

**Study of Some Applications in Univalent Function**

**Theory**

*A Thesis*

*Submitted to Council of the College of Education for Pure Sciences*

*University of Anbar*

*As a Partial Fulfillment of The Requirements for The*

*Degree of Master in Mathematics*

*By*

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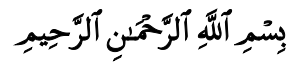
B.Sc. in *Mathematics –College of Education for Pure Sciences*

*University of Anbar -2017*

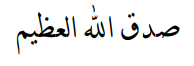
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**2020 A.D 1442 A.H**







**سورة العلق (5-1)**

الإهداء

**إلى من بلغ الرسالة و أدى الأمانة....و نصح الأمه.... إلى نبي الرحمة ونور العالمين ......**

**((نبينا محمد صلى ألله عليه وسلم ))**

**إلى رجل الكفاح ... إلى من زرع القيم و المبادئ الاسلامية ... إلى من افنى زهره شبابه في تربيه ابنائه ....**

**((والدي العزيز ))**

**إلى القلب النابض .... إلى رمز الحنان والحب والتضحية..... إلى من كانت دعواتها الصادقة سر نجاحي .......**

**(( والدتي الغالية ))**

**كما اهدي ثمرة جهدي إلى(( الاستاﺫ الدكتور عبد الرحمن سلمان جمعه)) لما منحهٌ لي من وقت وجهد**

**وتوجيهات ومعلومات قيمه....وازاح عني ضباب الجهل بعلمه .....**

**إلى القلوب الطاهرة والنفوس البريئة ... إلى من استمد منهم عزوتي واصراري... إلى رياحين حياتي ...**

**وبهم تكتمل فرحتي ...**

**((أخواني وأخواتي ))**

**أليهم جميعا اهدي ثمره هذا الجهد المتواضع ....**

***طيبه***

***Acknowledgements***

First and foremost , thanks to **Allah**  for his help and for giving me the ability to complete

this thesis .I would like to express my deep gratitude to my supervisor,

Prof .Dr. **Abdul Rahman Salman Juma** for his valuable suggestions ,extensive discussion ,

encouragement and continuous guidance.

I also extend my deepest thanks and respect to the College of Education for Pure Sciences

at University of Anbar ,represented by its dean and all its distinguished professors.

I extend my sincere thanks to all my distinguished professors in the Department of

Mathematics of the College of Education for Pure Sciences at the University of Anbar

I wish to express my deep gratitude to my family for their patience during my study.

Finally, I would also like to extend my thanks to everyone who helped me to complete my

study .

**Teba**

***Supervisor Certification***

*I certify that this dissertation entitled* ***" Study of Some Applications in Univalent Function Theory "*** *Submitted by the student* ***" Teba Rzaij Sabah "******,****was prepared under my supervision at the University of Anbar , College of Education for Pure Sciences / Department of Mathematics , as a partial fulfillment of the requirements for the degree of master in Mathematics .*

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***List of Symbols***

|  |  |
| --- | --- |
| **Meaning** | **Symbol** |
| Complex plane |  |
| \ |  |
| Set of all real numbers |  |
| Set of natural numbers. |  |
|  |  |
| Open unit disk |  |
| Punctured Open unit disk |  |
| Class of analytic functions in the open unit disk |  |
| Class of normalized analytic univalent functions in the open unit disk |  |
| Subclass of consisting of univalent function in |  |
| Class of the starlik functions in |  |
| Class of the starlik functions of order |  |
| Class of normalized convex functions in |  |

***List of Symbols***

|  |  |
| --- | --- |
| **Meaning** | **Symbol** |
| Class of normalized convex functions of order |  |
| Class of close-to- convex univalent function |  |
| Class of close-to-convex functions of order |  |
| Real part of a complex number |  |
| Hadamard product (or Convolution) of functions |  |
| Class of meromorphic univalent functions in | ∑ |
| Subclass of ∑ |  |
| The - neighborhood of a function |  |
| Class of Harmonic univalent functions |  |
| Radius of starlikess |  |
| Radius of convexity |  |
| Radius of close-to-convex |  |

**Abstract.**

The main aim of this thesis is to study some of applications in univalent function

theory and obtain expected results . It includes studying classof normalized

analytic functions in the open unit disk and class ∑ of

meromorphic functions in the unit punctured disk . We

introduced and studied the subclasses of univalent function with

negative coefficients and of univalent meromorphic functions defined by

new linear operator and differential operator respectively.We

obtain some geometric properties from classes and such as

coefficient inequality , growth theorem and distortion theorem , we get radii of

starlikeness , convexity and close-to-convexity of class the concept of

convolution investigate and neighborhoods of the elements for the are

obtained . Also, we studied of some results for differential subordination by generalized

differential operator . Also, we get the concept of Quasi-Subordination

and the estimates of the Fekete- Szego coefficient function for functions

belonging to classes and . By making use of the general liner

operator on the harmonic univalent function we give

some properties like coefficient condition, distortion bounds ,extreme points,

convolution and convex combinations. Also ,by utilizing generalization of the

Srivastava- Attiya operator on two classes and

for harmonic univalent functions we get sufficient coefficient conditions

of the classes ,in addition to that ,we obtain properties like distortion bounds and

extremepoints, so this proves that the classes studied in this section one is closed

under convolution and convex combination.

**Introducion**

The foundation stone for the geometric function theory is the theory of univalent

and multivalent functions and as yet plays a significant role contributes to new ideas

and results . The study for univalent function is of the most ravishing as pacts for the

theory of analytical functions of a complex variable . Several mathematicians have

contributed a lot in the present study and investigate in theory of univalent , Koebe

the first to study and investigaeon the theory of univalent in (1907) , Gronwall's proof

of the area theorem in (1914) and bieberbach's assessment for the second coefficient

of normalized univalent function in (1916) , also they introduce and study classes of

univalent functions defined in and the family of univalent function

introduced by this study . Alexander,Nevanlinna and others in traduced the starlike

function ,Gronwall , Loewner and others also studied the convex function.

This thesis is divided to four chapters .Outlined as follows :

**Chapter One** includes the many definitions, basic lemmas and standard theorems,

which we needed over the course of results discussion .

**In chapter Two** we have studied a linear operator on univalent functions

with negative coefficients by defining the class satisfying the inequality

for

We get many interesting results ,such as coefficient inequality ,growth and distortion

theorem. Hadmard product ,extreme point, closure theorem and radii of starlikeness, convexity

and close-to-convexity of functions belonging to our subclass.

Also ,we have introduced class belong to ∑ consisting function

Which are meromorphic univalent function in and satisfying the following

When the differential operator is defined .Some geometric properties such as the

coefficient estimates of the class growth and distortion theorems have be non

studied ,radii of starlikeness and convexity, convolution and neighborhoods of the

elements for the class .

**Chapter Three ,** this chapter is totally dedicated to the study of differential

subordination and Includes two sections ,section one studied some results for

differential subordination for analytic univalent function in by the generalized

differential operator .

Section two , included studied some subclasses of analytic univalent functions in

. obtain the concept of Quasi-Subordination and also get the estimates

of the Fekete-Szego coefficient function for functions belonging to

these subclasses .

**Chapter Four ,** This chapter studies the harmonic functions and includes two

sections. Section one, includes two classes and of

harmonic univalent function defined by generalization of the Srivastava – Attiya

operator and we get sufficient coefficient to condition of the two classes

,also we get some geometric properties distortion bounds ,extreme points, convolution

and convex combination from the study this classes c and

.In section two , defined the class of harmonic univalent

functions and satisfying the following

and making use of the general linear operator, we have obtain some geometric properties .

**CHAPTER ONE**

**Some Basic Definitions and Standard Theorems*.***

* 1. **Introduction**

The aim of this chapter is to present the definitions ,lemmas and the theorems

required during the course of investigation and review some general principles of

complex analysis emphasized in the univalent function theorems and the results of

these theorems and definitions mentioned in this chapter can be obtained from

standard texts such as Hayman , Darun , Goodman and other references in

analytic function theory and this chapter consists of two sections , section one reviews

the basic definitions and sections two several fundamental theorems and lemmas

to prove our principal results .

**1.2 Basic Definitions**

**Definition 1.2.1 :**

The function of complex variable is analytic at a point if its derivative exists not only at , but each point in some neighborhoods of . It is analytic in unit disk if it is analytic at every point in ,where .Also it is said to be analytic in punctured unit disk if it is analytic at every point in ,where  **.**

**Example 1.2.2 :**

The function is analytic wherever ,and since

The two functions

the partial derivatives are continuous in all the value

Then

Hence , the function is analytic wherever .

**Definition 1.2.3 :**

The analytic function in the open unit disk is said to be univalent if it does not take the same value twice that if for all .

**Example 1.2.4 :**

The function is univalent in .

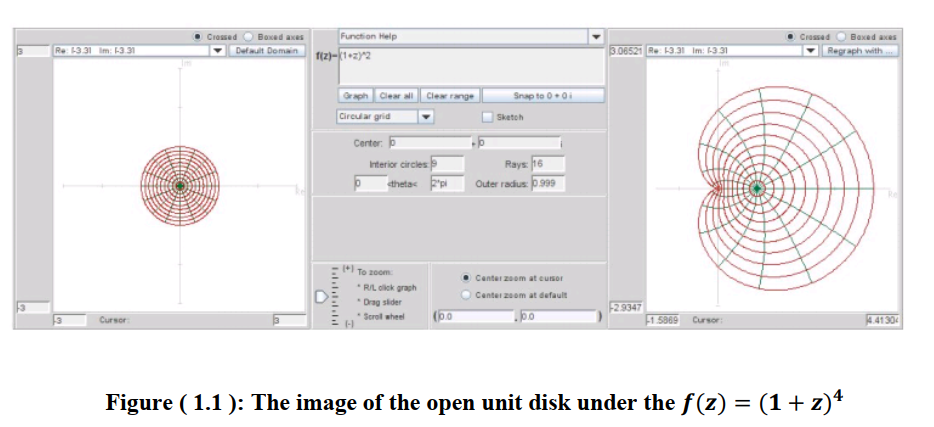
Let and suppose

Then

Since we know that

Hence

But , the function is not univalent in .



**Definition 1.2.5 :**

A set denotes the class of analytic functions in can be the subclass of

consisting of functions of the form

with and .

**Definition 1.2.6 :**

A set denotes the subclass of in the open unit disk where and of the following form

**Definition 1.2.7 :**

We say that is normalized if satisfies conditions

**Definition 1.2.8 :**

A starlike function is one which maps the unit disk

conformally onto starlike domain with respect to the origin and the function is to be starlike of order if

We use for the class of starlike function of order and for the class of

starlik functions , .

**Definition 1.2.9 :**

The convex function is a conformal mapping of the unit disk onto a convex domain and the function is said to be convex of order if

We shall use of the class of convex function for order and of the class of convex functions , .

**Definition 1.2.10 :**

If ,then is called close –to convex of order in the unit disk ,if there exists a convex function such as

we is said to be of the class for close-to convex functions of order and of the class of close-to- convex functions ,

We note that

**Definition 1.2.11 :**

Radius of starlikeness of function is the largest of which is starlike

in .

**Definition 1.2.12**

Radius of convexity of function is the largest of which is convex in .

**Definition 1.2.13**

Radius of close-to- convexity of function is the largest of which is close-to- convex in .

**Definition 1.2.14 :**

If be two analytic function in ,then is called subordinate to ,

written as ,if there exists a function analytic in with and

and such as . If is univalent ,then if and only if

.

**Definition 1.2.15:**

A function with continuous second partial derivatives is said to be harmonic in a domain everywhere in

**Definition 1.2.16 :**

A continuous function is a complex -valued harmonic function in a domain if both are real harmonic in . we can express

where are analytic functions in and we call are analytic part and co-analytic part of ,respectively.

**Example 1.2.17 :**

To show the image of under the harmonic function

enter this function in complex tool in the form : (see the following fig.)

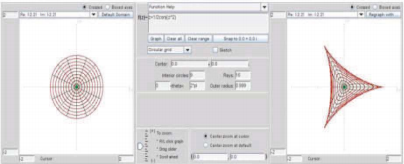


Figure (1.1) : Image of under the harmonic function .

Note that the harmonic function can be written in the form

Thus for the function above it can be written as

In complex tool you can also enter the harmonic function in form . where this function must be in the form

**Definition 1.2.18 :**

The harmonic function is locally injective and orientation –preserving in

if and only if

**Definition 1.2.19 :**

Let are two analytic functions belonging to the class given by

Then convolution denoted by is defined by

**Definition 1.2.20 :**

Let Y be a topological space over the field of and let the set . Then the point is said to be an extreme point of if it has no representation of the from

as a proper combination of two distinct points

**Definition 1.2.21 :**

Let Y e a topological vector space over the field of and let the set . Then the closed convex hull of defined by is the smallest convex set containing

**Definition 1.2.22**:

Let ∑ denote the class of analytic functions in punctured unit disk

of the form

Also , let be defined by the subclass of ∑ consisting of analytic and univalent functions in

**Definition 1.2.23 :**

Let of the form (1.8) belonging to ∑ is called meromorphically starlike of order

if

we shall denote the class of such all functions

**Definition 1.2.24**

Let of the form (1.8) belonging to ∑ is called meromorphically convex of order

if

we shall denote the class of such all functions

**Definition 1.2.2**

**Definition 1.2.26**

**Definition 1.2.27**

If we put , we have Salagean operator

**Definition 1.2.28**

Let be of the form(1.8)andbe real numbers with ,then the

analogue of the differential operator is defined as follows :

If ∑ is given by (1.8), then we have

(1.14)

where

**Definition 1.2.29**

We define the Neighborhood of a function by

For the identity function we get

**Definition 1.2.30**:

For  , the generalization of the Srivastava – Attiya Operator

is defined by

where

,

and

**Definition 1.2.31**:

is general linear operator

where

(

**1.3 Essential Results**

In this section , we review some famous such as lemmas and standard theorems result utilized in the thesis

**Lemma 1.3.1:**

If if and only if

**Lemma 1.3.2:**

With the same condition such as is given by to lemma (1.3.1), if and only if

(1.17)

**Lemma 1.3.3:**

Let the function be analytic and convex univalent in with =1 .Suppose that

the function given by

be analytic in .

If , ,

then

and is the best dominant.

**Lemma 1.3.4:**

Let be given by

If

then is convex .

**Lemma 1.3.5 (Schwarz Lemma)**

If be analytic in the unit disk with ,then

and in . the rigorous inequality holds in both estimates unless is a rotation of the disk ,

**Lemma 1.3.6:**

then

**Lemma 1.3.7:**

then

**Theorem 1.3.8**:

A function be non-constant in If attains its maximum value on at a point

**Theorem 1.3.9**: **(Growth Theorem)**

For every

Equality take place if and only if is proper rotation of the koebe function.

**Theorem 1.3.10**: **(Distortion Theorem)**

For every

Equality take place if and only if is proper rotation of the koebe function.

**Theorem 1.3.11**: **(Alexander's Theorem)**

If be an analytic function in with , then if and only if

.

**Theorem 1.3.12**:  **(Bieberbach Conjecture)**

For every coefficient of satisfy for the rigorous

inequality holds for all unless is the Koebe function or one of its rotation.

**Theorem 1.3.13**: **(Littlewood's Theorem )**

For constant c ,for every the coefficient satisfy for .

**CHAPTER TWO**

**Some Properties of Univalent Functions**

**2.1 Introduction**

Chapter two is dedicated to the study of univalent function with negative

coefficients and meromorphic univalent function in terms of different subclasses with

some operators. This chapter ,includes two sections.

Section one , contains the subclass of univalent function with negative

coefficients in the open unit disk and by using new linear

operator ,we obtainsome geometric properties such as coefficient inequality

,growth and distortion theorems. Hadamard product ,extreme point, closure theorem and radii

of starlikeness, convexity and close-to-convexity of functions .

Section two , deals with study of the class meromorphic univalent function

by making use of differential operator with study some geometric properties, such as

coefficients inequality , growth theorem and distortion theorem, radii of starlikeness

and convexity and the concept of convolution investigate and Neighborhoods of the

elements of class are obtained .

**2.2Some Properties of Univalent Function with Negative Coefficient**

**Defined by a Linear Operator in the Open Unit Disk**

therefore ,

**Definition 2.2.1:**

for

and studied by G.H.Esa and Darus

We characterize class

**Theorem 2.2 .1:**

the outcome (2.7)is sharp of the function

**Proof:** Suppose that (2.7) holds true and .Then

Since by maximum modulus principle.

Conversely ,

Hence

We take the values of

**Corollary 2.2.1:**

Let

**Remark 2.2 .1:**

and equality holds for

A growth and distortion property for is offered as follows.

**Theorem 2.2.2:**

The result is sharp for function

**Proof:**

is non decreasing and positive for

That is equivalent to

Using (2.2) and (2.11) ,we obtain

Similarly,

From (2.12)and(2.13),we have

**Theorem 2. 2 .3 :**

we obtain

The result is sharp for function given by

**Proof:**

So,

From (2.14)and(2.15),we obtain

In the following theorem, we obtain the Hadamard product (or convolution) result for

**Theorem 2 .2 .4 :**

for

where

**Proof:**

we must find the smallest number

By Cauchy Schwarz inequality ,we have

Thus it is sufficient to prove that

So ,

From (2.19) ,we obtain

It is sufficient to prove that

Next ,we obtain the radius of starlikeness , convexity and close-to-convexity for class by the following theorems.

**Theorem 2.2.5 :**

The estimate is sharp from

**Proof:** From definition 1.2.8 ,we have

to prove that

From corollary(2.2 .1),we have

Hence

The proof is complete.

**Theorem 2 .2. 6 :**

the estimate is sharp from

**Proof:**  From definition 1.2.9 ,we have

equivalent ,

Which is equal to ,

From corollary (2.2 .1),we obtain

Hence

The proof is complete.

**Theorem 2 .2.7 :**

**Proof: :** From definition 1.2.10 ,we have

which is equivalent to

which is simplified to

Hence

The proof is complete.

Now, in the following theorem , we obtain extreme points for class

**Theorem 2 .2.8 :**

**Proof:** Assume that

Conversely , assume that

Put ,

and

it is enough to obtain

is the result of the theorem .

Now , we ought to show that the following closure theorems belongs to

**Theorem 2 .2.9 :**

where,

**Proof:** We obtain

such that

Hence ,

in (2.30) ,

Now,

**Theorem 2 .2.10 :**

**Proof:**

Using (2.34)and(2.35),we have

Hence ,we must show

where

Which simplifies

The proof is complete.

**2.3 Certain Subclasses of Meromorphic Univalent Function**

**Involving Differential Operator**

Denote by ∑ the class of functions of the form

which are analytic and univalent in the punctured open unit disk

.

Let be function of the form(2.38)andbe real numbers with .

Then the analogue of the differential operator given in [57] is defined as follows :

If ∑ is given by (2.38), then we have

where

Note that for =0 and =1, we obtain the differential operator introduced by [ 15 ].

In this section, we define the following subclasses of meromorphic function by utilizing the operator

**Definition 2.3.1:**

A function of the form (2.38) is in the class if it satisfies the following inequality

This class was studied by many researchers ( for example Juma ,Juma and Zirar

and Mahmoud .

We derive the coefficient inequality for class in the next theorem.

**Theorem 2.3.1:**

The function given by (2.38) is in the class , if and only if

The result is sharp for function given by

**Proof:** Suppose (2 .41) holds , and if Then by (2.40) , we have

Thus,

.

Let . Then we get

Since ,we have

then

Thus by the maximam modulus theorem, we get .

Conversely, if of the form (2.38) is in class then by (2.40) we get

Thus,

Since for all, we have

Now, we take the worth of on the real axis so that the worth is real , then the denominate of (2.43)and through positive real value , we have the inequality (2.41) .

The result is sharp for ,defined by

**Corollary 2.3. 1:**

Let.Then

where and

We derive some properties distortion and growth of in the next theorems.

**Theorem 2.3.2:**

If of the form (2.38), then for ,we get

with equality for

**Proof :**By hypothess , we get from Theorem 2.3.1

that is,

then

Thus ,for , we have

So,

Thus, the proof is completed.

**Theorem 2.3.3:**

If of the form (2.38), then for we get

with equality for

**Proof:** Utilizing Theorem 2.3.1, we get

then

Thus , for

Also,

Now, the radius of starlikeness and convexity for is given by the following theorems:

**Theorem2.3. 4:**

If of the form (2.38),then is meromorphically starlike of

order in the disk , wehere

The result is sharp for is given by the following

**Proof:** It is sufficient to prove that

Thus,

by Theorem 2.3.1, we have

Therefore,

**Theorem 2.3.5:**

If of the form (2.38), then is meromorphically convex of order

in ,where

**Proof:** It is sufficient to prove that

if

that is ,if

**Theorem 2.3.6:**

If and ,then for

where

**Proof :** Hence then by using Theorem 2.3.1, we get

and,

We need to find the largest such that

By Cauchy – Schwarz inequality , we have

To proof Theorem 2.3.6, it is sufficient to prove that

which is equivalent to

From (2.48) , we get

We must prove that

such that

which gives

**Theorem 2.3. 7:**

If functions defined by

is in ,then function defined by

is in the class ,where

**Proof:** Hence . Then utilizing Theorem 2.3.1,we get

since

and

Therefore, we need to find the largest such that

**Theorem 2.3.8:**

Let , and with ,is

in the class . Then .

**Proof :** From Theorem 2.3.1, we get

Hence

Thus ,

**Corollary 2.3. 2:**

Let , andfor

is in the class .Then

In this section ,the concept of neighborhood of analytic function was first introduced by

Goodman and then generalized by Ruscheweyh.Lin and Srivastava,

investigated this concept for the elements of several famous subclass of analytic

function .We define the Neighborhood of a function by

For the identity function we get

**Definition 2.3.2:**

The function is said to be in the class if there exists a function

such that

**Theorem 2.3.9:**

Ifand

then

**Proof:** Assume that Then we have from (2.49) that

which suggests coefficient inequality

Hence we get from Corollary 2.3.1

From (2.50),we get

Since ,by definition 2.3.2 , for given by (2.51).

**CHAPTER THREE**

**Some Results of Differential Subordination on Univalent Function**

**3.1 Introduction**

Chapter three is totally dedicated to the study of differential subordination and

subclasses. A analytic univalent function associated with quasi- subordination are

defined and the Fekete – Szego coefficient functional for function

belonging to these subclasses are derived . The differential subordination in field of

complex plane is the generalization of inequality on the real line . The differential

subordination have been utilized in variant fields like differential equations

,meromorphic functions and harmonic etc . Differential implications were presented in

1935 by G.M. Goluzin and R. m .Robinson and also there a lot of literature on

differential subordination is available such as Protter and Weinberger ,Walter

and S. S .Miller and P.T. Mocanu etc.

This chapter includes two sections .Section one, includes some results for differential

subordination for analytic univalent function in open unit disk by the generalized

differential operator .

Section two , introduces and investigates some subclasses of analytic univalent function

defined in obtain the concept of Quasi-Subordination, the estimates of the Fekete-Szego coefficient function for functions belonging to these

subclasses are derived.

**3.2 Subordination Results Involving a Generalized**

**Differential Operator**

Let denote the class of analytic function in the unit disk and

. We can let

with

Letdenote the class of function in of the form

**Definition 3.2.1**

Let function be in the class .For

the generalized differential operator is defined by

Some of the special cases of the operator defined by (3.1)can be seen in

To show our main results ,we need Lemma 3.2.1 , 1.3.3 and 1.3.4.

**Lemma 3.2.1:**

If and the operator

be defined by (3.1) ,then

**Proof :**  From Lemma 3.2.1 ,we have

which establishes the identity (3.2).

Next , we assume throughout this paper that

**Theorem 3.2.1 :**

Let with which checks the inequality

If and checks the differential subordination

then

where

function is convex and is the best dominant.

**Proof**: From Lemma 3.2.1 , we get

differentiating (3.6) with regard to ,we get

If we know function

then (3.7) becomes

Utilize (3.9)and (3.4) we get

where

by utilizing Lemma 1.3.3 for we get

is the best dominant.

By using Lemma 1.3.4 of the function given by (3.5) and function with the property in (3.3) of , we notice that function is convex .

As a result of Theorem 3.2.1 ,we obtain the following corollary . Put

, in Theorem 3.2.1 ,we get the following corollary.

**Corollary 3.2. 1 :**

If function which checks the inequality

Let the function and checks the differential subordination

where

The function is convex and is the best dominant .

**Theorem 3.2. 2:**

Function with ,which checks the inequality

Let the function and the checks differential subordination

Then

where

**Proof:** If we know function

and get

Then we get from (3.11)

where

By utilizing Lemma 1.3.3 of

where

This is the best dominant .

By using Lemma 1.3.4 of function of the from (3.13) and function with the

property in (3.11) of , we notice that function is convex .

As a result for Theorem 3.2.2 and put in

Theorem 3.2.2 , we obtain the following corollary

**Corollary 3.2.2:**

The function which checks the in equality

Let and checks the differential subordination

where

The function is convex and is the best dominant .

**3.3 Coefficient Estimates for Subclass of Analytic Function**

**Using Quasi- Subordination**

.

then

**Definition 3.3.1** :

**Definition 3.3.2 :**

To prove our results , we need Lemmas 1.3.6 and 1.3.7.

Now.

**Theorem 3.3.1:**

**Proof :**

Utilize(3.26)and(3.27)in(3.25),we obtain

Applying Lemmas 1.3.6 and 1.3.7 with (3.29),we get

**Corollary 3.3.1:**

Following, if we utilize the Schwarz function of the following form

Next, we have the following results.

**Theorem 3.3.2:**

**Proof:**

we have

Utilize (3.26)and (3.31)in (3.30),we obtain

Thus (3.32) give

Also ,

Utilize this fact and

and applying Lemma1.3.6,we get

That is the required result.Further setting

The proof is complete.

**Theorem 3.3.3:**

Then the following inequalities hold

**Proof:** The results follow by taking

Thus, the proof is complete.

**Corollary 3.3.2 :**

**Theorem 3.3.4:**

**Proof:**

A computation shows that

Utilize by (3.35),(3.36)and (3.37)in(3.34) and equating coefficients in both sides ,we obtain

From

Further ,

Again applying

Applying Lemma 1.3.6 to

Results into

Therefore ,we have

The proof is complete.

**Corollary 3.3.3 :**

**Theorem 3.3.5 :**

and for some

**Proof:** The required result obtained by setting in the proof of Theorem 3.3.4 .

**CHAPTER FOUR**

**Some Results of Harmonic Univalent Function**

**4.1 Introduction**

Last chapter is totally dedicated to the study for harmonic univalent functions .

Many of mathematic participated, such as O.P . Ahuja and J.M. Jahangiri in the study

of harmonic function of the form ,Were ( are analytic functions in the unit disk) and have contributed a lot in developed .As well . Ahuja , Jahangiri and

Silverman they have share about the contraction of harmonic univalent mappings

.the researchers also studied a number of properties in their research about the

harmonic univalent function such as distortion theorem , extreme points , convolution

condition and convex combination ,these researchers have get good results . Also the

contribution by Jahangiri and Silverman in the study of thorough class of complex

valued harmonic univalent functions with differing arguments . T. Rosy , B.A. Stephen

,K.G. Subramanian and Jahangiri have also endowed in the study of harmonic

functions.This chapter includes two sections .

Section one, consists of study two new classes and

.We get sufficient

coefficient to Conditions of the class of harmonic univalent functions and also

obtain properties like distortion bounds and extreme points ,so prove that the class

studied in this section is closed under convolution and convex combination .

In section, two we introduce the class of functions harmonic univalent

functions and making use of the general linear operator obtain to

some properties like coefficient condition ,distortion bounds ,extreme points,

convolution and convex combinations .

**4.2 On a Classes of Harmonic Univalent Function Defined by**

**Generalization of The Srivastava – Attiya Operator**

A complex valued continuous function defined in a simply connected

convex domain is said to be harmonic in if both are real valued

harmonic function in.We can write , where are analytic in .We

call the analytic part and the Co-analytic part of .A substantial and sufficient

condition of to be locally univalent and orientation-preserving in is that

(see Clunie and Sheil –Small ) .Let be class of

function that are harmonic univalent and sense-preserving in the open unit

disk where are analytic in and is normalized by

.Then for ,we may express the analytic

functionsas

, , (4.1)

several authors (see had studied different

subclasses of the class of harmonic function.For given by (4.1) and

is generalization of the Srivastava – Attiya Operator for and given

by

where

,

and

Now ,introduce and study two new classes and

of harmonic univalent function in

**Definition 4.2.1:**

Let be a harmonic function where

, , (4.3)

Function , if it satisfies

where is defined by

**Definition 4.2. 2:**

Let be a harmonic function , where

, , (4.5)

Then , if it satisfies

where is defined by

**Theorem 4.2.1:**

Let where are given by (4.1) . If

where

Then is harmonic univalent function and sense-preserving in and .

**Proof:** For ,we have by (4.7)

Consequently, is univalent in . It is observed that is sense-preserving in , since by using (4.7) ,

Now , itis enough to prove that using the fact if and only if

Substituting from (4.3) and using (4.7) , it follows that

The harmonic function

where

Shows that the coefficient bound given by (4.7) is sharp

**Theorem 4.2.2:**

If ,where are offered by (4.5) , then if and only if

where

**Proof:** Assume that . Then from (4.4) with given by (4.5) ,

it can be found that

Now choose to be real and let ,then

therefore

which is the assertion (4.9) of Theorem 4.2.2 .

Conversely , assume that (4.9) holds true. Then from (4.4) with given by (4.5)

and on using (4.9) for ,it can be found that

This shows that

Next, we will get distortion bounds for .

**Theorem 4.2.3:**

Let .Then for ,

and

These bounds are sharp.

**Proof:** Let where are defined as in (4.5) .Then

The proof of upper bounds of is similar.

Hence ,these bounds are sharp and equalities occur if

and

**Corollary 4.2.1:**

Let .Then

Now ,we determine a representation theorem for

**Theorem 4.2. 4:**

Let where are defined as in (4.5) .Then if and only if

where

and

In particular, the extreme points of are

**Proof:** Let

where ,and

Hence

and so .

Conversely ,assume that . Let

and .

Then note that by Theorem 4.2.2 ,

and

Consequently , as required

Now , we prove that the class is invariant under convolution and convex

Combination for it is member .

**Definition 4.2. 3:**

Then the convolution for are given by

This can be written as

**Theorem 4.2.5:**

If and and , then

.

**Proof:** The convolution of is defined by (4.10) .It is enough to show that the

coefficients of satisfy the condition given in Theorem 4.2.2 .

For

Now of the convolution , it follows that

Yielding the desired result .

**Theorem 4.2.6:**

The family is closed under convex combination .

**Proof:**  Let , where

Then for the convex combination of may be written as

Utilizing (4.9) ,one can see that

Therefore

The proof of the theorem is completed.

**4.3 On Subclass of Harmonic Univalent Functions Involving**

**general linear Operator**

Let denote the class for all harmonic the functions that are univalent and

orientation- preserving in for which is normalized by

with denotes partial derivative of and we

call the analytic part and the co-analytic part of .

We may express the analytic functions

Now , we introduce the class of functions satisfying

(4.12)

where are real (see A. R. S. Juma ), such that

and is general linear operator of and given by

Where

(

Also ,

Moreover , let is the subfamily for consist of functions ,which are

harmonic and are of the form

Let be the subclass for and (4.12) holds true.

Many researchers such as J.M. Jahangiri ,S. M. Khairnar and

G. Murugusundaramoorthy and K. Vijaya. Studied the harmonic univalent defined

by different operators.

In this section, we necessitate the following theorem due to J. M. Jahangiri .

**Theorem 4.3. 1**

Let where

.

Further ,Let

Where

Leads to be the harmonic univalent function in U and

is the subclass for consisting of harmonic convex function of order

**Theorem 4.3.2:**

Ifwith

where

and

Then is harmonic univalent in *and*.

**Proof:** From Theorem 4.3.1, we get is harmonic , since

depend on the fact that by if and only if

To verify that it enough to show that (4.12) holds. Let

We must prove that

Substituting equation(4.16) and equation(4.17)in equation(4.18), we have

Thus .

For sharpness we take into account the harmonic function by the form

where

.

**Theorem 4.3. 3:**

If the functions of the form (4.14) ,then it is necessary and enough

condition of the function belong to the class is that

where

.

**Proof:** Hence , then the substantial part follows from

Theorem 4.3.2.

For the enough part, we prove that (4.20) does not hold good implies that

Now ,a function if and only if

Thus,

The last inequality must hold for all

Taking the worth of on the positive real axis where we must get

)

We note that if the condition (4.20) does not hold then the numerator in (4.21) where

goes to 1 is negative . this contradiction for and the proof is complete

**Definition 4.3. 1 :**

If be a topological vector space over the field of complex numbers ,and if be a subset of then the closed convex hull of denoted by is the smallest convex set containing.

**Theorem 4.3.4 :**

if and only if ,

where,

In specific the extreme points of

**Proof:** Let be written as (4.22) , Then we have

Thus ,

Also

Conversely , suppose that . Imposing

and

where

=1 ,

then we get by simple calculation

**Theorem 4.3.5:**

If the function ,then

where

and

where

**Proof:** Let Taking the absolute value of ,we obtain

Also ,

**Definition 4.3.2:**

Let

and

We define the convolution of and as

**Theorem 4.3.6:**

If the functions and ,then for we have

**Proof:** Hence and ,then both of them satisfy

(4.20) and hencewe will write

The right hand side of the last inequality is bounded above by 1, then

**Theorem 4.3.7:**

The classis closed beneath convex combinations.

**Proof:** Assume (j=1,2,3,…….) are defined by

Utilizing (4.20) we get

For the convex combination of may be written as

Therefore ,

+

Then

**Conclusions**

The obtain a new concepts for some analytical univalent function by using linear

operator in the open unit disc and some geometric properties such as coefficient

inequality ,growth and distortion theorem. Hadmard product,extreme point, closure

theorem and radii of starlikeness, convexity and close-to-convexity of functions

belonging to our subclass.. Also obtained some important results of meromorphic

analytical univalent function by using the differential operator such as the coefficient

estimates of the class growth and distortion theorems ,radii of starlikeness

and convexity, convolution and neighborhoods of the elements for the class

.Moreover, That was obtained some important results differential

subordination of analytical univalent function by usingthe generalized differential

operator . And obtainobtain the concept of Quasi-Subordination and also get the

estimates of the Fekete-Szego coefficient function for functions belonging to

these subclasses . Also we obtain some properties like coefficient condition ,distortion

bounds ,extreme points, convolution and convex combinations by making use of the

general linear operator on the class of harmonic univalent functions in

And, we have obtain some geometric properties . by utilizing

generalization of the Srivastava- Attiya operator on two classes for harmonic univalent

function in

**Future works**

Based on the results obtained in study of some applications in univalent function

theory ,which are represented in some engineering properties in the open disk ,such as

coefficient inequality , growth theorem and distortion theorem, radii of starlikeness and

convexity etc using linear operation , I suggest studying these properties ,with the

problems of the following

1-Study other properties of univalent function associated with generalized multiplier

transformations

2-Define new subclass of harmonic univalent functions involving new operator

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**المستخلص**

الهدف الرئيسي من الرساله هو دراسة بعض التطبيقات في نظرية الدوال أحادية التكافؤ والحصول على بعض النتائج .

والتي تتضمن دراسة فئة *للدوال التحليلية في قرص الوحدة المفتوحةو الفئة*  ∑

*للدوال الميرومورفية احادية التكافؤ في قرص الوحدة المثقوب .نقدم و ندرس*

*الفئات الفرعية للدالة أحادية التكافؤ دات المعاملات السالبة و*  *للدوال الميرومورفية*

*احادية التكافؤ المحددة بواسطة مؤثرتشغيل خطي والمؤثر التفاضلي على التوالي*

*.والحصول على بعض الخصائص الهندسية من دراسة الفئات و*   *مثل معامل عدم*

*المساواة ,نضريه النمو ونضريه التشويه ونحصل على انصاف الاقطار التشابه بالنجوم , والتحدب و القريب من التحدب*

*للفئه وكذلك تحقيق مفهوم الألتفاف والحصول على الجوارات ل . ايضا ,درسنا بعض*

*نتائج التبعية التفاضلية بواسطة مؤثرالتفاضل ,ايضا نحصل على مفهوم شبه التبعية وتقديرات*

*معامل دالة* Fekete-Szego *في الفئة للدوال التوافقية احاديه التكافؤ نحصل على*

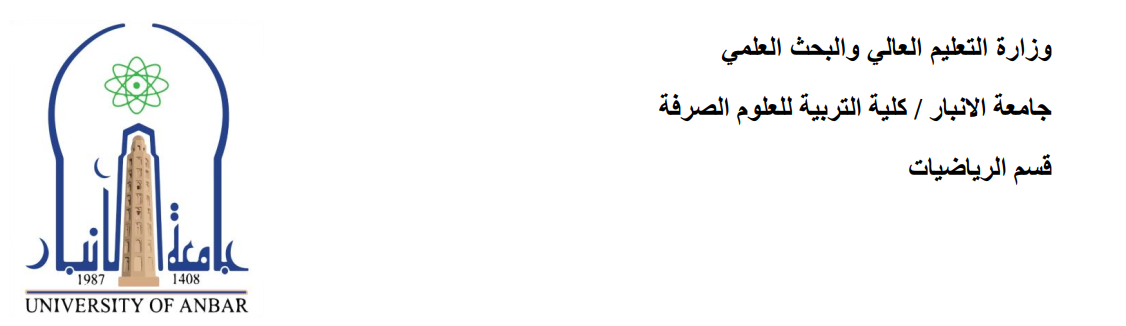
*بعض الخصائص مثل شرط المعامل , حدود التشوية ,النقاط المتطرفة ,الألتفاف و مجموعات محدبة, و من خلال*

*استخدام مؤثر* the Srivastava- Attiya *على فئتين و*

*للدوال التوافقية احادية التكافؤ نحصل على شرط معامل كافية للفئات , بالأضافة الى ذلك ,*

*نحصل على خصائص مثل حدود التشوية و النقاط المتطرفة ,ذلك اثبت ان الفئة التي تمت دراستة في هذا*

*الفصل مغلق تحت الالتفاف والمحدب مزيج.*



**دراسة بعض التطبيقات في نظرية الدوال أحادية التكافؤ**

**رسالة مقدمة**

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**كجزء من متطلبات نيل درجة الماجستير في الرياضيات**

**من قبل**

**طيبة رزيج صباح**

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**جامعة الانبار 2017-**

**بإشراف**

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**م2020 ﮬ1442**