Available online at http://qu.edu.iq/journalsc/index.php/JOPS

Al-Qadisiyah Journal Of Pure Science (QJPS)

Vol. 25, No. 1, pp Math. 16 –22, Year2020

ISSN 2411-3514 ONLINE

ISSN 1997-2490PRINTED

Received : 23/1/2020 Accepted :18/2/2020

Some Properties on a Class of Analytic Functions Involving Generalized linear operator

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ABSTRACT:

In this paper, we introduce generality the linear operator $\varnothing^{s,a,\lambda}_{(\lambda p),(\mu q),\varsigma}$ defined on the open unit disc $U = \{Z \in \mathbb{C} : |Z| < 1\}$.By using this linear operator $\varnothing^{s,a,\lambda}_{(\lambda p),(\mu q),\varsigma}$, we introduce a subclass of analytic functions $\Im^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$ Moreover, We obtain some geometric characterization like coefficient estimates, distortion and growth theorems closure theorems and integral operators, radii of close –to– convexity, convexity and starlikeness for functions in the class $\Im^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$.

KEYWORDS: Analytic functions, Close-to-convex functions, Linear operator, Integral operator

1. Introduction

let A symbol to the class of analyticfunctions that from

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \qquad (1.1)$$

which are analytic and normalized in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ in thecomplex plane. For functions $f \in \mathfrak{A}$ .Next we will provide generalized the linear operator drawn upand introduced by Srivastava and Gaboury [1] as follows :

$$\varnothing^{s,a,\lambda}_{(\lambda p),(\mu q),\varsigma}(f) : A \to A,$$

when characterized by

$$\varnothing^{s,a,\lambda}_{(\lambda p),(\mu q),\varsigma} f(z) = \zeta^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(z) * f(z) \qquad (1.2)$$

Such that $\zeta^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(z)$ is defined by

$$\zeta^{s,a,\lambda}_{(\lambda p),(\mu q),\varsigma}(z)$$

$$:= \frac{\lambda \prod_{j=1}^{q}(\mu_j)(1+a)^s \Gamma(s).\Lambda\left[1+a,\varsigma,s,\lambda\right]^{-1}}{\prod_{j=1}^{p}(\lambda_j)}$$

$$.\left[\Phi^{(1,\dots,1,1,\dots,1)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}\left(z,s,a;\varsigma,\lambda\right) - \frac{a^{-s}}{\lambda\Gamma(s)}\Lambda(a,\varsigma,s,\lambda)\right]$$

$$= z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\frac{a+1}{a+n}\right)^s \frac{z^n}{n!} \qquad (1.3)$$

Where

$$\Lambda(a,\varsigma,s,\lambda) := \mathrm{H}^{2,0}_{0,2}\left[\varsigma^{\frac{1}{\lambda}}(n+a) \,\middle|\, \overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]. \qquad (1.4)$$

linking (1.2) and (1.3), we Obtain

$$\text{Ø}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} := z+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^{s} a_n \frac{z^n}{n!}. \quad (1.5)$$

$\big(\lambda_j \in \mathbb{C}\ (j=1,\cdots,p);\ \mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^-(j=1,\cdots,q); z\in U;$ $p \leq q+1;\ \min\{\Re(a),\Re(s)\}>0; \lambda>0$ when $\Re(\varsigma)>0$ and $s\in\mathbb{C}; a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ when $\varsigma=0\big)$

for more details see[2]

Defintion 1.1 : A function $f\in A$ be given by (1.1) is said to be in the class $T^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$ if the following condition holds:

$$\operatorname{Re}\left\{1+\frac{1}{d}\left[(1-\delta)\frac{\text{Ø}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)}{z}+\delta\left(\text{Ø}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)\right)'-1\right]\right\}>0. \quad (1.6)$$

Or, equivalently:

$$\left|\frac{(1-\delta)\frac{\text{Ø}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)}{z}+\delta\left(\text{Ø}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)\right)'-1}{(1-\delta)\frac{\text{Ø}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)}{z}+\delta\left(\text{Ø}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)\right)'-1+2d}\right|<1, \quad (1.7)$$

where $z\in U, \delta\geq 0, d\in\mathbb{C}-\{0\}$.

Some special cases of the above class can be found in [3],[4]

Let T denote the subclass of A consisting of function of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n, \quad a_n\geq 0 \quad (1.8)$$

Now we define the class $\mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$ by:

$$\mathfrak{I}^{s,a,\lambda}_{(\lambda_p)(\mu_q)\varsigma}(\delta,d)=T^{s,a,\lambda}_{(\lambda_p)(\mu_q),\varsigma}(\delta,d)\cap T \quad (1.9)$$

The class $\mathfrak{I}^{s,a,\lambda}_{(\lambda_p)(\mu_q),b}(\delta,d)$ is introduced and studied by Al-Hawary et al. [5], Darus and Faisal [6], and Amourah et al.[7,8,9].

In our present paper, we obtain some interesting geometric properties in the class $\mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$

2.Coefficient Inequalities

Theorem 2.1. A function $f\in A$ given by (1.1) is in the class $\mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$ if and only if

$$\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^{s}\frac{a_n}{n!}\leq|d|, \quad (2.1)$$

$\big(\lambda_j\in\mathbb{C}\ (j=1,\cdots,p);\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^-(j=1,\cdots,q);z\in U;\ p\leq q+1;$ $\min\{\Re(a),\Re(s)\}>0;\lambda>0$ when $\Re(\varsigma)>0$ and $s\in\mathbb{C};a\in\mathbb{C}\setminus\mathbb{Z}_0^-$ when $\varsigma=0\big)$

Proof: Let $f\in\mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$. Then for $z\in U$ we have

$$\left|(1-\delta)\frac{\text{Ø}^{s,a,\lambda}_{(\lambda_p)(\mu_q)\varsigma} f(z)}{z}+\delta\left(\text{Ø}^{s,a,\lambda}_{(\lambda_p)(\mu_q)\varsigma} f(z)\right)'-1\right|-$$

$$\left|(1-\delta)\frac{\text{Ø}^{s,a,\lambda}_{(\lambda_p)(\mu_q)\varsigma} f(z)}{z}+\delta\left(\text{Ø}^{s,a,\lambda}_{(\lambda_p)(\mu_q)\varsigma} f(z)\right)'-1+2d\right|=$$

$$\left|\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^{s}a_n z^{n-1}\right|-$$

$$\left|2d-\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^{s}a_n z^{n-1}\right|$$

$$\leq\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^{s}a_n\left|z^{n-1}\right|-2|d|$$

$$+\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^{s}a_n\left|z^{n-1}\right| \quad (1.8)$$

$$\leq\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^{s}a_n-|d|\leq 0.$$

This implies

$$\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^{s}a_n\leq|d|,$$

Contrariwise, let inequality (2.1) is satisfied. Then

$$\left|\frac{(1-\delta)\dfrac{\varnothing^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)}{z}+\delta\left(\varnothing^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)\right)'-1}{(1-\delta)\dfrac{\varnothing^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)}{z}+\delta\left(\varnothing^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma} f(z)\right)'-1+2d}\right|<1.$$

This Completes the proof of Theorem 2.1.

Corollary 2.2. If f in $\Im^{s,a,\lambda}_{(\lambda_p)(\mu_q),\varsigma}(\delta,d)$ is given by (1.1), then

$$a_n \le \frac{|d|}{[1+\delta(n-1)]\dfrac{\prod_{j=1}^{p}(\lambda_j+1)_{n-1}}{\prod_{j=1}^{q}(\mu_j+1)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}, \; n \ge 2.$$

3.Distortion and Growth Theorems

we give distortion and growth bounds for the functions f belonging to the class $\Im^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$ is contained in the following theorem.

Theorem 3.1. Let $f \in \Im^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$ which is defined by (1.8). Then $|z|=r<1$, we have for

$$r-\frac{|d|}{[1+\delta]\dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}r^2 \le |f(z)|$$

$$\le r+\frac{|d|}{[1+\delta]\dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}r^2$$

and

$$1-\frac{2|d|}{[1+\delta]\dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}r \le |f'(z)|$$

$$\le 1+\frac{2|d|}{[1+\delta]\dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}r.$$

Proof: Since $f \in \Im^{s,a,\lambda}_{(\lambda_p)(\mu_q),\varsigma}(\delta,d)$, from Theorem 2.1 we can write

$$\sum_{n=2}^{\infty} a_n \le \frac{|d|}{[1+\delta(n-1)]\dfrac{\prod_{j=1}^{p}(\lambda_j+1)_{n-1}}{\prod_{j=1}^{q}(\mu_j+1)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}. \quad (3.1)$$

Thus, for $|z|=r<1$, and making use of (3.1) we have

$$|f(z)| \le |z|+\sum_{n=2}^{\infty} a_n\left|z^n\right| \le r+r^2\sum_{n=2}^{\infty} a_n$$

$$\le r+\frac{|d|}{[1+\delta)]\dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}r^2$$

and

$$|f(z)| \ge |z|-\sum_{n=2}^{\infty} a_n\left|z^n\right| \ge r-r^2\sum_{n=2}^{\infty} a_n$$

$$\ge r-\frac{|d|}{[1+\delta)]\dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}r^2.$$

As well from Theorem 2.1, it follows that

$$\frac{[1+\delta)]\dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}{2}\sum_{n=2}^{\infty} na_n \le$$

$$\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s a_n \le |d|.$$

Hence

$$\left|f'(z)\right| \le 1+\sum_{n=2}^{\infty} na_n\left|z^n\right| \le 1+r\sum_{n=2}^{\infty} na_n \le$$

$$1+\frac{2|d|}{[1+\delta)]\dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}}\left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}r.$$

and

$$\left|f'(z)\right| \geq 1 - \sum_{n=2}^{\infty} n a_n \left|z^n\right| \geq 1 - r \sum_{n=2}^{\infty} n a_n \geq$$

$$1 - \frac{2|d|}{[1+\delta)]\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{n-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{n-1}}\left(\frac{\Lambda(a+n,ç,s,\lambda)}{\Lambda(a+1,ç,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s} r.$$

Then, the proof of Theorem 3.1.is complete

4. Closure Theorems

Let the functions $g_i(z)$, $i = 1, 2, \ldots, I$ be defined by

$$g_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \quad, a_{n,i} \geq 0 \qquad (4.1)$$

for z in U.

Closure theorems for the class $\Im_{(\lambda_p)(\mu_q),b}^{s,a,\lambda}$ are given by the following

.

Theorem 4.1. Let the functions $g_i(z)$ which is defined by (4.1) be in the class $\Im_{(\lambda_p),(\mu_q),ç}^{s,a,\lambda}(\delta, d)$

$\left(\lambda_j \in \mathbb{C}\ (j=1,\cdots,p);\ \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-(j=1,\cdots,q); i = 1, 2, ..., I; z \in U;\right.$

$p \leq q+1$; $\min\{\Re(a),\Re(s)\}>0; \lambda>0$ when $\Re(ç)>0$ and $s \in \mathbb{C}$;

$a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ when $ç = 0)$. Then the function E(z) defined by

$$E(z) = z - \sum_{n=2}^{\infty} q_n z^n, \quad q_n \geq 0 \qquad (4.2)$$

is a member of the class $g_i(z)$ in $\Im_{(\lambda_p),(\mu_q),ç}^{s,a,\lambda}(\delta, d)$,where

$$q_n = \frac{1}{I}\sum_{i=1}^{I} a_{n,i} \quad (n \geq 2)$$

Proof: Since $g_i(z) \in \Im_{(\lambda_p),(\mu_q),ç}^{s,a,\lambda}(\delta, d)$ it follows from Theorem 2.1 that

$$\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{n-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{n-1}}\left(\frac{\Lambda(a+n,ç,s,\lambda)}{\Lambda(1+a,ç,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s a_{n,i} \leq |d|$$

for every $i = 1,2, \ldots, I$. Hence

$$\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{n-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{n-1}}\left(\frac{\Lambda(a+n,ç,s,\lambda)}{\Lambda(1+a,ç,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s q_n$$

$$\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{n-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{n-1}}\left(\frac{\Lambda(a+n,ç,s,\lambda)}{\Lambda(1+a,ç,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s \left\{\frac{1}{I}\sum_{i=1}^{I} a_{n,i}\right\}$$

$$= \frac{1}{I}\sum_{i=1}^{I}\left(\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{n-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{n-1}}\left(\frac{\Lambda(a+n,ç,s,\lambda)}{\Lambda(1+a,ç,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s a_{n,i}\right)$$

$\leq \frac{1}{I}\sum_{i=1}^{I}|d| = |d|$, which implies that E(z)$\in \Im_{(\lambda_p)(\mu_q),b}^{s,a,\lambda}(\delta, d)$.

Theorem 4.2. The class $\Im_{(\lambda_p)(\mu_q),ç}^{s,a,\lambda}(\delta, d)$ is closed under convex linear combination, where

$\left(\lambda_j \in \mathbb{C}\ (j=1,\cdots,p);\ \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-(j=1,\cdots,q); z \in U;\ p \leq q+1;\right.$

$\min\{\Re(a),\Re(s)\}>0; \lambda>0$ when $\Re(ç)>0$ and $s \in \mathbb{C}$; $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ when $ç=0)$.

Proof: Suppose that the functions $g_i(z)(i = 1, 2)$ defined by (4.1) are in the class $\Im_{(\lambda_p)(\mu_q),ç}^{s,a,\lambda}(\delta, d)$, it is suffices to prove that the function

$$K(z) = \varphi g_1(z) + (1-\varphi) g_2(z) \quad , \quad (0 \leq \varphi \leq 1) \qquad (4.3)$$

is also in the class $\Im_{(\lambda_p)(\mu_q),ç}^{s,a,\lambda}(\delta, d)$.

Since, for $0 \leq \varphi \leq 1$

$$K(z) = z + \sum_{n=2}^{\infty}\left\{\varphi a_{n,1} + (1-\varphi) a_{n,2}\right\} z^n,$$

we observe that

$$\sum_{n=2}^{\infty}[1+\delta(n-1)]\frac{\prod_{j=1}^{p}\left(1+\lambda_j\right)_{n-1}}{\prod_{j=1}^{q}\left(1+\mu_j\right)_{n-1}}\left(\frac{\Lambda(a+n,ç,s,\lambda)}{\Lambda(1+a,ç,s,\lambda)}\right)$$

$$\bullet \left(\frac{1+a}{a+n}\right)^s \left\{\varphi a_{n,1} + (1-\varphi) a_{n,2}\right\}$$

$$= \varphi \sum_{n=2}^{\infty} [1+\delta(n-1)] \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s a_{n,1}$$

$$+(1-\varphi) \sum_{n=2}^{\infty} [1+\delta(n-1)] \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s a_{n,2}$$

$$\le \varphi|d| + (1-\varphi)|d| = |d|.$$

Hence K(z) $\in \mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}$. This completes the proof of Theorem 4.2

5. Integral Operators

In this part, we review integral transforms of functions in the class $\mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$

Theorem 5.1. If the function f defined by (1.6) is in the class $\mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$ Where

$\Big(\lambda_j \in \mathbb{C}\ (j=1,\cdots,p,) ; \mu_j \in \mathbb{C} \setminus \overline{\mathbb{Z}}_0 (j=1,\cdots,q); z \in U;\ p \le q+1;$

$\min\{\Re(a),\Re(s)\}>0; \lambda>0$ when $\Re(\varsigma)>0$ and $s \in \mathbb{C}; a \in \mathbb{C} \setminus \overline{\mathbb{Z}}_0$

when $\varsigma=0\Big)$.

defined by

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt, \quad (c>-1) \qquad (5.1)$$

also belongs to the class $\mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$.

Proof: from (5.1), it follows that

$F(z) = z - \sum_{n=2}^{\infty} Q_n z^n$, where $Q_n = \left(\frac{c+1}{c+n}\right) a_n$.

Therefore,

$$\sum_{n=2}^{\infty} [1+\delta(n-1)] \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s Q_n$$

$$= \sum_{n=2}^{\infty} [1+\delta(n-1)] \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s \left(\frac{c+1}{c+n}\right) a_n$$

$$\le \sum_{n=2}^{\infty} [1+\delta(n-1)] \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s a_n \le |d|,$$

since $f(z) \in \mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$. Hence by Theorem 2.1,

F(z) $\in \mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$

6. Radii of Close-to-Convexity, Starlikeness and Convexity

A function $f \in A$ is said to be close-to-convex of order η if it satisfies

$$\mathcal{R}e\{f'(z)\} > \eta, \qquad (6.1)$$

for some $\eta (0 \le \eta \le 1)$ and for all $z \in U$ Also a function $f \in A$ is said to be starlike of order η if it satisfies

$$\mathcal{R}e\left\{\frac{z f'(z)}{f(z)}\right\} > \eta, \qquad (6.2)$$

for some $(0 \le \eta \le 1)$ and for all $z \in U$. Further, a function $f \in A$ is said to be convex of order η, if and only if $z f'(z)$ is starlike of order η, that is If

$$\mathcal{R}e\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \eta, \qquad (6.3)$$

for every η $(0 \le \eta \le 1)$ and for all z in U.

Theorem 6.1. The function f belong to be the class $\mathfrak{I}^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$ is close-to-convex of order η in $|z|<h_1(\mu,\delta,d,\eta)$, where

$$h_1(\mu,\delta,d,\eta) = \inf_n \left\{\frac{(1-\eta)[1+\delta(n-1)] \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s}{n|d|}\right\}^{\frac{1}{n-1}}$$

Proof: It is sufficient to show that

$$\left|f'(z)-1\right| < \sum_{n=2}^{\infty} n a_n |z|^{n-1} \le 1-\eta \qquad (6.4)$$

and

$$\sum_{n=2}^{\infty} [1+\delta(n-1)] \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(1+a,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s a_n \le |d|.$$

Observe that (6.4) is true if

$$\frac{n|z|^{n-1}}{1-\eta} \le \frac{[1+\delta(n-1)] \frac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\frac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\frac{1+a}{a+n}\right)^s}{|d|}. \qquad (6.5)$$

Solving (6.5) for $|z|$, we get

$$|z| \leq \left\{ \frac{(1-\eta)[1+\delta(n-1)] \dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}{n|d|} \right\}^{\frac{1}{n-1}} \quad (n\geq 2).$$

Theorem 6.2. If f belong to be the class $\Im^{s,a,\lambda}_{(\lambda_p),(\mu_q),\varsigma}(\delta,d)$,then $f(z)$ is starlike of order η in $|z|<$ where

$$h_2(\mu,\delta,d,\eta) = \inf_n \left\{ \frac{(1-\eta)[1+\delta(n-1)] \dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}{(n-\eta)|d|} \right\}^{\frac{1}{n-1}}$$

Proof: We must show that $\left|\frac{zf''(z)}{f'(z)} - 1\right| < 1 - \eta$ for $|z|<h_2(\mu,\delta,b,\eta)$ since

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{k=2}^{\infty} a_n|z|^{n-1}}$$

If

$$\frac{(n-\eta)|z|^{n-1}}{1-\eta} \leq$$

$$\frac{[1+\delta(n-1)] \dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}{|d|},$$

$f(z)$ is starlike of order η

Corollary 5.3. If f belong to be the class $\Im^{s,a,\lambda}_{(\lambda_p)(\mu_q),\varsigma}(\delta,d)$.

Then f is convex of order η in $|z|< h_4(\mu,\delta,d,\eta)$, where

$$h_4(\mu,\delta,d,\eta) = \inf_n \left\{ \frac{(1-\eta)[1+\delta(n-1)] \dfrac{\prod_{j=1}^{p}(1+\lambda_j)_{n-1}}{\prod_{j=1}^{q}(1+\mu_j)_{n-1}} \left(\dfrac{\Lambda(a+n,\varsigma,s,\lambda)}{\Lambda(a+1,\varsigma,s,\lambda)}\right)\left(\dfrac{1+a}{a+n}\right)^s}{n(n-\eta)|d|} \right\}^{\frac{1}{n-1}}$$

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