[7],[8],[12],[15]) which have investigated the related work on the subject of meromorphic functions.

Our aim in this paper is to introduce new subclass of Bazilević functions $\mathcal{Q}_{q,\xi}^{m,\lambda}(\alpha,\delta)$ involving the fractional q–calculus operators and obtain some sufficient conditions for functions belonging to this class.

2. Preliminaries

To establish our main results we shall need each of the following.

Lemma 2.1. (Jack, [14]) Let the nonconstant function $w(z)$ be analytic in $\mathcal{U}$ ($\mathcal{U}=\mathcal{U}^*\cup\{0\}$) with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathcal{U}$, then

$$z_0w'(z_0)=kw(z_0),$$

where $k\geq 1$ is a real number.

Lemma 2.2. (Miller and Mocanu, [17]) Let S be a set in the complex plane $\mathbb{C}$ and suppose that $\varphi(z)$ is a mapping from $\mathbb{C}^2\times\mathcal{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix,y;z)\notin S$ for all $z\in\mathcal{U}$, and for all real x,y such that $y\leq-(1+x^2)/2$. If the function $q(z)=1+q_1z+q_2z^2+\cdots$ is analytic in $\mathcal{U}$ such that $\varphi(q(z),zq'(z);z)\in S$ for all $z\in\mathcal{U}$, then $\mathrm{Re}\{q(z)\}>0$.

3. Main Results

The first result of this section is the following.

Theorem 3.1. *Let $\gamma\geq 0$, $\alpha\geq 0$, $0\leq\delta<p$ and $p\in\mathbb{N}$. If $f(z)\in\Sigma_p(n)$ satisfies the following inequality*

$$\left|p+\frac{z(\Omega_{q,\lambda}^{\xi,m}f(z))''}{(\Omega_{q,\lambda}^{\xi,m}f(z))'}-(1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m}f(z))'}{(\Omega_{q,\lambda}^{\xi,m}f(z))}-\alpha\frac{z(\Omega_{q,\lambda}^{\xi,m}g(z))'}{(\Omega_{q,\lambda}^{\xi,m}g(z))}-\gamma\left(p+\frac{z^p(\Omega_{q,\lambda}^{\xi,m}f(z))'}{(\Omega_{q,\lambda}^{\xi,m}f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m}g(z))^{\alpha}}\right)\right|$$
$$<\frac{(p-\delta)(\gamma(2p-\delta)+1)}{2p-\delta},$$

then $f(z)\in\mathcal{Q}_{q,\xi}^{m,\lambda}(\alpha,\delta)$.

Proof . We define the function $w(z)$ by

$$\frac{z^p(\Omega_{q,\lambda}^{\xi,m}f(z))'}{(\Omega_{q,\lambda}^{\xi,m}f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m}g(z))^{\alpha}}=-p+(\delta-p)w(z). \tag{3.1}$$

Then $w(z)$ is analytic in $\mathcal{U}^*$ and $w(0) = 0$. Differentiating logarithmically both sides of (3.1) with respect to z, we get

$$p + \frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))''}{(\Omega_{q,\lambda}^{\xi,m} f(z))'} - (1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))} - \alpha\frac{z(\Omega_{q,\lambda}^{\xi,m} g(z))'}{(\Omega_{q,\lambda}^{\xi,m} g(z))} = \frac{(p-\delta)zw'(z)}{p+(p-\delta)w(z)}. \tag{3.2}$$

Using (3.1) in (3.2), we get

$$p + \frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))''}{(\Omega_{q,\lambda}^{\xi,m} f(z))'} - (1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))} - \alpha\frac{z(\Omega_{q,\lambda}^{\xi,m} g(z))'}{(\Omega_{q,\lambda}^{\xi,m} g(z))} - \gamma\left(p + \frac{z^p(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} g(z))^{\alpha}}\right)$$

$$= \gamma(p-\delta)w(z) + \frac{(p-\delta)zw'(z)}{p+(p-\delta)w(z)}. \tag{3.3}$$

Suppose that there exists $z_0 \in \mathcal{U}$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)|$$

and using Lemma 2.1, we find that

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).$$

Writing $w(z) = e^{i\theta}$ $(0 \leq \theta < 2\pi)$ and putting $z = z_0$ in (3.3), we have

$$\begin{aligned}
&\Bigg| p + \frac{z_0(\Omega_{q,\lambda}^{\xi,m} f(z_0))''}{(\Omega_{q,\lambda}^{\xi,m} f(z_0))'} - (1-\alpha)\frac{z_0(\Omega_{q,\lambda}^{\xi,m} f(z_0))'}{(\Omega_{q,\lambda}^{\xi,m} f(z_0))} - \alpha\frac{z_0(\Omega_{q,\lambda}^{\xi,m} g(z_0))'}{(\Omega_{q,\lambda}^{\xi,m} g(z_0))} \\
&-\gamma\left(p + \frac{z_0^p(\Omega_{q,\lambda}^{\xi,m} f(z_0))'}{(\Omega_{q,\lambda}^{\xi,m} f(z_0))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} g(z_0))^{\alpha}}\right)\Bigg| \\
&= \left|\gamma(p-\delta)e^{i\theta} + \frac{(p-\delta)ke^{i\theta}}{p+(p-\delta)e^{i\theta}}\right| \\
&\geq Re\Big(\gamma(p-\delta) + \frac{(p-\delta)k}{p+(p-\delta)e^{i\theta}}\Big) \\
&> \gamma(p-\delta) + \frac{(p-\delta)}{2p-\delta} \\
&= \frac{(p-\delta)(\gamma(2p-\delta)+1)}{2p-\delta},
\end{aligned}$$

which contradicts our assumption (1.4). Therefor, we have $|w(z)| < 1$ in $\mathcal{U}$. Finally, we have

$$\left| p + \frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))''}{(\Omega_{q,\lambda}^{\xi,m} f(z))'} - (1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))} - \alpha\frac{z(\Omega_{q,\lambda}^{\xi,m} g(z))'}{(\Omega_{q,\lambda}^{\xi,m} g(z))} - \gamma\left(p + \frac{z^p(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} g(z))^{\alpha}}\right)\right|$$

$$= |(p-\delta)w(z)| = (p-\delta)|w(z)| < p-\delta \quad (z \in \mathcal{U})$$

that is $f(z) \in \mathcal{Q}_{q,\xi}^{m,\lambda}(\alpha,\delta)$. This completes the proof. $\square$

Theorem 3.2. *Let $\alpha \geq 0$, $0 \leq \beta < p$ and $p \in \mathbb{N}$. If $f \in \Sigma_p(n)$ satisfies the following inequality*

$$\mathrm{Re}\Big\{\frac{z^p(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} g(z))^{\alpha}}\Big(\frac{z^p(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} g(z))^{\alpha}} + (1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))} + \alpha\frac{z(\Omega_{q,\lambda}^{\xi,m} g(z))'}{(\Omega_{q,\lambda}^{\xi,m} g(z))}$$

$$-\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))''}{(\Omega_{q,\lambda}^{\xi,m} f(z))'}\Big)\Big\} > \beta(\beta+\frac{1}{2}) + p(\beta-\frac{1}{2}),$$

then $f \in \mathcal{Q}_{q,\xi}^{m,\lambda}(\alpha,\delta)$.

Proof . We define the function $h(z)$ by

$$\frac{z^p(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} f(z))^{\alpha}} = -\beta + (\beta - p)h(z). \tag{3.4}$$

Then we see that $h(z) = 1 + h_1 z + h_2 z^2 + \cdots$ is analytic in $\mathcal{U}$. Now differentiating logarithmically both sides of (3.4) with respect to z, we obtain

$$p + \frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))''}{(\Omega_{q,\lambda}^{\xi,m} f(z))'} - (1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))} - \alpha\frac{z(\Omega_{q,\lambda}^{\xi,m} g(z))'}{(\Omega_{q,\lambda}^{\xi,m} g(z))} = \frac{(p-\beta)zh'(z)}{\beta + (p-\beta)h(z)},$$

or equivalently

$$[\beta + (p-\beta)h(z)]\left(\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))''}{(\Omega_{q,\lambda}^{\xi,m} f(z))'} - (1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))} - \alpha\frac{z(\Omega_{q,\lambda}^{\xi,m} g(z))'}{(\Omega_{q,\lambda}^{\xi,m} g(z))}\right)$$

$$= (p-\beta)zh'(z) + p[\beta + (p-\beta)h(z)]. \tag{3.5}$$

From (3.4) and (3.5), we get

$$-\frac{z^p(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} g(z))^{\alpha}}\left(\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))''}{(\Omega_{q,\lambda}^{\xi,m} f(z))'} - (1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))} - \alpha\frac{z(\Omega_{q,\lambda}^{\xi,m} g(z))'}{(\Omega_{q,\lambda}^{\xi,m} g(z))}\right)$$

$$= (p-\beta)zh'(z) + p[\beta + (p-\beta)h(z)]. \tag{3.6}$$

Again using (3.4) in (3.6), we get

$$\Big(\frac{z^p(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} g(z))^{\alpha}}\Big)^2 - \frac{z^p(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))^{1-\alpha}(\Omega_{q,\lambda}^{\xi,m} g(z))^{\alpha}}\Big(\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))''}{(\Omega_{q,\lambda}^{\xi,m} f(z))'} - (1-\alpha)\frac{z(\Omega_{q,\lambda}^{\xi,m} f(z))'}{(\Omega_{q,\lambda}^{\xi,m} f(z))}$$

$$-\alpha\frac{z(\Omega_{q,\lambda}^{\xi,m} g(z))'}{(\Omega_{q,\lambda}^{\xi,m} g(z))}\Big) = (p-\beta)zh'(z) + (p-\beta)^2h^2 + (p-\beta)[2\beta+p]h(z) + p\beta + \beta^2.$$

$$= \varphi(h(z), zh'(z); z),$$

where

$$\varphi(r,s;z) := (p-\beta)s + (p-\beta)^2r^2 + (p-\beta)[2\beta+p]r + p\beta + \beta^2.$$

Assume that $y \leq -(1+x^2)/2$ for all real numbers x and y, we have

$$\begin{aligned}
\mathrm{Re}(\varphi(ix, y; z)) &= (p-\beta)y - (p-\beta)^2 x^2 + p\beta + \beta^2 \\
&\leq -\frac{1}{2}(p-\beta)(1+x^2) - (p-\beta)^2 x^2 + p\beta + \beta^2 \\
&= -\frac{1}{2}(p-\beta) - (p-\beta)(\frac{1}{2} + p - \beta)x^2 + p\beta + \beta^2 \\
&\leq p\beta + \beta^2 - \frac{1}{2}(p-\beta) \\
&= \beta(\beta + \frac{1}{2}) + p(\beta - \frac{1}{2}).
\end{aligned}$$

Let $S = \{w : Re > \beta(\beta + \frac{1}{2}) + p(\beta - \frac{1}{2})\}$. Then $\varphi(h(z), zh'(z); z) \in S$ and $\varphi(ix, y; z) \notin S$ for all $z \in \mathcal{U}$ and real numbers x and y such that $y < -(1+x^2)/2$. By applying Lemma 2.2, we have $\mathrm{Re}\{h(z)\} > 0$, that is $f(z) \in \mathcal{Q}_{q,\xi}^{m,\lambda}(\alpha, \delta)$. This completes the proof. □

Remark 3.3. (*i*) If $p = \gamma = 1$ and $m = \alpha = 0$ in Theorem 3.1, then we obtain the result obtained by Goyal and Prajapat, [12, Corollary 5].

(*ii*) If $p = \gamma = 1$ and $m = \alpha = \delta = 0$ in Theorem 3.1, then we obtain the result obtained by Goyal and Prajapat, [12, Corollary 6].

Putting $p = 1$ and $m = \alpha = \beta = 0$ in Theorem 3.2, we obtain:

Corollary 3.4. *If $f \in \Sigma$ satisfies the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{2zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2},$$

then $f \in \Sigma s^*$.

Putting $p = \alpha = \gamma = 1$ and $m = 0$ in Theorem 3.1, we obtain:

Corollary 3.5. *If $f \in \Sigma$ satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} - \frac{zf'(z)}{g(z)}\right| < \frac{(1-\delta)(3-\delta)}{2-\delta},$$

then $f(z) \in \Sigma_k(\delta)$.

Further setting $\delta = 0$ in Corollary 3.5, we have:

Corollary 3.6. *If $f \in \Sigma$ satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} - \frac{zf'(z)}{g(z)}\right| < \frac{3}{2},$$

then $f(z) \in \Sigma_k$.

Further setting $p = \alpha = 1$ and $m = \beta = 0$ in Theorem 3.2, we get:

Corollary 3.7. *If $f \in \Sigma$ satisfies the condition*

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\left(\frac{zf'(z)}{g(z)} - \frac{zg'(z)}{g(z)} - \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2},$$

then $f(z) \in \Sigma_k$.

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