Two and Three point Implicit Second Derivative Block Methods For Solving First Order Ordinary Differential Equations

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Abstract

In this research we developed implicit block methods which make used of the first and second derivatives of the problems. The aim is to give a more accurate as well as faster numerical results for solving first order ordinary differential equations. The methods are then used to solve a set of first order initial value problems. Numerical results clearly show that the new proposed methods performed better than other well-known existing methods in solving the set of test problems.

Keywords— Second derivative, implicit block methods

1 Introduction

Many researchers have focused on the block method for solving first order ordinary differential equations (ODEs). Such as Majid et al. (2003) developed two point implicit method for solving a system of ODEs . Majid et al. (2006) derived three point block method to solve first order ODEs.Ibrahim et al. (2007) used block backward diffference formula to solve first order ODEs. This is by computing two or three points simultaneously using x_{n-1} and x_n as the back values of each block. Ibrahim et al. (2008) developed the block method by adding a fixed coefficients block backward differentiation formules to solve first order ODEs. Ibrahim et al. (2011) also considered the property of convergence two point block backward differentiation formulas. Mehrkanoon et al. (2012) used the Gauss-seidel technique for the implementation of the three point 3-step block multistep method for solving system of first order ODEs. However, all the work mentioned earlier used onlythe first derivative of the problems in the derivation of the methods. thus, work on extra derivatives in the derivation of the methods have been done by Sahi et al. (2012), where they implemented Simpson's types second derivative block method for solving first order ODEs. A four-step block generalized Adam's type second derivative method had been modified by Kumleng and Sirisena (2014) to solve first order ODEs. Akinfenwa et al. (2015) used a family of continuous third derivative block methods derived from the collocation and interpolation technique to solve first order ODEs. This paper considered initial value problems (IVPs) for first-order ODEs in the form:

$$y^{'} = f(x, y), y(x_0) = y_0. \tag{1}$$

The second derivative with respect to x gives

$$g(x, y) = y^{''} = f'(x, y) = f_x + f\ f_y$$

In this work, a new two and three point second derivative implicit block methods were derived.

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The new methods are derived using Hermite Interpolating Polynomial P, which can be defined by :

$$P(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_{i-1}} f_i^{(k)} L_{i,k}(x), \qquad (2)$$

where $f_i = f(x_i), x_i = a + ih, i = 0, 1, ..., n$ and $h = \frac{b-a}{n}$, n is a positive integer. $L_{i,k}(x)$ is the generalized Lagrange polynomial which can be defined by

$$L_{i,m_i}(x) = \ell_{i,m_i}(x), \; i = 0, 1, ..., n,$$

$$\ell_{i,k}(x) = \frac{(x-x_i)^k}{k!} \prod_{j=0, j\neq i}^{n} (\frac{x-x_j}{x_i-x_j})^m j,$$

$$i = 0, 1, ..., n, k = 0, 1, ..., m_i.$$

And recursively for $k = m_i - 2, m_i - 3, ..., 0$.

$$L_{i,k}(x) = \ell_{i,k}(x) - \sum_{v=k+1}^{m_i-1} \ell_{i,k}^{(v)}(x_i) L_{i,v}(x).$$

The purpose of including the derivatives in the formula is that, more accurate numerical results can be obtained. Some numerical examples were given to show the effectiveness of these new methods compared with the existing methods.

2 Derivation of The New Methods

In order to evaluate the approximate solution in each block, the interval [a,b] is divided into a series of blocks that each block contains k points. The approximate solution at the point x_n is used to start the i th block while the approximate solution at the point x_{n+k} is the last point in the i th block. Then, the evaluation information at the last point in the i th block will be restored as the approximate solution at the point x_n to start the $(i+1)$ th block and the process continues for the next block.

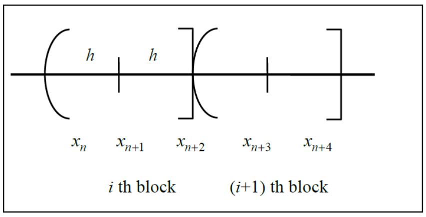

Figure 1: 2-point block method.

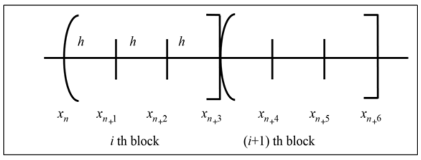

Figure 2: 3-point block method.

2.1 Two - Point Second Derivative Implicit Block Method

In two point block method, the interval [a,b] contains two points for each block with the step size $2h$ (refer to Figure1). The two values of y_{n+1} and y_{n+2} are calculated concurrently in a block. For the evaluation of y_{n+1} take $x_{n+1} = x_n + h$ and integrating (1) over the interval $[x_n, x_{n+1}]$ gives :

$$\int_{x_n}^{x_{n+1}} y' dx = \int_{x_n}^{x_{n+1}} f(x, y) dx,$$

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y) dx. \qquad (3)$$

Then, $f(x, y)$ in (3) will be replaced by Hermite Interpolating Polynomial in (2) and define $p_2(x)$ as follows,

$$p_2(x) = [(\frac{x-x_{n+1}}{x_n-x_{n+1}})(\frac{x-x_{n+2}}{x_n-x_{n+2}})^2 - ((\frac{2}{x_n-x_{n+2}})$$

$$+(\frac{1}{x_n-x_{n+1}}))(x-x_n)(\frac{x-x_{n+1}}{x_n-x_{n+1}})(\frac{x-x_{n+2}}{x_n-x_{n+2}})^2]f_0$$

$$+[(\frac{x-x_n}{x_{n+1}-x_n})^2(\frac{x-x_{n+2}}{x_n-x_{n+2}})^2]f_1 + [(\frac{x-x_n}{x_{n+2}-x_n})^2$$

$$(\frac{x-x_{n+1}}{x_{n+2}-x_{n+1}}) - ((\frac{1}{x_{n+2}-x_{n+1}}) + (\frac{2}{x_{n+2}-x_n}))$$

$$(x-x_{n+2})(\frac{x-x_n}{x_{n+2}-x_n})^2(\frac{x-x_{n+1}}{x_{n+2}-x_{n+1}})]f_2$$

$$+[(x-x_n)(\frac{x-x_{n+1}}{x_n-x_{n+1}})(\frac{x-x_{n+2}}{x_n-x_{n+2}})^2]g_0$$

$$+[(x-x_{n+2})(\frac{x-x_n}{x_{n+2}-x_n})^2(\frac{x-x_{n+1}}{x_{n+2}-x_{n+1}})]g_2. \quad (4)$$

Let $x = x_{n+2} + s\ h$ and

$$s = \frac{x-x_{n+2}}{h}. \quad (5)$$

Replace $dx = h\ ds$ and change the limit of integration from -2 to -1 in (3) we obtain:

$$y(x_{n+1}) = y(x_n) + \int_{-2}^{-1} [f_0 L_{0,0}(s) + f_1 L_{1,0}(s)$$

$$+f_2 L_{2,0}(s) + g_0 L_{0,1}(s) + g_2 L_{2,1}(s)]\ h\ ds. \quad (6)$$

where

$$L_{0,0}(s) = -\frac{1}{4}(s^3+s^2) - \frac{1}{2}(s^3+2s^2)(s+1),$$
$$L_{1,0}(s) = s^2(s+2)^2,$$
$$L_{2,0}(s) = \frac{1}{4}(s+2)^2(s+1) - \frac{1}{2}(s^2+s)(s+2)^2,$$

$$L_{1,0}(s) = -\frac{h}{4}\ s^2\ (s+2)\ (s+1),$$
$$L_{2,1}(s) = \frac{h}{4}\ s\ (s+2)^2(s+1).$$

Evaluating the integral in (6) by using MAPLE produces the first formula of the two-point implicit block method as follows,

$$y_{n+1} = y_n + \frac{h}{240}[131f_n + 128f_{n+1} - 19f_{n+2}]$$

$$+\frac{h^2}{240}[23g_n + 7g_{n+2}]. \quad (7)$$

Now, integrating (1) over the interval $[x_n, x_{n+2}]$ to obtain the approximate solutions of y_{n+2} we have

$$\int_{x_n}^{x_{n+2}} y^{'} dx = \int_{x_n}^{x_{n+2}} f(x,y)dx,$$

$$y(x_{n+2}) = y(x_n) + \int_{x_n}^{x_{n+2}} f(x,y)dx. \quad (8)$$

Then, $f(x,y)$ in (8) will be replaced by Hermite Interpolating Polynomial in (4). Also, by replacing (5) letting $dx = h\ ds$ and changing the limit of integration from -2 to 0 in (8) we obtain:

$$y(x_{n+2}) = y(x_n) + \int_{-2}^{0} [f_0 L_{0,0}(s) + f_1 L_{1,0}(s)$$

$$+f_2 L_{2,0}(s) + g_0 L_{0,1}(s) + g_2 L_{2,1}(s)]\ h\ ds. \quad (9)$$

Evaluating the integral in (9) by using MAPLE produces the second formula of the two-point Implicit block method as follows,

$$y_{n+2} = y_n + \frac{h}{15}[7f_n + 16f_{n+1} + 7f_{n+2}]$$

$$+\frac{h^2}{15}[g_n - g_{n+2}]. \quad (10)$$

2.2 Three - Point Second Derivative Implicit Block Method

In the three points block, the interval [a,b] contains three points for each block with the step size $3h$ (refer to Figure 2). The three approximation values of y_{n+1}, y_{n+2} and y_{n+3} at the point x_{n+1}, x_{n+2} and x_{n+3} are calculated concurrently in a block. The derivations of the three point block method are similar to the previous derivations of the two point block method.

Equation (1) will be integrated over the interval $[x_n, x_{n+1}]$, $[x_n, x_{n+2}]$ and $[x_n, x_{n+3}]$ to obtian the approximate solutions of y_{n+1}, y_{n+2} and y_{n+3}, defined $p_3(x)$ as follows,

$$p_3(x) = [(\frac{x-x_{n+1}}{x_n-x_{n+1}})(\frac{x-x_{n+1}}{x_n-x_{n+1}})(\frac{x-x_{n+3}}{x_n-x_{n+3}})^2$$

$$+((\frac{-1}{x_n-x_{n+1}}) + (\frac{-1}{x_n-x_{n+2}}) + (\frac{-2}{x_n-x_{n+3}}))$$

$$((x-x_n)(\frac{x-x_{n+1}}{x_n-x_{n+1}})(\frac{x-x_{n+2}}{x_n-x_{n+2}})(\frac{x-x_{n+3}}{x_n-x_{n+3}})^2)]f_0 +$$

$$[(\frac{x-x_n}{x_{n+1}-x_n})^2(\frac{x-x_{n+2}}{x_{n+1}-x_{n+2}})(\frac{x-x_{n+3}}{x_{n+1}-x_{n+3}})^2]f_1$$

$$+[(\frac{x-x_n}{x_{n+2}-x_n})^2(\frac{x-x_{n+1}}{x_{n+2}-x_{n+1}}) \quad (\frac{x-x_{n+3}}{x_{n+2}-x_{n+3}})^2]f_2$$

$$+[(\frac{x-x_n}{x_{n+3}-x_n})^2(\frac{x-x_{n+1}}{x_{n+3}-x_{n+1}})(\frac{x-x_{n+2}}{x_{n+3}-x_{n+2}})$$

$$-((\frac{2}{x_{n+3}-x_n}) + (\frac{1}{x_{n+3}-x_{n+1}}) + (\frac{1}{x_{n+3}-x_{n+2}}))$$

$$((x-x_{n+3})(\frac{x-x_n}{x_{n+3}-x_n})^2(\frac{x-x_{n+1}}{x_{n+3}-x_{n+1}})$$

$$(\frac{x-x_{n+2}}{x_{n+3}-x_{n+2}}))]f_3+((x-x_n)(\frac{x-x_{n+1}}{x_n-x_{n+1}})$$
$$(\frac{x-x_{n+2}}{x_n-x_{n+2}})(\frac{x-x_{n+3}}{x_n-x_{n+3}})^2)g_0+((x-x_{n+3})$$
$$(\frac{x-x_n}{x_{n+3}-x_n})^2(\frac{x-x_{n+1}}{x_{n+3}-x_{n+1}})(\frac{x-x_{n+2}}{x_{n+3}-x_{n+2}}))]g_3. \quad (11)$$

Then, Hermite Interpolating Polynomial in (11) will interpolate $f(x,y)$ and let $x = x_{n+3} + sh$ and $s = \frac{x-x_{n+3}}{h}$. For each evaluation of y_{n+1}, y_{n+2} and y_{n+3}, we take $x_{n+1} = x_n + h$, $x_{n+2} = x_{n+1} + h$ and $x_{n+3} = x_{n+2} + h$ respectively.

The first , second and third point can be written as follows,

$$y_{n+1} = y_n + \frac{h}{6480}[3463f_n + 3537f_{n+1} - 783f_{n+2}$$
$$+263f_{n+3}] + \frac{h^2}{1080}[97g_n - 17g_{n+3}].$$
$$y_{n+2} = y_n + \frac{h}{405}[181f_n + 459f_{n+1} + 189f_{n+2}$$
$$-19f_{n+3}] + \frac{h^2}{135}[8g_n + 2g_{n+3}].$$
$$y_{n+3} = y_n + \frac{h}{80}[39f_n + 81f_{n+1} + 81f_{n+2}$$
$$+39f_{n+3}] + \frac{h^2}{40}[3g_n - 3g_{n+3}]. \quad (12)$$

3 Order Conditions And Error Constant Of The New Methods

This section presents a definition of the order of the two and three point block methods that have been derived in this paper.
According to Fatunla (1991) and Lambert (1991), the local truncation error associated with normalized form of the new method can be defined as the linear difference operator

$$L[Z(x);h] = \sum_{i=0}^{k} \alpha_i Z(x+jh) - \sum_{i=0}^{k} h\beta_i Z'(x+jh)$$
$$-\sum_{i=0}^{k} h^2\gamma_i Z''(x+jh). \quad (13)$$

Assuming that $Z(x)$ is sufficiently differentiable, (13) can be expanded as a Taylor series expansion about the point x to obtain the expression $L[Z(x);h] = C_0Z(x) + C_1hZ'(x) + ... + C_phpZ^p(x) + ...$, where the constant coefficients $Cp,\ p = 0,1,...$ are given as follows:

$$C_0 = \sum_{i=0}^{k} \alpha_j,$$
$$C_1 = \sum_{i=0}^{k} j\alpha_j - \sum_{i=0}^{k} \beta_j,$$
$$Cp = \frac{1}{p!}\sum_{i=0}^{k} j^p\alpha_j - \frac{1}{(p-1)!}\sum_{i=0}^{k} j^{p-1}\beta_j$$
$$-\frac{1}{(p-2)!}\sum_{i=0}^{k} j^{p-2}\gamma_j,\ p = 2,3,.. \quad (14)$$

According to Henrici (1962), it can be said that the new method has order p if $C_0 = C_1 = ... = C_p = 0, C_{p+1} \neq 0$.

Therefore, C_{p+1} is the error constant and $C_{p+1}h^{p+1}Z^{(p+1)}(x_n)$ is the principal local truncation error at the point x_n.
The formulae of a new two point block method is given by (7) and (10) and the formulae is written into a matrix as follows:

$$\alpha Y_m = +h\beta F_m + h^2\gamma G_m \quad (15)$$

where $\alpha,\ \beta$ and γ are the coefficients with the m-vector $Y_m,\ F_m$ and G_m be defind as,

$$\alpha = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \beta = \begin{bmatrix} \frac{131}{240} & \frac{128}{240} & \frac{-19}{240} \\ \frac{7}{15} & \frac{16}{15} & \frac{7}{15} \end{bmatrix},$$
$$\gamma = \begin{bmatrix} \frac{23}{240} & 0 & \frac{7}{240} \\ \frac{1}{15} & 0 & \frac{-1}{15} \end{bmatrix},$$
$$Y_m = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \end{bmatrix}, F_m = \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \end{bmatrix}, G_m = \begin{bmatrix} g_n \\ g_{n+1} \\ g_{n+2} \end{bmatrix}.$$

$$\alpha_0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \alpha_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\beta_0 = \begin{bmatrix} \frac{131}{240} \\ \frac{7}{15} \end{bmatrix}, \beta_1 = \begin{bmatrix} \frac{128}{240} \\ \frac{16}{15} \end{bmatrix}, \beta_2 = \begin{bmatrix} \frac{-19}{240} \\ \frac{7}{15} \end{bmatrix},$$

$$\gamma_0 = \begin{bmatrix} \frac{23}{240} \\ \frac{1}{15} \end{bmatrix}, \gamma_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \gamma_2 = \begin{bmatrix} \frac{7}{240} \\ \frac{-1}{15} \end{bmatrix}.$$

For $p = 0$,

$$C_0 = \sum_{i=0}^{2} \alpha_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

For $p = 1$,

$$C_1 = \sum_{i=0}^{2} j\alpha_j - \sum_{i=0}^{2} \beta_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

For $p = 2$,

$$C_2 = \frac{1}{2!}\sum_{i=0}^{2} j^2\alpha_j - \sum_{i=0}^{2} j\beta_j - \sum_{i=0}^{2} \gamma_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

For $p = 3$,

$$C_3 = \frac{1}{3!}\sum_{i=0}^{2} j^3\alpha_j - \frac{1}{(2)!}\sum_{i=0}^{2} j^2\beta_j$$

$$-\sum_{i=0}^{2} j\gamma_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

For $p = 4$,

$$C_4 = \frac{1}{4!}\sum_{i=0}^{2} j^4\alpha_j - \frac{1}{(3)!}\sum_{i=0}^{2} j^3\beta_j -$$

$$\frac{1}{(2)!}\sum_{i=0}^{2} j^2\gamma_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

For $p = 5$,

$$C_5 = \frac{1}{5!}\sum_{i=0}^{2} j^5\alpha_j - \frac{1}{(4)!}\sum_{i=0}^{2} j^4\beta_j$$

$$-\frac{1}{(3)!}\sum_{i=0}^{2} j^3\gamma_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

For $p = 6$,

$$C_6 = \frac{1}{6!}\sum_{i=0}^{2} j^6\alpha_j - \frac{1}{(5)!}\sum_{i=0}^{2} j^5\beta_j$$

$$-\frac{1}{(4)!}\sum_{i=0}^{2} j^4\gamma_j = \begin{bmatrix} \frac{-1}{720} \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then, the 2-point implicit block method has order $p = 5$ and error constant $C_6 = [\frac{-1}{720}, 0]^T$.

The formulae of a new three point block method is given by (12) and the formulae is written into a matrix from (15) as follows:

$$\alpha = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

$$\beta = \begin{bmatrix} \frac{3463}{6480} & \frac{3537}{6480} & \frac{-783}{6480} & \frac{263}{6480} \\ \frac{181}{405} & \frac{459}{405} & \frac{189}{405} & \frac{-19}{405} \\ \frac{39}{80} & \frac{81}{80} & \frac{81}{80} & \frac{39}{80} \end{bmatrix},$$

$$\gamma = \begin{bmatrix} \frac{97}{1080} & 0 & 0 & \frac{-17}{1080} \\ \frac{8}{135} & 0 & 0 & \frac{2}{135} \\ \frac{3}{40} & 0 & 0 & \frac{-3}{40} \end{bmatrix},$$

$$Y_m = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix}, F_m = \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}, G_m = \begin{bmatrix} g_n \\ g_{n+1} \\ g_{n+2} \\ g_{n+3} \end{bmatrix}.$$

$$\alpha_0 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\beta_0 = \begin{bmatrix} \frac{3463}{6480} \\ \frac{181}{405} \\ \frac{39}{80} \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} \frac{3537}{6480} \\ \frac{459}{405} \\ \frac{81}{80} \end{bmatrix}, \beta_2 = \begin{bmatrix} \frac{-783}{6480} \\ \frac{189}{405} \\ \frac{81}{80} \end{bmatrix},$$

$$\beta_3 = \begin{bmatrix} \frac{263}{6480} \\ \frac{-19}{405} \\ \frac{39}{80} \end{bmatrix}, \gamma_0 = \begin{bmatrix} \frac{97}{1080} \\ \frac{8}{135} \\ \frac{3}{40} \end{bmatrix}, \gamma_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \gamma_3 = \begin{bmatrix} \frac{-17}{1080} \\ \frac{2}{135} \\ \frac{-3}{40} \end{bmatrix}.$$

For $p = 0$,

$$C_0 = \sum_{i=0}^{3} \alpha_j = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

For $p = 1$,

$$C_1 = \sum_{i=0}^{3} j\alpha_j - \sum_{i=0}^{3} \beta_j = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

For $p = 2$,

$$C_2 = \frac{1}{2!}\sum_{i=0}^{3} j^2\alpha_j - \sum_{i=0}^{3} j\beta_j - \sum_{i=0}^{3} \gamma_j = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

For $p = 3$,

$$C_3 = \frac{1}{3!}\sum_{i=0}^{3} j^3\alpha_j - \frac{1}{(2)!}\sum_{i=0}^{3} j^2\beta_j - \sum_{i=0}^{3} j\gamma_j = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

For $p = 4$,

$$C_4 = \frac{1}{4!}\sum_{i=0}^{3} j^4\alpha_j - \frac{1}{(3)!}\sum_{i=0}^{3} j^3\beta_j - \frac{1}{(2)!}\sum_{i=0}^{3} j^2\gamma_j = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

For $p = 5$,

$$C_5 = \frac{1}{5!}\sum_{i=0}^{3} j^5\alpha_j - \frac{1}{(4)!}\sum_{i=0}^{3} j^4\beta_j - \frac{1}{(3)!}\sum_{i=0}^{3} j^3\gamma_j = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

For $p = 6$,

$$C_6 = \frac{1}{6!}\sum_{i=0}^{3} j^6\alpha_j - \frac{1}{(5)!}\sum_{i=0}^{3} j^5\beta_j - \frac{1}{(4)!}\sum_{i=0}^{3} j^4\gamma_j = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

For $p = 7$,

$$C_7 = \frac{1}{7!}\sum_{i=0}^{3} j^7\alpha_j - \frac{1}{(6)!}\sum_{i=0}^{3} j^6\beta_j - \frac{1}{(5)!}\sum_{i=0}^{3} j^5\gamma_j = \begin{bmatrix}\frac{97}{100800}\\\frac{-1}{6300}\\\frac{9}{11200}\end{bmatrix} \neq \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Then, the 3-point implicit block method has order $p = 6$ and error constant $C_7 = [\frac{97}{100800}, \frac{-1}{6300}, \frac{9}{11200}]^T$.

4 The Zero-Stability Of The Methods

In this section, the zero-stability of the 2-point and 3-point implicit block method are discussed.

Two point implicit block method

The general form of (7) and (10) can be written in the matrix form :

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix} = \begin{bmatrix}0&1\\0&1\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_n\end{bmatrix} + h\begin{bmatrix}\frac{131}{240}&\frac{128}{240}&\frac{-19}{240}\\\frac{7}{15}&\frac{16}{15}&\frac{7}{15}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\end{bmatrix} + h^2\begin{bmatrix}\frac{23}{240}&0&\frac{7}{240}\\\frac{1}{15}&0&\frac{-1}{15}\end{bmatrix}\begin{bmatrix}g_n\\g_{n+1}\\g_{n+2}\end{bmatrix}.$$

The first characteristic polynomial of the 2-point implicit block method is given as follows,
$\rho(R) = \det\,[RA^{(0)} - A^{(1)}] = 0$,
where

$$A^{(0)} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \text{ and } A^{(1)} = \begin{bmatrix}0&1\\0&1\end{bmatrix}.$$

$\rho(R) = \det\begin{bmatrix}R&-1\\0&R-1\end{bmatrix} = 0$, $R(R-1) = 0$, $R = 0, 1,$ $|\,R\,| \leq 1$.

Three point implicit block method

The general form of (12) can be written in the matrix form :

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix} = \begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-1}\\y_n\end{bmatrix} + h\begin{bmatrix}\frac{3463}{6480}&\frac{3537}{6480}&\frac{-783}{6480}&\frac{263}{6480}\\\frac{181}{405}&\frac{459}{405}&\frac{189}{405}&\frac{-19}{405}\\\frac{39}{80}&\frac{81}{80}&\frac{81}{80}&\frac{39}{80}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix} + h^2\begin{bmatrix}\frac{97}{1080}&0&0&\frac{-17}{1080}\\\frac{8}{135}&0&0&\frac{2}{135}\\\frac{3}{40}&0&0&\frac{-3}{40}\end{bmatrix}\begin{bmatrix}g_n\\g_{n+1}\\g_{n+2}\\g_{n+3}\end{bmatrix}.$$

The first characteristic polynomial of the 3-point implicit block method is given as follows,
$\rho(R) = \det\,[RA^{(0)} - A^{(1)}] = 0$,
where

$$A^{(0)} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \text{ and } A^{(1)} = \begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}.$$

$\rho(R) = \det \begin{bmatrix} R & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & R-1 \end{bmatrix} = 0,\ R^2(R-1) = 0,\ R = 0, 0, 1\ , \mid R \mid \leq 1.$

According to Fatunla (1991), the two point and three point implicit block methods are zero-stable, since the first characteristic polynomial $\rho(R) = 0$ satisfy $\mid R_j \mid \leq 1, j = 0, ..., k$. Also, the two point and three point implicit block methods are consistent as they have order p greater than one. Following Henrici (1962), we can say that the two point and three point block methods are convergence because they are zero-stable and consistent.

5 Implementation

This section focuses on the explanation of the implementation of the two and three point implicit second derivative block methods.

Two point implicit second derivative block method

The values of y_{n+1} and y_{n+2} in (7) and (10) will be approximated by using the predictor-corrector equations.

The predictor equations:

$$y^p_{n+m} = y^c_n + m\ h\ f^c_n,\quad m = 1, 2, \tag{16}$$

$$f^p_{n+m} = f(x_{n+m}, y^p_{n+m}),$$
$$g^p_{n+m} = f'(x_{n+m}, y^p_{n+m}).$$

The corrector equations:

$$y^c_{n+1} = y^c_n + \frac{h}{240}[131 f^c_n + 128 f^p_{n+1} - 19 f^p_{n+2}] + \frac{h^2}{240}[23 g^c_n + 7 g^p_{n+2}].$$

$$y^c_{n+2} = y^c_n + \frac{h}{15}[7 f^c_n + 16 f^p_{n+1} + 7 f^p_{n+2}] + \frac{h^2}{15}[g^c_n - g^p_{n+2}].$$

And the next corrector equations will be taken as follows:

$$y^c_{n+1} = y^c_n + \frac{h}{240}[131 f^c_n + 128 f^c_{n+1} - 19 f^c_{n+2}] + \frac{h^2}{240}[23 g^c_n + 7 g^c_{n+2}].$$

$$y^c_{n+2} = y^c_n + \frac{h}{15}[7 f^c_n + 16 f^c_{n+1} + 7 f^c_{n+2}] + \frac{h^2}{15}[g^c_n - g^c_{n+2}].$$

$$f^c_{n+m} = f(x_{n+m}, y^c_{n+m}),$$
$$g^c_{n+m} = f'(x_{n+m}, y^c_{n+m}),\ m = 1, 2.$$

Three point implicit second derivative block method

The values of y_{n+1}, y_{n+2} and y_{n+3} in (12) will be approximated by using the predictor-corrector equations.

The predictor equations:
Define (16) as the predictor equations and let $m = 1, 2, 3$.
The corrector equations:

$$y^c_{n+1} = y^c_n + \frac{h}{6480}[3463 f^c_n + 3537 f^p_{n+1} - 783 f^p_{n+2} + 263 f^p_{n+3}] + \frac{h^2}{1080}[97 g^c_n - 17 g^p_{n+3}].$$

$$y^c_{n+2} = y^c_n + \frac{h}{405}[181 f^c_n + 459 f^p_{n+1} + 189 f^p_{n+2} - 19 f^p_{n+3}] + \frac{h^2}{135}[8 g^c_n + 2 g^p_{n+3}].$$

$$y^c_{n+3} = y^c_n + \frac{h}{80}[39 f^c_n + 81 f^p_{n+1} + 81 f^p_{n+2} + 39 f^p_{n+3}] + \frac{h^2}{40}[3 g^c_n - 3 g^c_{n+3}].$$

And the next corrector equations will be taken as follows:

$$y^c_{n+1} = y^c_n + \frac{h}{6480}[3463 f^c_n + 3537 f^c_{n+1} - 783 f^c_{n+2} + 263 f^c_{n+3}] + \frac{h^2}{1080}[97 g^c_n - 17 g^c_{n+3}].$$

$$y^c_{n+2} = y^c_n + \frac{h}{405}[181 f^c_n + 459 f^c_{n+1} + 189 f^c_{n+2} - 19 f^c_{n+3}] + \frac{h^2}{135}[8 g^c_n + 2 g^c_{n+3}].$$

$$y^c_{n+3} = y^c_n + \frac{h}{80}[39 f^c_n + 81 f^c_{n+1} + 81 f^c_{n+2} + 39 f^c_{n+3}] + \frac{h^2}{40}[3 g^c_n - 3 g^c_{n+3}].$$

$$f^c_{n+m} = f(x_{n+m}, y^c_{n+m}),$$
$$g^c_{n+m} = f'(x_{n+m}, y^c_{n+m}),\ m = 1, 2, 3.$$

6 Numerical Experiments

In this section, based on the new methods we developed C codes for solving first - order ordinary differential equation problems and compared the numerical results when the same set of problems are solved by using the existing methods .

Problem 1:

$$y' = y - x^2 + 1, \quad y(0) = \frac{1}{2}, \quad [0,5].$$

$$Exact\ solution : y(x) = (1+x)^2 - \frac{1}{2}e^x.$$

Source Yaacob and Sanugi (1995).

Problem 2:

$$y' = xy^3 - y, \quad y(0) = 1, \quad [0,10].$$

$$Exact\ solution : y(x) = \frac{2}{\sqrt{2+4x+2e^{2x}}}.$$

Source: Famurewa et al. (2011).

Problem 3:

$$\begin{aligned} y_1' &= y_3, & y_1(0) &= 1, \quad [0,\pi]. \\ y_2' &= y_4, & y_2(0) &= 1, \\ y_3' &= -e^{-x}y_2, & y_3(0) &= 0, \\ y_4' &= 2e^x y_3, & y_4(0) &= 1. \end{aligned}$$

$$\begin{aligned} Exact\ solution : y_1(x) &= cos(x), \\ y_2(x) &= e^x cos(x), \\ y_3(x) &= -sin(x), \\ y_4(x) &= e^x cos(x) - e^x sin(x). \end{aligned}$$

Source : Abdul Majid et al. (2012).

Problem 4:

$$y_i' = -\beta_i y_i + y_i^2, \; i = 1,2,3,4, \quad y_i(0) = -1, \; [0,20].$$
$$with\ \beta_1 = 0.2, \; \beta_2 = 0.2, \; \beta_3 = 0.3, \; \beta_4 = 0.4.$$

$$Exact\ solution : y_i(x) = \frac{\beta_i}{1 + c_i e^{\beta_i x}}, \quad c_i = -(1+\beta_i).$$

Source : Johnson and Barney (1976).

Notations used are as follows.

- h: step size.
- Time: seconds.
- Max Error: maximum error $|y(x_i) - y_i|$.
- New 2P: The new 2-point implicit second derivative block method derived in this paper.
- New 3P: The new 3-point implicit second derivative block method derived in this paper.
- method 2A: 2-point Implicit third derivative block method proposed by Akinfenwa et al. (2015).
- method 3A: 3-point Implicit third derivative block method proposed by Akinfenwa et al. (2015).
- method 2M : 2-point implicit block one-step method half Gauss-Seidel proposed by Majid et al. (2003).
- method S : A simpson's-type second derivative block method proposed by Sahi et al. (2012).

Table 1: Numerical Results of the New 2P, 2A and Majid Methods for solving Problem 1.

h	Methods	MAXE	Time
0.1	New 2P	3.097580(-7)	0.008
	2A	9.075473(-6)	0.009
	Majid	1.028331(-3)	0.007
0.05	New 2P	3.096409(-9)	0.027
	2A	2.336125(-7)	0.033
	Majid	4.335581(-5)	0.026
0.025	New 2P	4.814321(-11)	0.072
	2A	6.980545(-9)	0.074
	Majid	2.708640(-6)	0.070
0.0125	New 2P	6.593552(-13)	0.117
	2A	5.621164(-10)	0.119
	Majid	6.505416(-7)	0.115

Table 2: Numerical Results of the New 2P, 2A and Majid Methods for solving Problem 2.

h	Methods	MAXE	Time
0.1	New 2P	8.068236(-7)	0.036
	2A	2.407103(-5)	0.038
	Majid	5.322828(-5)	0.035
0.05	New 2P	1.852646(-8)	0.079
	2A	6.901808(-6)	0.081
	Majid	4.287203(-6)	0.078
0.025	New 2P	3.577403(-10)	0.130
	2A	1.925143(-6)	0.132
	Majid	3.081526(-7)	0.128
0.0125	New 2P	6.248941(-12)	0.155
	2A	5.110584(-7)	0.158
	Majid	2.073222(-8)	0.154

Table 3: Numerical Results of the New 2P, 2A and Majid Methods for solving Problem 3.

h	Methods	MAXE	Time
0.1	New 2P	4.803249(-6)	0.046
	2A	6.021320(-6)	0.047
	Majid	7.434949(-5)	0.045
0.05	New 2P	1.380934(-7)	0.094
	2A	9.966465(-7)	0.096
	Majid	4.751417(6)	0.093
0.025	New 2P	4.138981(-9)	0.141
	2A	8.262003(-8)	0.143
	Majid	3.003513(-7)	0.140
0.0125	New 2P	1.266727(-10)	0.175
	2A	5.800818(-9)	0.177
	Majid	1.887970(-8)	0.174

Table 4: Numerical Results of the New 2P, 2A and Majid Methods for solving Problem 4.

h	Methods	MAXE	Time
0.1	New 2P	1.085710(-5)	0.062
	2A	3.864142(-4)	0.063
	Majid	4.230879(-4)	0.060
0.05	New 2P	7.773816(-7)	0.171
	2A	3.396333(-5)	0.173
	Majid	3.499478(-5)	0.169
0.025	New 2P	3.393665(-8)	0.296
	2A	3.873437(-6)	0.298
	Majid	2.530419(-6)	0.294
0.0125	New 2P	1.239543(-9)	0.483
	2A	4.865963(-7)	0.485
	Majid	1.703761(-7)	0.481

Table 5: Numerical Results of the New 3P, 3A and Sahi Methods for solving Problem 1.

h	Methods	MAXE	Time
0.1	New 3P	8.462243(-8)	0.007
	3A	2.311753(-6)	0.007
	Sahi	1.605447(-6)	0.010
0.05	New 3P	1.309864(-9)	0.025
	3A	5.083822(-8)	0.026
	Sahi	2.476227(-8)	0.034
0.025	New 3P	2.071335(-11)	0.065
	3A	2.913834(-9)	0.069
	Sahi	3.812920(-10)	0.075
0.0125	New 3P	2.957443(-12)	0.098
	3A	2.473823(-11)	0.114
	Sahi	1.740532(-11)	0.122

Table 6: Numerical Results of the New 3P, 3A and Sahi Methods for solving Problem 2.

h	Methods	MAXE	Time
0.1	New 3P	1.368469(-7)	0.032
	3A	1.897087(-5)	0.034
	Sahi	1.350523(-7)	0.036
0.05	New 3P	5.097240(-9)	0.072
	3A	6.888210(-6)	0.074
	Sahi	9.439226(-9)	0.080
0.025	New 3P	8.308907(-11)	0.126
	3A	2.049793(-6)	0.127
	Sahi	6.071470(-10)	0.131
0.0125	New 3P	1.311735(-12)	0.151
	3A	5.595309(-7)	0.153
	Sahi	3.812597(-11)	0.157

Table 7: Numerical Results of the New 3P, 3A and Sahi Methods for solving Problem 3.

h	Methods	MAXE	Time
0.1	New 3P	2.941061(-6)	0.040
	3A	4.800874(-5)	0.041
	Sahi	7.541150(-5)	0.046
0.05	New 3P	8.019770(-8)	0.089
	3A	6.981759(-7)	0.091
	Sahi	3.244254(-6)	0.095
0.025	New 3P	2.349645(-9)	0.136
	3A	1.057838(-8)	0.137
	Sahi	1.087780(-7)	0.141
0.0125	New 3P	7.114069(-11)	0.172
	3A	1.630123(-10)	0.174
	Sahi	3.662937(-9)	0.175

Table 8: Numerical Results of the New 3P, 3A and Sahi Methods for solving Problem 4.

h	Methods	MAXE	Time
0.1	New 3P	5.100885(-6)	0.046
	3A	1.469320(-3)	0.047
	Sahi	2.407790(-5)	0.062
0.05	New 3P	3.832369(-7)	0.109
	3A	4.396974(-5)	0.110
	Sahi	2.257036(-6)	0.171
0.025	New 3P	1.772670(-8)	0.218
	3A	1.770248(-6)	0.220
	Sahi	1.573703(-7)	0.296
0.0125	New 3P	6.679853(-10)	0.405
	3A	5.820560(-7)	0.407
	Sahi	1.023594(-8)	0.483

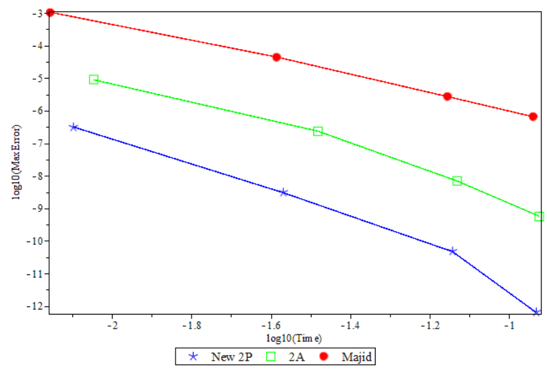

Figure 3: The Efficiency curves for Problem 1 (2-point block method) with step size $h = 0.1, 0.05, 0.025, 0.00125$.

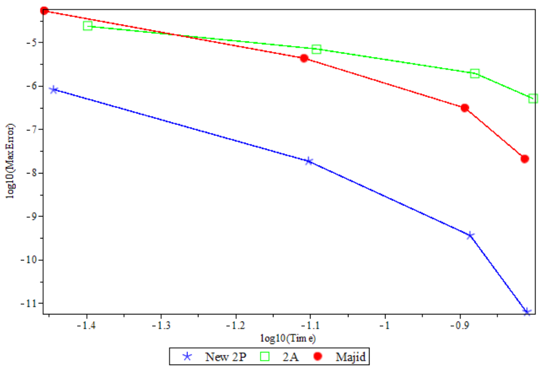

Figure 4: The Efficiency curves for Problem 2 (2-point block method) with step size $h = 0.1, 0.05, 0.025, 0.0125$.

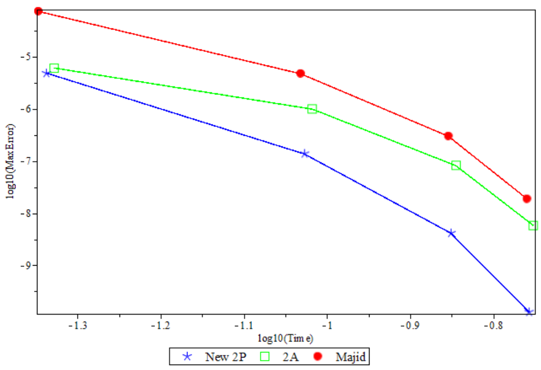

Figure 5: The Efficiency curves for Problem 3 (2-point block method) with step size $h = 0.1, 0.05, 0.025, 0.0125$.

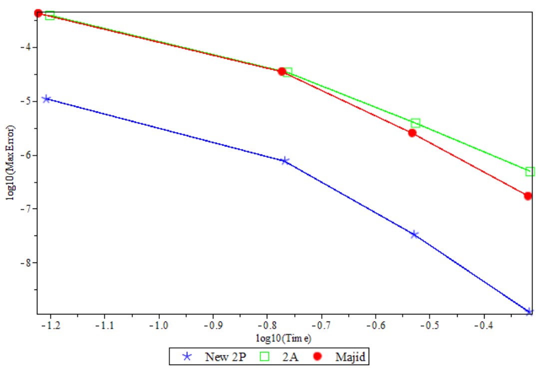

Figure 6: The Efficiency curves for Problem 4 (2-point block method) with step size $h = 0.1, 0.05, 0.025, 0.0125$.

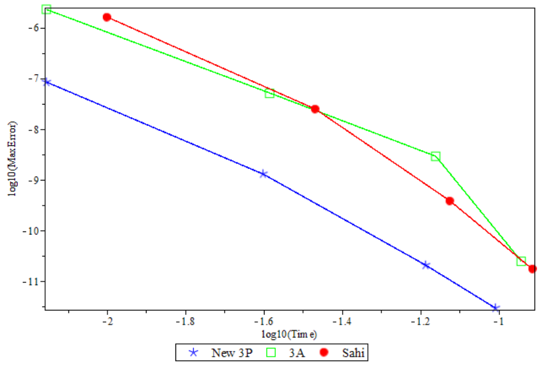

Figure 7: The Efficiency curves for Problem 1 (3-point block method) with step size $h = 0.1, 0.05, 0.025, 0.0125$.

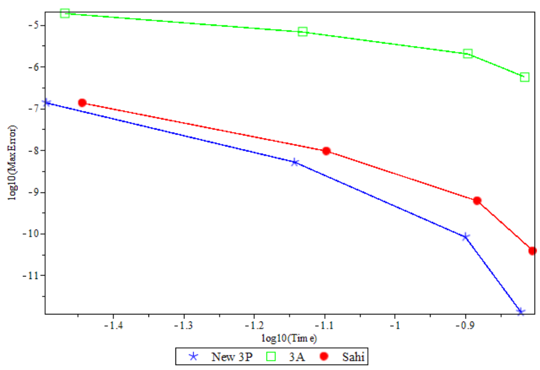

Figure 8: The Efficiency curves for Problem 2 (3-point block method) with step size $h = 0.1, 0.05, 0.025, 0.0125$.

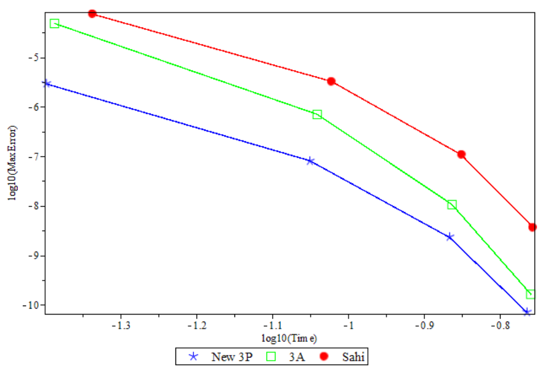

Figure 9: The Efficiency curves for Problem 3 (3-point block method) with step size $h = 0.1, 0.05, 0.025, 0.0125$.

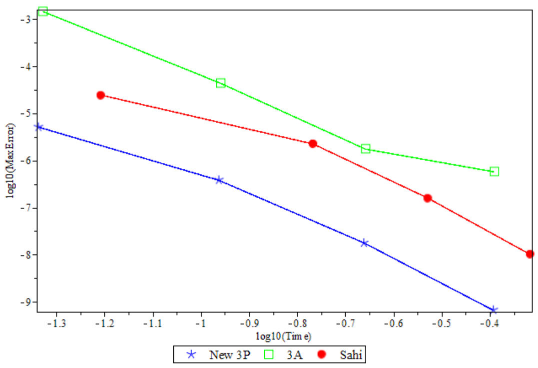

Figure 10: The Efficiency curves for Problem 4 (3-point block method) with step size $h = 0.1, 0.05, 0.025, 0.0125$.

7 Results and Discussion

In this paper, we presented the derivation of two and three point second derivatives implicit block methods for solving first-order ODEs. The numerical results are tabulated in Tables 1-8 and are plotted in Figures 3-10. Those figures showed the efficiency curves, where the common logarithm of the maximum global errors were plotted versus the computational time. Figures 3-6 revealed that 2P (2- point order 5 second derivative block method derived in this paper) is the most efficient compared to 2A (2-point one-step order 5 implicit third derivative block multistep method) and Majid (2-point implicit block method). Tables 1-4 showed that the new 2P method has less maximum error compared with 2A and Majid methods. Figures 7-10 showed that the new 3P (3- point order 6 second derivative block method derived in this paper) is the most efficient compared to 3A (order 6, 3-point implicit third derivative block multistep method) and Sahi (Simpson's type order 6 second derivative block method). Tables 5-8 showed that the new 3P method has less maximum error and less computational time compared to 3A and Sahi's methods.

Numerical results revealed that the new 2P and 3P methods are more efficient as compared to the existing methods and they also illustrated that the new second derivative block methods are more accurate and competent for solving first order ODEs.

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