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Second Derivative Multistep Method For Solving First-Order Ordinary Differential Equations

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Abstract. In this paper, a new second derivative multistep method was constructed to solve first order ordinary differential equations (ODEs). In particular, we used the new method as a corrector method and 5-steps Adam's Bashforth method as a predictor method to solve first order (ODEs) . Numerical results were compared with the existing methods which clearly showed the efficiency of the new method.

INTRODUCTION

This paper deals with numerical solution of the initial value problems (IVPs) for first-order ODEs in the form:

$$y'(x) = f(x, y),\ y(x_0) = y_0 \quad . \tag{1}$$

The second derivative with respect to x gives.

$$y'' = f'(x, y) = fx + ffy = g(x, y) \quad . \tag{2}$$

A lot of fields in applied science such as mathematics, electricity, chemistry, nuclear and physics are related with one another by the use of this type of problems.

Ismail and Ibrahim [1], derived a special class of second derivative multistep method and discussed the stability analysis of this class which depend on free parameters. Famurewa et.al [2], focused on the development, analysis, implementation and the comparative study of Implicit multi-derivative linear multistep methods for numerical solution of non-stiff and stiff Initial Value Problems of first order ordinary differential equations. Hojjati et.al [3], derived a new class of second derivative multistep methods , discussed the stability analysis and the improvement in stability region.Ezzeddine and Hojjati [4], presented a class of multistep methods for the numerical solution of stiff ordinary differential equations. In these methods the first, second and third derivatives of the solution were used to improve the accuracy and absolute stability regions of the methods. Khalsaraei et.al [5], presented details of a new class of implicit formulas of linear multistep methods to integrate ordinary differential equations numerically.Hamid et.al [6], modified second derivative multistep methods which were constructed to solve ordinary differential equations.Sharifi and Seif [7],derived a new family of multistep numerical integration methods based on Hermite interpolation.

Multistep method is one of the useful techniques with fast convergence rate and small calculation error, see Lapidus and Seinfeld [8]. This is because single step methods are inefficient since they do not use full information from the calculation, see Yaacob and Chang [9].

In this work we derived a new second derivative multistep method. Some numerical examples were given to

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show the effectiveness of our new method compared with the existing methods .The new derivation used Hermite Interpolating Polynomial.

DERIVATION OF THE NEW METHOD

Integrating equation (1) over the interval $[x_n, x_{n+4}]$

$$\int_{x_n}^{x_{n+4}} y' dx = \int_{x_n}^{x_{n+4}} f(x, y(x))dx,$$

$$y(x_{n+4}) - y(x_n) = \int_{x_n}^{x_{n+4}} f(x, y(x))dx, \tag{3}$$

$f(x, y(x))$ can be replaced by Hermite Interpolating Polynomial P which is gven by:

$$P(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_{i-1}} f_i^{(k)} L_{i,k}(x), \tag{4}$$

where $f_i = f(x_i), \xi_i = x_i = a + ih, i = 0, 1, ..., n$ and $h = \frac{b-a}{n}$ n is a positive integer

$$L_{i,m_i}(x) = \ell_{i,m_i}(x),\ i = 0, 1, ..., n,$$

$$\ell_{i,k}(x) = \frac{(x - \xi_i)^k}{k!} \prod_{j=0, j\neq i}^{n} (\frac{x - \xi_j}{\xi_i - \xi_j})^m j, i = 0, 1, ..., n, k = 0, 1, ..., m_i.$$

And recursively for $k = m_i - 2, m_i - 3, ..., 0$.

$$L_{i,k}(x) = \ell_{i,k}(x) - \sum_{v=k+1}^{m_i-1} \ell_{i,k}^{(v)}(\xi_i) L_{i,v}(x).$$

By substituting $n = 4$ and $m_0 = 2, m_1 = m_2 = m_3 = 1$ and $m_4 = 2$ for each $i = 0, 1, 2, 3$ and 4 in equation (4) we have:

$$P(x) = \sum_{i=0}^{4} \sum_{k=0}^{m_{i-1}} f_i^{(k)} L_{i,k}(x) = f_0 L_{0,0}(x) + f_0' L_{0,1}(x) + f_1 L_{1,0}(x) + f_2 L_{2,0}(x) + f_3 L_{3,0}(x) + f_4 L_{4,0}(x) + f_4' L_{4,1}(x),$$

where $f_0 = f(0), f_1 = f(\frac{1}{4}), f_2 = f(\frac{1}{2}), f_3 = f(\frac{3}{4}), f_4 = f(1), f_0' = f'(0)$ and $f_4' = f'(1)$.

$$L_{0,1}(x) = \ell_{0,1}(x) = \frac{(x - \xi_0)^1}{1!} \prod_{j=0, j\neq 0}^{4} (\frac{x - \xi_j}{\xi_0 - \xi_j})^m j = \frac{-32}{3}(x^6 - \frac{14}{4}x^5 + \frac{75}{16}x^4 - \frac{95}{32}x^3 + \frac{7}{8}x^2 - \frac{3}{32}x).$$

$$L_{0,0}(x) = \ell_{0,0}(x) - \sum_{v=1}^{1} \ell_{0,0}^{(v)}(\xi_2) L_{0,v}(x) = \ell_{0,0}(x) - \ell_{0,0}^{(1)}(0) L_{0,1}(x) = \frac{(x - \xi_0)^0}{0!} \prod_{j=0, j\neq 0}^{4} (\frac{x - \xi_j}{\xi_0 - \xi_j})^m j - \ell_{0,0}^{(1)} L_{0,1}(x)$$

$$= (\frac{-896}{9}x^6 + \frac{3040}{9}x^5 - \frac{1288}{3}x^4 + \frac{2210}{9}x^3 - \frac{4992}{9}x^2 + 1).$$

$$L_{1,0}(x) = \frac{(x - \xi_1)^0}{0!} \prod_{j=0, j\neq 1}^{4} (\frac{x - \xi_j}{\xi_1 - \xi_j})^m j = \frac{2048}{9}(x^6 - \frac{13}{4}x^5 + \frac{31}{8}x^4 - 2x^3 + \frac{3}{8}x^2).$$

$$L_{2,0}(x) = \frac{(x - \xi_2)^0}{0!} \prod_{j=0, j\neq 2}^{4} (\frac{x - \xi_j}{\xi_2 - \xi_j})^m j = -256(x^6 - 3x^5 + \frac{51}{16}x^4 - \frac{11}{8}x^3 + \frac{3}{16}x^2).$$

$$L_{3,0}(x)=\frac{(x-\xi_3)^0}{!}\prod_{j=0,j\neq 3}^{4}(\frac{x-\xi_j}{\xi_3-\xi_j})^m j=\frac{2048}{9}(x^6-\frac{11}{4}x^5+\frac{21}{8}x^4-x^3+\frac{1}{8}x^2).$$

$$L_{4,1}(x)=\frac{(x-\xi_4)^1}{1!}\prod_{j=0,j\neq 4}^{4}(\frac{x-\xi_j}{\xi_4-\xi_j})^m j=\frac{32}{3}(x^6-\frac{5}{2}x^5+\frac{35}{16}x^4-\frac{25}{32}x^3+\frac{3}{32}x^2).$$

$$L_{4,0}(x)=\ell_{4,0}(x)-\sum_{v=1}^{1}\ell_{4,0}^{(v)}(\xi_4)L_{4,v}(x)=\ell_{4,0}(x)-\ell_{4,0}^{(1)}(1)L_{4,1}(x)=\frac{(x-\xi_4)^0}{0!}\prod_{j=0,j\neq 4}^{4}(\frac{x-\xi_j}{\xi_4-\xi_j})^m j-\ell_{4,0}^{(1)}L_{4,1}(x)$$
$$=(\frac{-896}{9}x^6+\frac{2336}{9}x^5-\frac{2104}{9}x^4+\frac{766}{9}x^3-\frac{31}{3}x^2).$$

$$\int_0^1 P(X)dx=\int_0^1[f_0L_{0,0}(x)+f_1L_{1,0}(x)+f_2L_{2,0}(x)+f_3L_{3,0}(x)+f_4L_{4,0}(x)+f_0^{'}L_{0,1}(x)+f_4^{'}L_{4,1}(x)]dx$$
$$=\frac{31}{270}f_0+\frac{256}{945}f_1+\frac{8}{35}f_2+\frac{256}{945}f_3+\frac{31}{270}f_4+\frac{1}{252}[f_0^{'}-f_4^{'}]. \tag{5}$$

Now , by using the transformation $x=a+t(b-a)$ and equation (5) we obtain:

$$\int_a^b P(X)dx=\int_0^1(b-a)[f_0L_{0,0}(t)+f_1L_{1,0}(t)+f_2L_{2,0}(t)+f_3L_{3,0}(t)+f_4L_{4,0}(t)]dt$$

$$+\int_0^1(b-a)^2[f_0^{(')}L_{0,1}(t)+f_4^{(')}L_{4,1}(t)]dt,$$

then

$$\int_a^b f(x)dx=4h[\frac{31}{270}f(a)+\frac{256}{945}f(a+h)+\frac{8}{35}f(a+2h)+\frac{256}{945}f(a+3h)+\frac{31}{270}f(b)]$$

$$+\frac{16h^2}{252}[f^{'}(a)-f^{'}(b)],h=\frac{b-a}{n},n=4. \tag{6}$$

Now by substituting equation (2) in equation (6) we have:

$$\int_a^b f(x)dx=4h[\frac{31}{270}f(a)+\frac{256}{945}f(a+h)+\frac{8}{35}f(a+2h)+\frac{256}{945}f(a+3h)+\frac{31}{270}f(b)]$$

$$+\frac{16h^2}{252}[g(a)-g(b)],h=\frac{b-a}{n},n=4. \tag{7}$$

Integrating (7) over the interval $[x_n,x_{n+4}]$ and substituting it in equation (3) we obtian:

$$y_(x_{n+4})-y_(x_n)=4h[\frac{31}{270}f(x_n)+\frac{256}{945}f(x_{n+1})+\frac{8}{35}f(x_{n+2})+\frac{256}{945}f(x_{n+3})+\frac{31}{270}f(x_{n+4})]$$

$$+\frac{16h^2}{252}[g(x_n)-g(x_{n+4})]. \tag{8}$$

Equation (8) is called the new second derivative multistep method.

Order Conditions And Error Constant Of The New Method

According to Fatunla [10] and Lambert [11], we can define the local truncation error associated with normalized form of the new method as the linear difference operator

$$L[Z(x);h] = \sum_{i=0}^{k}[\alpha_i Z(x+jh) - h\beta_i Z'(x+jh) - h^2\gamma_i Z''(x+jh)]. \tag{9}$$

Assuming that Z(x) is sufficiently differentiable, we can expand the terms in (9) as a Taylor series about the point x to obtain the expression $L[Z(x);h] = C_0 Z(x) + C_1 hZ'(x) + ... + C_q hqZ^q(x) + ...$, where the constant coefficients Cq , q= 0, 1, ... are given as follows:

$$C_0 = \sum_{i=0}^{k}\alpha_j,$$

$$C_1 = \sum_{i=0}^{k} j\alpha_j - \sum_{i=0}^{k}\beta_j,$$

$$\vdots$$

$$Cq = \frac{1}{q!}\sum_{i=0}^{k} j^q\alpha_j - \frac{1}{(q-1)!}\sum_{i=0}^{k} j^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{i=0}^{k} j^{q-2}\gamma_j \ .$$

According to Henrici [12], we say that the new method has order p if $C_0 = C_1 = ... = C_p = 0, C_{p+1} \neq 0$.

Therefore, C_{p+1} is the error constant and $C_{p+1}h^{p+1}Z^{(p+1)}(x_n)$ the principal local truncation error at the point x_n. The new method has order p=8 and error constant $C_9 = \frac{16}{99225}$.

PROBLEMS TESTED AND NUMERICAL RESULTS

In this section, we applied the new method to solve first- order ordinary differential equation problems and compared it with the existing methods .

Problem 1: Source : [7].

$$y' = 1 + y, \quad y(0) = 0, \quad 0 \leq x \leq 2 \ .$$

The exact solution :

$$y = -1 + e^x \ .$$

Problem 2: Source : [6].

$$y' = ycos(x), \quad y(0) = 1, \quad 0 \leq x \leq 2 \ .$$

The exact solution :

$$y = e^{sin(x)} \ .$$

Problem 3: Source : [6].

$$y' = x + y, \quad y(0) = 0, \quad 0 \leq x \leq 1 \ .$$

The exact solution :

$$y = e^x - x - 1 \ .$$

Problem 4: Source : [7].

$$y' = -\frac{1}{2}y^3 \ , \quad y(0) = 1 \ , \quad 0 \leq x \leq 1 \ .$$

The exact solution :

$$y = \frac{1}{\sqrt{x+1}} \ .$$

Notations used :

* h - step size.
* Time - seconds.
* Max Error - maximum error $|y(x_i) - y_i|$.
* New method - the new four-step second derivative multistep method derived in this paper .
* method R - new second derivative multistep method proposed from Hamid et.al [6].
* method S - the new four-step implicit method proposed from Sharifi and Seif [7] .

TABLE 1. Maximum Error and Time for problem 1.

h	New method	Time	method R	Time	method S	Time
0.1	$4.414 * 10^{-5}$	0.330	$2.371 * 10^{-3}$	0.330	$4.843 * 10^{-5}$	0.330
0.02	$8.171 * 10^{-10}$	0.340	$2.334 * 10^{-5}$	0.337	$1.049 * 10^{-9}$	0.343
0.004	$1.91 * 10^{-13}$	0.372	$1.949 * 10^{-8}$	0.370	$2.708 * 10^{-13}$	0.374
0.0001	$1.673 * 10^{-21}$	3.452	$3.077 * 10^{-13}$	3.280	$2.475 * 10^{-21}$	3.617

TABLE 2. Maximum Error and Time for problem 2.

h	New method	Time	method R	Time	method S	Time
0.1	$9.907 * 10^{-5}$	0.330	$1.003 * 10^{-3}$	0.330	$9.674 * 10^{-5}$	0.330
0.02	$9.63 * 10^{-10}$	0.343	$1.99 * 10^{-7}$	0.340	$1.009 * 10^{-9}$	0.343
0.001	$7.42 * 10^{-17}$	1.136	$1.045 * 10^{-11}$	1.130	$9.153 * 10^{-17}$	1.157
0.0001	$6.35 * 10^{-22}$	8.674	$9.802 * 10^{-15}$	8.564	$8.074 * 10^{-22}$	8.955

TABLE 3. Maximum Error and Time for problem 3.

h	New method	Time	method R	Time	method S	Time
0.1	$8.826 * 10^{-7}$	0.330	$3.329 * 10^{-4}$	0.330	$1.16 * 10^{-5}$	0.330
0.01	$7.815 * 10^{-12}$	0.350	$5.401 * 10^{-8}$	0.350	$9.803 * 10^{-12}$	0.350
0.001	$7.043 * 10^{-17}$	0.468	$5.636 * 10^{-11}$	0.440	$9.141 * 10^{-17}$	0.500
0.0001	$6.959 * 10^{-22}$	1.789	$5.66 * 10^{-14}$	1.775	$9.081 * 10^{-22}$	1.913

TABLE 4. Maximum Error and Time for problem4 .

h	New method	Time	method R	Time	method S	Time
0.1	$2.545 * 10^{-5}$	0.330	$4.152 * 10^{-5}$	0.330	$2.428 * 10^{-5}$	0.330
0.01	$5.498 * 10^{-12}$	0.350	$1.64 * 10^{-8}$	0.350	$5.333 * 10^{-12}$	0.350
0.004	$9.96 * 10^{-18}$	0.549	$1.794 * 10^{-11}$	0.546	$8.189 * 10^{-18}$	0.590
0.0003	$1.288 * 10^{-21}$	1.158	$4.879 * 10^{-13}$	1.045	$1.069 * 10^{-20}$	1.220

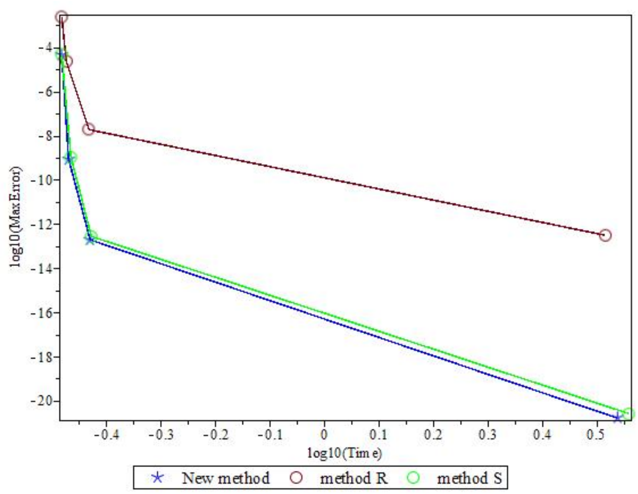

FIGURE 1. Efficiency curves of methods for problem 1 with h= 0.1, 0.02, 0.004 and 0.0001

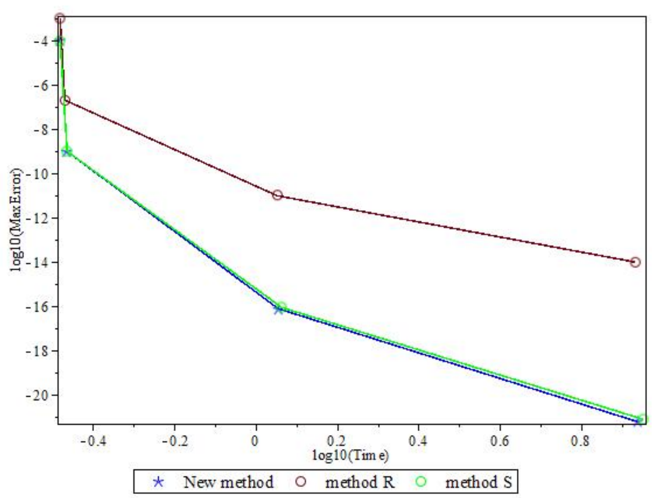

FIGURE 2. Efficiency curves of methods for problem 2 with h= 0.1, 0.02, 0.001 and 0.0001

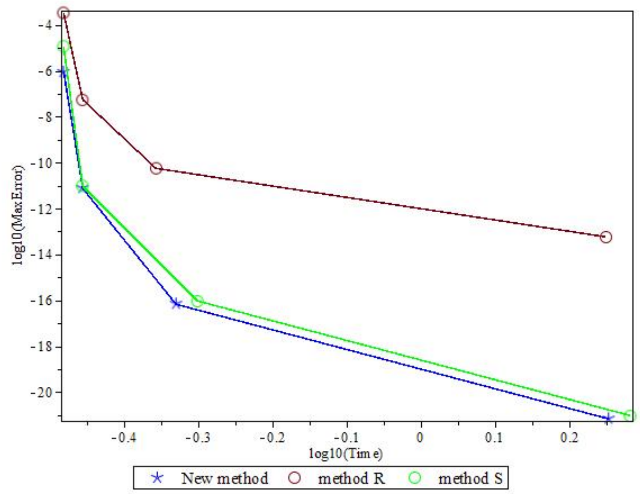

FIGURE 3. Efficiency curves of methods for problem 3 with *h*= 0.1, 0.01, 0.001 and 0.0001

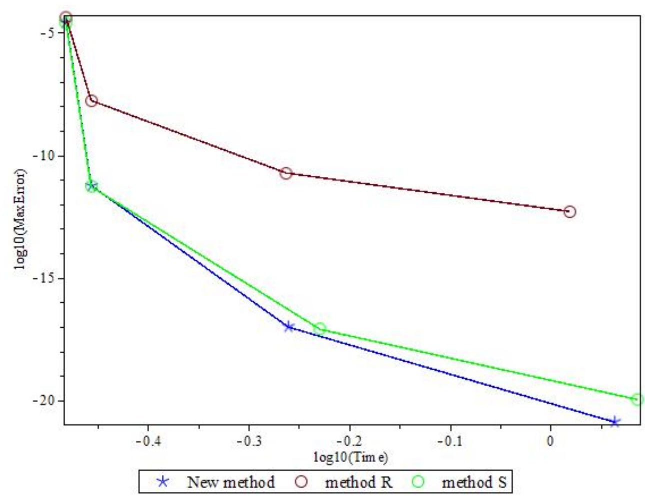

FIGURE 4. Efficiency curves of methods for problem 4 with *h*= 0.1, 0.01, 0.001 and 0.0003

In analyzing the numerical results,methods with almost the same number of steps were compared.The results were plotted in Figures 1-4.We presented efficiency curves where the common logarithm of the maximum global error was plotted versus the computed time.From Figures 1-4 ,we observed that the new method was more efficient for solving first-order ODEs compared with the existing methods.

CONCLUSION

In this work, a new second derivative multistep method with $k = 4$ was derived. From the results shown in figures 1-4,we noticed that the new method was more efficient for solving first-order ODEs when compared with the existing methods .We concluded that the new method was more accurate for solving first-order ODEs.

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