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Some Properties of univalent function with negative coefficients defined by a linear operator in the open unit disk.

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ABSTRACT

In this paper ,we introduce and study the subclass $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$ of univalent function with negative coefficients by using new linear operator$I^{\gamma,\delta}(c,b,\lambda)$ in the open unit disk. We obtain some geometric properties like coefficient inequality ,growth and distortion theorem. Hadmard product ,extreme point, closure theorem and radii of starlikeness, convexity and close-to-convexity of functions belonging to our subclass.

MSC: 30C45, 30C50

1.Introduction

Let $T(m)$ denote the class of functions normalized by

$$f(z) = z + \sum_{l=m+1}^{\infty} a_l z^l \qquad (l \in N, m = 1,2,3, \ldots\ldots), \tag{1}$$

which are analytic and univalent in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $S(m)$ be the subclass of $T(m)$, consisting of functions of the form

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$$f(z) = z - \sum_{l=m+1}^{\infty} a_l z^l \qquad (m \in N, a_l \geq 0). \qquad (2)$$

For function $f(z) \in T(m)$ defined by (1)and $h(z) \in T(m)$ defined by

$$h(z) = z + \sum_{l=m+1}^{\infty} d_l z^l \qquad (m \in N),$$

the convolution of $f(z)$ and $h(z)$ defined by

$$f(z) * h(z) = z + \sum_{l=m+1}^{\infty} a_l d_l z^l \qquad (z \in U). \qquad (3)$$

Now, the function $\Psi(c, k; z)$ given by

$$\Psi(c, k; z) = z + \sum_{l=m+1}^{\infty} \frac{(c)_{l-1}}{(d)_{l-1}} z^j \qquad (c \in R, k \in R - \{0, -1, -2, \dots \dots\}),$$

where$(c)_l$ is the pochhammer symbol defined by

$$(c)_l = \frac{\Gamma(c+l)}{\Gamma(c)} = \begin{cases} 1, & \text{if } l=0 \\ c(c+1)(c+2)\dots\dots\dots(c+l-1), & \text{if } l \in N. \end{cases}$$

also, consider a function $\theta(c, k; z)$defined by the convolution

$$\Psi(c, k; z) * \theta(c, k; z) = \frac{z}{(1-z)^{\gamma+1}}, \qquad \text{where } \gamma > -1, z \in U$$

Therefore ,

$$L^{\gamma}(c, k; z)\, f(z) = \theta(c, k; z) * f(z), \qquad z \in U,$$

where $c, k \in -\{0, -1, -2, \dots \dots \dots\}$. For a function $f \in S(m)$, it follows that for $\gamma > -1$

$$L^{\gamma}(c, k; z) f(z) = z - \sum_{l=m+1}^{\infty} \frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}} a_l z^l, \qquad (4)$$

is the Cho – Kown – Srivastava integral operator[7].

Moreover, for a function f in $T(m)$, introduced the following operatorby AL – Oboudi[2] ,

and studied by several authors like A. R. S. Juma and S. R. Kulkarni[5],

$$H^{\delta}(\lambda) f(z) = z + \sum_{l=m+1}^{\infty} [1 + (l-1)\lambda]^{\delta} a_l z^l, \qquad (\lambda > -1, \delta \in N \cup \{0\}). \qquad (5)$$

Now , for $f \in S(m)$, we introduce the new operator as:

$$I^{\gamma,\delta}(c,k,\lambda)f(z) = L^{\gamma}(c,k;z)f(z) * H^{\delta}(\lambda)f(z)$$

$$= z - \sum_{l=m+1}^{\infty} \frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}} [1+(l-1)\lambda]^{\delta} a_l z^l, \quad (6)$$

where $\gamma, \lambda > -1, \delta \in N \cup \{0\}, z \in U$.

Note that, there are many special cases of the our operator as following:

(i) $I^{0,0}(1,1,1) \equiv S^n$ is the salagean derivative operator, see[9]

(ii) $I^{0,\delta}(1,1,\lambda) \equiv S_{\lambda}^n$ is the salagean derivative operator introduced by AL − Oboudi [2]

(iii) $I^{\gamma,0}(c,b,1) \equiv R^n$ is the Ruscheweyh derivative operator , see[8]

(iv) $I^{\gamma,\delta}(1,1,\lambda) \equiv R_{\lambda}^n$ is the generalized Ruscheweyh derivative operator , see[1] .

Definition(1.1): Let the function $f \in S(m)$ be of the from (2) is in the class $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$ if it satisfies the inequality

$$\left| \frac{\left(I^{\gamma,\delta}(c,k,\lambda)f(z)\right)' - 1}{\tau\left(I^{\gamma,\delta}(c,k,\lambda)f(z)\right)' + (\sigma-\varphi)} \right| < \beta \quad , \qquad z \in U$$

for $0 \le \tau < 1,\ 0 \le \varphi < 1,\ 0 < \sigma \le 1, 0 < \beta < 1$.

In the case $\lambda = 0\ \sigma = 1, \gamma, \delta = N \cup \{0\}$, we have $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi) \equiv R_{\lambda}^{\gamma,\delta}(\beta,\tau,1,\varphi)$ introduced and studied by G.H.Esa and Darus [4].

Also the a bove class studied by Serap Bulut [3] and S. M. Khairnar and meena More [6].

We characterize the class $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, prove the following our main result .

2.Coefficient inequality.

Theorem(2 .1):

If $f(z) \in S(m)$, then $f \in R_{\lambda}^{\gamma,\delta,}(\beta,\tau,\sigma,\varphi)$ if and only if satisfies

$$\sum_{l=m+1}^{\infty} l(1-\beta\tau) \frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}} [1+(l-1)\lambda]^{\delta} a_l \le \beta(\tau+\sigma-\varphi), \quad (7)$$

$(0 \le \tau < 1, 0 \le \varphi < 1, 0 < \sigma \le 1, 0 < \beta < 1, k, c \in R - \{0.-1,-2,\dots\}, \lambda, \gamma > -1, \delta \in N \cup \{0\})$.

The outcome (7) is sharp of the function

$$f(z)=z-\frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}\, z^{l}, \qquad l\geq m+1. \tag{8}$$

Proof: Suppose that (7) holds true and $|z|=1$.Then

$$\left|\frac{(I^{\gamma,\delta}(c,k,\lambda)f(z))'-1}{\tau\left(I^{\gamma,\delta}(c,k,\lambda)f(z)\right)'+(\sigma-\varphi)}\right|<\beta \quad , \qquad z\in U$$

$$\left|\left(I^{\gamma,\delta}(c,k,\lambda)f(z)\right)'-1\right|-\beta\left|\tau\left(I^{\gamma,\delta}(c,k,\lambda)f(z)\right)'+(\sigma-\varphi)\right|$$

$$\leq\left|\left(z-\sum_{l=m+1}^{\infty}\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}a_l z^{l}\right)'-1\right|$$

$$-\beta\left|\tau\left(z-\sum_{l=m-1}^{\infty}\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}a_l z^{l}\right)'+(\sigma-\varphi)\right|$$

$$\leq\left|1-\sum_{l=m+1}^{\infty}\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}\, l a_l z^{l-1}-1\right|$$

$$-\beta\left|\tau\left(1-\sum_{l=m+1}^{\infty}\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}\, l a_l z^{l-1}\right)+(\sigma-\varphi)\right|$$

$$\leq\left|-\sum_{l=m+1}^{\infty}l\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}a_l z^{l-1}\right|$$

$$-\beta\left|\tau-\sum_{l=m+1}^{\infty}\tau l\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}a_l z^{l-1}+(\sigma-\varphi)\right|$$

$$\leq\sum_{l=m+1}^{\infty}l\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}a_l|z|^{l-1}$$

$$-\beta\tau-\sum_{l=m+1}^{\infty}\beta\tau l\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}a_l|z|^{l-1}-\beta(\sigma-\varphi)$$

$$\leq\sum_{l=m+1}^{\infty}l(1-\beta\tau)\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}[1+(l-1)\lambda]^{\delta}a_l-\beta(\tau+\sigma-\varphi)\leq 0.$$

Since by maximum modulus principle.

$$f(z)\in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi).$$

Conversely , assume that $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$. Then we have

$$\left| \frac{\left(I^{\gamma,\delta}(c,k,\lambda)f(z)\right)' - 1}{\tau\left(I^{\gamma,\delta}(c,k,\lambda)f(z)\right)' + (\sigma-\varphi)} \right| < \beta \quad , \qquad z \in U$$

$$\frac{\left| -\sum_{l=m+1}^{\infty} l \frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}} [1+(l-1)\lambda]^{\delta} a_l z^{l-1} \right|}{\left| -\sum_{l=m+1}^{\infty} \tau l \frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}} [1+(l-1)\lambda]^{\delta} a_l z^{l-1} + (\tau+\sigma-\varphi) \right|} < \beta \, . \tag{9}$$

Hence $\left| Re\left(f(z)\right)' \right| \leq \left| f(z) \right|$, for all z we obtain

$$Re\left\{ \frac{\sum_{l=m+1}^{\infty} l \frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}} [1+(l-1)\lambda]^{\delta} a_l |z|^{l-1}}{\sum_{l=m+1}^{\infty} \tau l \frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}} [1+(l-1)\lambda]^{\delta} a_l |z|^{l-1} + (\tau+\sigma-\varphi)} \right\} < \beta. \tag{10}$$

We taking the values of z on real axis such that $\left(I^{\gamma,\delta}f(z)\right)'$ is real and upon clearing ,

the denominator of the above expression and letting $z \to 1^{-}$through real values , we obtain

$$\sum_{l=m+1}^{\infty} l(1-\beta\tau) \frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}} [1+(l-1)\lambda]^{\delta} a_l \leq \beta(\tau+\sigma-\varphi).$$

■

Corollary(2 .1). Let $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$. Then

$$a_l \leq \frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} \, , \qquad (l \geq m+1, m \in N)$$

$$where\ 0 \leq \tau < 1, 0 \leq \varphi < 1, 0 < \sigma \leq 1, 0 < \beta < 1, c, k, \in R - \{0,,-1,-2,\dots\}, \gamma, \lambda > -1, \delta \in N \cup \{0\} \, .$$

Remark (2 .1). If $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,1,\varphi)$, then

$$a_l \leq \frac{\beta(\tau+1-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} \, , \qquad (l \geq m+1, m \in N)$$

and equality holds for

$$f(z) = z - \frac{\beta(\tau+1-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} z^{l}.$$

3.Growth and Distortion Theorem

A growth and distortion property for $f \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$ is offered as follows.

Theorem(3.1):

Let $f(z)$denoted by (2) belong to $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$. Then for $|z| = r < 1$, we obtain

$$r-\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m+1} \le |f(z)| \le r+\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m+1}.$$

The result is sharp for the function $f(z)$given by

$$f(z) = z-\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}z^{m+1}.$$

Proof: Assum that $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$. By the inequality(7)

$$(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}\frac{(k)_m(\gamma+1)_m}{(1)_m(c)_m},$$

is non decreasing and positive for $l \ge m+1$, we obtain

$$(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}\frac{(k)_m(\gamma+1)_m}{(1)_m(c)_m}\sum_{l=m+1}^{\infty} a_l$$

$$\le \sum_{l=m+1}^{\infty} l(1-\beta\tau)[1+(l-1)\lambda]^{\delta}\frac{(k)_{l-1}(\gamma+1)_{l-1}}{(1)_{l-1}(c)_{l-1}}a_l \le \beta(\tau+\sigma-\varphi).$$

That is equivalent to

$$\sum_{l=m+1}^{\infty} a_l \le \frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}. \quad (11)$$

Using (2) and (11) ,we obtain

$$f(z) = z-\sum_{l=m+1}^{\infty} a_l z^l \quad ,$$

$$|f(z)| \le |z| + \sum_{l=m+1}^{\infty} a_l |z|^l \le r+\sum_{l=m+1}^{\infty} a_l r^{m+1} \le r+r^{m+1}\sum_{l=m+1}^{\infty} a_l$$

$$\le r+\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m+1} \quad . \quad (12)$$

Similarly,

$$|f(z)| \ge r-\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)(1-\beta\tau)\left[1_{(m)\lambda}\right]^{\delta}(k)_m(\gamma+1)_m}r^{m+1} \quad . \quad (13)$$

From (12)and(13),we have

$$r-\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m+1}\leq |f(z)|$$

$$\leq r+\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m+1}.$$

■

Theorem(3 .2):

Let $f(z)$ defined(2) belong to $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$. Then for $|z|=r<1$,we obtain

$$1-\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m}\leq |f'(z)|$$

$$\leq 1+\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m}.$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z-\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(m+1)(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m+1}.$$

Proof: Using (2)and(11), we obtain

$$f'(z)=1-\sum_{l=m+1}^{\infty}la_l z^{l-1},$$

$$|f'(z)|\leq 1+\sum_{l=m+1}^{\infty}la_l|z|^{l-1}\leq 1+\sum_{l=m+1}^{\infty}la_l r^{l-1}\leq 1+\sum_{l=m+1}^{\infty}la_l r^{m}\leq 1+r^m\sum_{l=m+1}^{\infty}la_l$$

$$\leq 1+\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m}. \qquad (14)$$

So,

$$|f'(z)|\geq 1-\sum_{l=m+1}^{\infty}la_l|z|^{l=1}\geq 1-\sum_{l=m+1}^{\infty}la_l r^{l-1}\geq 1-\sum_{l=m+1}^{\infty}l\,a_l r^{m}\geq 1-r^m\sum_{l=m+1}^{\infty}la_l$$

$$\geq 1-\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(1-\beta\tau)[1+(m)\lambda]^{\delta}(k)_m(\gamma+1)_m}r^{m}. \qquad (15)$$

From (14)and(15),we obtain

$$1-\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(1-\beta\tau)[1+(m)\lambda]^\delta(k)_m(\gamma+1)_m}r^m \le \left|f'(z)\right|$$
$$\le 1+\frac{\beta(\tau+\sigma-\varphi)(1)_m(c)_m}{(1-\beta\tau)[1+(m)\lambda]^\delta(k)_m(\gamma+1)_m}r^m.$$

■

4. Hadamard product

In the following theorem, we obtain the Hadamard product (or convolution) result for $f \in R_\lambda^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$.

Theorem(4 .1):

If $f,h \in R_\lambda^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, then

$$(f*h)(z)=z-\sum_{l=m+1}^{\infty} a_l d_l z^l,$$

for

$$f(z)=z-\sum_{l=m+1}^{\infty} a_l z^l \qquad , \qquad f(z)=z-\sum_{l=m+1}^{\infty} d_l z^l \ ,$$

where

$$\rho \ge \frac{\beta^2(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{\tau\beta^2(\tau+\sigma-\varphi)+l(1-\beta\tau)^2(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^\delta}.$$

Proof: Let $f,h \in R_\lambda^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$. Then we have

$$\sum_{l=m+1}^{\infty}\frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^\delta}{(1)_{l-1}(c)_{l-1}\beta(\tau+\sigma-\varphi)}a_l \le 1 \ , \qquad (16)$$

$$\sum_{l=m+1}^{\infty}\frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^\delta}{(1)_{l-1}(c)_{l-1}\beta(\tau+\sigma-\varphi)}d_l \le 1 \ , \qquad (17)$$

we must find the smallest number ρ such that

$$\sum_{l=m+1}^{\infty}\frac{l(1-\rho\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^\delta}{(1)_{l-1}(c)_{l-1}\rho(\tau+\sigma-\varphi)}a_l d_l \le 1 . \qquad (18)$$

By Cauchy Schwarz inequality ,we have

$$\sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{(1)_{l-1}(c)_{l-1}\beta(\tau+\sigma-\varphi)} \sqrt{a_l d_l} \leq 1 \quad . \qquad (19)$$

Thus it is sufficient to prove that

$$\sum_{l=m+1}^{\infty} \frac{l(1-\rho\tau)(k)_{l-1}(\gamma+1)_{l-1}([1+(l-1)\lambda])^{\delta}}{(1)_{l-1}(c)_{l-1}\rho(\tau+\sigma-\varphi)} a_l d_l$$

$$\leq \sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{(1)_{l-1}(c)_{l-1}\beta(\tau+\sigma-\varphi)} \sqrt{a_l d_l}.$$

So ,

$$\sqrt{a_l d_l} \leq \frac{\rho(1-\beta\tau)}{\beta(1-\rho\tau)} . \qquad (20)$$

From (19) ,we obtain

$$\sqrt{a_l d_l} \leq \frac{(1)_{l-1}(c)_{l-1}\beta(\tau+\sigma-\varphi)}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l}[1+(l-1)\lambda]^{\delta}} . \qquad (21)$$

It is sufficient to prove that

$$\frac{(1)_{l-1}(c)_{l-1}\beta(\tau+\sigma-\varphi)}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} \leq \frac{\rho(1-\beta\tau)}{\beta(1-\rho\tau)},$$

$$\rho \geq \frac{\beta^2(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{\tau\beta^2(\tau+\sigma-\varphi)+l(1-\beta\tau)^2(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} .$$

■

5. Radius of Starlikeness , Close-to-Convexity and Convexity.

Next ,we obtain the radius of starlikeness , convexity and close-to-convexity for the class $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$ by the following theorems.

Theorem(5 .1):

If $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, then f is starlik of order $\alpha(0 \leq \alpha < 1)$in the disk $|z| < R$, where

$$R = inf_l \left\{ \frac{(1-\alpha)l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{(l-\alpha)\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} \right\}^{\frac{1}{l-1}} .$$

The estimate is sharp from

$$f(z)=z-\frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}z^{l}\,, \qquad (l\geq m+1, m\in N).$$

Proof: From (2),we have

$$R\left(\frac{zf'(z)}{f(z)}\right)>\alpha\ , \qquad (0\leq\alpha<1)\,. \tag{22}$$

To prove that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq 1-\alpha,$$

$$\left|\frac{z(z-\sum_{l=m+1}^{\infty}a_l z^l)'-(z-\sum_{l=m+1}^{\infty}a_l z^l)}{z-\sum_{l=m+1}^{\infty}a_l z^l}\right|\leq 1-\alpha,$$

$$\left|\frac{z-\sum_{l=m+1}^{\infty}l a_l z^l-z+\sum_{l=m+1}^{\infty}a_l z^l}{z-\sum_{l=m+1}^{\infty}a_l z^l}\right|\leq 1-\alpha\,,$$

$$\left|\frac{-\sum_{l=m+1}^{\infty}(l-1)a_l z^l}{z-\sum_{l=m+1}^{\infty}a_l z^l}\right|\leq\frac{\sum_{l=m+1}^{\infty}(l-1)a_l|z|^{l-1}}{1-\sum_{l=m+1}^{\infty}a_l|z|^{l-1}}\leq 1-\alpha,$$

$$\frac{\sum_{l=m+1}^{\infty}(l-\alpha)a_l|z|^{l-1}}{(1-\alpha)}\leq 1-\alpha\,. \tag{23}$$

From corollary(2 .1),we have

$$\frac{\sum_{l=m+1}^{\infty}(l-\alpha)\frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}|z|^{l-1}}{(1-\alpha)}\leq 1.$$

Hence

$$|z|^{l-1}\leq\frac{(1-\alpha)l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{(l-\alpha)\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\,,$$

$$|z|\leq R_1=\inf_l\left\{\frac{(1-\alpha)l(1-\beta\tau)(k)_{l-1}(\gamma-1)_{l-1}[1+(l-1)\lambda]^{\delta}}{(l-\alpha)\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right\}^{\frac{1}{l-1}}.$$

The proof is completes.

Theorem(5 .2):

If $f(z)\in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, then f is convx of order ϑ $(0\leq\vartheta<1)$ in $|z|<R_2$, where

$$R_2=inf_l\left\{\frac{(1-\vartheta)l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{l(l-\vartheta)\beta((\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right\}^{\frac{1}{l-1}}, \quad (l\geq m+1, m\in N).$$

the estimate is sharp from

$$f(z) = z - \frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} z^{l}\,.$$

Proof: From(2),we have

$$Re\left(\frac{zf''(z)}{f'(z)}\right) > \vartheta\ , \tag{24}$$

which is equal to

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\vartheta.$$

Which is equal to

$$\frac{\sum_{l=m+1}^{\infty} l(l-\vartheta)a_l|z|^{l-1}}{(1-\vartheta)} \le 1\,.$$

From corollary (2 .1),we obtain

$$\frac{\sum_{l=m+1}^{\infty} l\,(l-\vartheta)\frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}|z|^{l-1}}{(1-\vartheta)} \le 1. \tag{25}$$

Hence

$$|z|^{l-1} \le \frac{(1-\vartheta)l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{l(l-\vartheta)\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\ ,$$

$$|z| \le R_2 = inf_l\left\{\frac{(1-\vartheta)l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}\left[1_{(l-1)\lambda}\right]^{\delta}}{l(l-\vartheta)\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right\}^{\frac{1}{l-1}}\ ,\ \ (l \ge m+1, m \in N).$$

The proof is completes.

Theorem(5 .3):

If $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, then f is close − to convex of order $\mu(0 \le \mu < 1) in\ |z| < R_3$, we where

$$R_3 = inf_l\left\{\frac{(1-\mu)(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right\}^{\frac{1}{l-1}}, \qquad (l \ge m+1, m \in N).$$

Proof: From (2),we have

$$Re\{f'(z)\} > \mu\ , \tag{26}$$

which is equivalent to

$$\left|f'(z)-1\right| \le 1-\mu,$$

which is simplifies to

$$\frac{\sum_{l=m+1}^{\infty} l\,\frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}\,|z|^{l-1}}{(1-\mu)} \le 1. \tag{27}$$

Hence

$$\left|z\right|^{l-1} \le \frac{(1-\mu)(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}},$$

$$\left|z\right| \le R_3 = inf_l\left\{\frac{(1-\mu)(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right\}^{\frac{1}{l-1}}.$$

The proof is complete.

6. Extreme point

Now, in the following theorem , we obtain extreme point for the class $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$.

Theorem(6 .1):

$$If\ f_1(z) = z\ and$$

$$f_n(z) = z - \frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} z^l \quad ,(\, l \ge m+1, m \in N).$$

then $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, if and only if given in the form

$$f(z) = \sum_{l=m+1}^{\infty} \omega_l\, f_l(z) \quad \text{where } \omega_l \ge 0 \ \text{ and } \sum_{l=m+1}^{\infty} \omega_l = 1.$$

Proof: Assume that

$$f(z) = \sum_{l=m+1}^{\infty} \omega_l f_l\,(z),$$

$$= \sum_{l=m+1}^{\infty} \omega_l \left(z - \frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} z^l\right),$$

$$= z - \sum_{l=m+1}^{\infty} \frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} \omega_l z^l . \quad (28)$$

So $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, hence

$$\sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} \cdot \frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} \omega_l$$

$$= \sum_{l=m+1}^{\infty} \omega_l = 1 - \omega_l \leq 1.$$

Conversely ,assume that $f(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$from (7), we have

$$a_l \leq \frac{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}} , \quad (l \geq m+1, m \in N)$$

put ,

$$\omega_l = \frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} a_l, \quad (l \geq m+1, m \in N) \quad (29)$$

and

$$\omega_1 = 1 - \textstyle\sum_{l=m+1}^{\infty} \omega_l,$$

it is enough we obtain

$$f(z) = \sum_{l=m+1}^{\infty} \omega_l f_l(z) \quad .$$

Is the result of the theorem .

7. Closur Theorem

Now , we ought to show that the following closure theorems belong $R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$.

Theorem(7 .1):

Suppose $f_i \in \mathrm{R}_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, $\quad (i = 1,2,3 \ldots \ldots r)$. Then

$$q(z) = \sum_{i=1}^{r} e_i f_i(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi),$$

where,

$$\sum_{i=1}^{r} e_i = 1 \quad , \qquad \text{and} \;\; f_i = z - \sum_{l=m+1}^{\infty} a_l z^l \, .$$

Proof: We obtain

$$q(z) = \textstyle\sum_{i=1}^{r} e_i \left(z - \sum_{l=m+1}^{\infty} a_l z^l\right),$$

$$= z \sum_{i=1}^{r} e_i - \sum_{i=1}^{r} \sum_{l=m+1}^{\infty} e_i a_{l,i} z^l \, ,$$

$$= z - \sum_{l=m+1}^{\infty} \left(\sum_{i=1}^{r} e_i a_{l,i} \right) z^l , \qquad (30)$$

$$= z - \sum_{l=m+1}^{\infty} w_l z^l \quad , \qquad (31)$$

such that

$$w_l = \sum_{i=1}^{r} e_i a_{l,i} \, .$$

Hence , $f_i \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$ from Theorem(2 .1), we have

$$\sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(b)_l(\gamma+1)_l\left[1_{(l-1)\lambda}\right]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} \, a_{l,i} \le 1, \qquad (32)$$

in (30) , $g(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$. Then

$$\sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(k)_l(\gamma+1)_l[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} w_l \le 1.$$

Now,

$$\sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(k)_l(\gamma+1)_l[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} w_l$$

$$= \sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} \sum_{i=1}^{r} e_i a_{l,i}$$

$$= \sum_{i=1}^{r} e_i \sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} a_{l,i}$$

$$\le \sum_{i=1}^{r} e_i\,, \quad \text{employ}(33)$$

$$= 1\,, \qquad hence\ g(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi).$$

Theorem(7 .2):

If $f(z), g(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$, then

$$p(z) = z - \sum_{l=m+1}^{\infty} (a_l^2 + v_l^2)\, z^l,$$

belongs to $R_{\lambda}^{\gamma,\delta}(\eta,\tau,\sigma,\varphi)$, where $\eta = \frac{2\beta}{\beta\tau+1}$.

Proof: Suppose $f(z), g(z) \in R_{\lambda}^{\gamma,\delta}(\beta,\tau,\sigma,\varphi)$and since

$$\sum_{l=m+1}^{\infty} \left[\frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right]^2 a_l^2 \le \left[\sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} a_l\right]^2 \le 1\,. \qquad (34)$$

$$\sum_{l=m+1}^{\infty} \left[\frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right]^2 v_l^2 \le \left[\sum_{l=m+1}^{\infty} \frac{l(1-\beta\tau)(b)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} v_l\right]^2 \le 1. \qquad (35)$$

Using (34)and(35),we have

$$\sum_{l=m+1}^{\infty} \frac{1}{2}\left[\frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right]^2 (a_l^2+v_l^2) \le 1. \qquad (36)$$

Hence ,we must show

$p(z) \in R_{\lambda}^{\gamma,\delta}(\eta,\tau,\sigma,\varphi)$, there is

$$\sum_{l=m+1}^{\infty}\left[\frac{l(1-\eta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\eta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}\right]^{2}(a_l^2+v_l^2)\leq 1, \qquad (37)$$

where $0 \leq \eta < 1$, from (36)and (37), we obtain

$$\frac{l(1-\eta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\eta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}} \leq \frac{1}{2}\frac{l(1-\beta\tau)(k)_{l-1}(\gamma+1)_{l-1}[1+(l-1)\lambda]^{\delta}}{\beta(\tau+\sigma-\varphi)(1)_{l-1}(c)_{l-1}}.$$

Which simplifies

$$\eta \leq \frac{2\beta}{\beta\tau+1}.$$

The proof is complete.

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