The Direct and Converse Theorems for Best Approximation of Algebraic Polynomial in $L_{p,\alpha}(X)$

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Abstract. The direct and inverse algebraic polynomials approximation theorems in weighted spaces of unbounded functions are proved by using modulus of smoothness. Also, we obtain sharp Jackson (direct) inequality of algebraic approximation of unbounded functions in terms modulus of smoothness . In addition, constructive characterization of modulus of smoothness are considered.

1. Introduction

Approximation problems concerning algebraic polynomials was recently studied in various spaces of algebraic polynomials for example , in the papers [6] , [9] , [12] , [14] , [15] and [17] .

Approximation problems for functions of one variable were also studied by many mathematicians. Some of these results can be found in [10] , [11] , [16] and [18] . For more general doubling weighted direct and converse algebraic approximation problems was investigated in [2] ,[8] and [13]. For a general discussion of weighted polynomial approximation was can refer to the [1] and [7].

Some direct and converse approximation by relational algebraic polynomials of some weighted bounded functions spaces defined on sufficiently modulus of smooth are investigates in [3] and [4] . In the present work we consider the improved direct and inverse approximation theorems by algebraic polynomials by using modulus of smoothness in the weighted space $L_{p,\alpha}(X)$, $1 \leq p < \infty$.

For formulation of the problem we need some further notations properties.

Let $X = [0,1]$ and $L_p(x)$, $1 \leq p < \infty$ be the space of all bounded functions with norm equipped :

$$\|f\|_p = \left(\int_X |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \qquad (1.1)$$

Let α be a weight function defined by :

$\alpha : X \to \mathbb{R}^+$, and $L_{p,\alpha}(x)$ the space of all unbounded functions with norm equipped :

$$\|f\|_{p,\alpha} = \left(\int_X |f(x).\alpha(x)|^p dx \right)^{\frac{1}{p}} < \infty \qquad (1.2)$$

For k = 1 , 2 , ….. the modulus of smoothness of the function $f \in L_{p,\alpha}(X)$ is defined by :

$$\omega_k(f,\delta)_{p,\alpha} = \sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(x)\right\|_{p,\alpha} \quad , \ \delta > 0\right\} \qquad (1.3)$$

where

$$\Delta_h^k f(x) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(x+ih) \qquad (1.4)$$

Let $P_n(n = 0,1, \dots\dots)$ be the set of algebraic polynomial of degree at most less than or equal n and let $E_n(f)_{p,\alpha}$ be the degree of best approximation of $f \in L_{p,\alpha}(X)$ by the polynomial p_n in P_n given by :

$$E_n(f)_{p,\alpha} = \inf_{p_n \in P_n}\|f - p_n\|_{p,\alpha} \qquad (1.5)$$

There are many results on approximation of functions belong to $L_{p,\alpha}(X)$ spaces , $1 \le p < \infty$. Especially , the classical Jackson theorem (direct theorem) .

$$E_n(f)_{p,\alpha} \le c\,\omega_r\left(f, \frac{1}{n}\right)_{p,\alpha} \quad , \qquad n = 1,2, \dots\dots \qquad (1.6)$$

and its weak converse

$$\omega_r\left(f, \frac{1}{n}\right)_{p,\alpha} \le \frac{c}{n^{2r}}\sum_{k=0}^{n}(k+1)^{2r-1}E_k(f)_{p,\alpha} \qquad (1.7)$$

Where n = 1 , 2, …….

2. Auxiliary results

In this section, we will mention some of the lemmas that we will need to proving the theorems of the main results, As well as we will prove some properties of the modulus of smoothness .

Lemma 2.1 : [5]
Let $\{y_i\}$ be a sequence of the real numbers be satisfy
$|y_i| \le \mathcal{K}$, $\sum_{i=2^{k-1}}^{2^k-1}|y_i - y_{i+1}| \le \mathcal{K}$ *for all* $i, k \in \mathbb{N}, \mathcal{K} > 0$
If $1 < p < \infty$, $\beta \in \mathcal{K}_p$ *and* $f \in L_{p,\beta}(X)$, then there is a function $\mathcal{F} \in L_{p,\beta}(X)$ such that
$\|\mathcal{F}\|_{p,\beta} \le c\,\mathcal{K}\|\mathcal{F}\|_{p,\beta}$

Lemma 2.2 : [5]
Let $n \in \mathbb{N}, 1 \le p < \infty$ *and* $f \in L_{p,\alpha}(X)$, then :

$$c\left\|\left(\sum_{u=1}^{\infty}|\Delta_u|^2\right)^{\frac{1}{2}}\right\|_{p,\alpha} \le \left\|\sum_{u=1}^{\infty}C_u p_n^{iux}\right\|_{p,\alpha} \le c\left\|\left(\sum_{u=1}^{\infty}|\Delta_u|^2\right)^{\frac{1}{2}}\right\|_{p,\alpha}$$

Lemma 2.3 : Let $f \in L_{p,\alpha}(X), 1 \le p < \infty, k \in \mathbb{N}, \delta > 0$, Then

$$\omega_k(f,\delta)_{p,\alpha} \ge 0$$

Proof **:** we have

$$\omega_k(f,\delta)_{p,\alpha} = \sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha}\right\}$$

$$= \sup_{|h|\le\delta}\left\{\left\|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(.+ih)\right\|_{p,\alpha}\right\}$$

since $\left\|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(.+ih)\right\|_{p,\alpha} \ge 0$

implies $\left\|\Delta_h^k f(.)\right\|_{p,\alpha} \ge 0$

Hence $\sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha}\right\} \ge 0$

$$\omega_k(f,\delta)_{p,\alpha} \geq 0 \quad .$$

Lemma 2.4 : Let $f \in L_{p,\alpha}(X), 1 \leq p < \infty, k \in \mathbb{N}, \delta > 0$, Then

$$\omega_k(f,\delta)_{p,\alpha} \to 0 \quad as \quad \delta \to 0$$

Proof :

let $\delta = \frac{1}{n}$

$$\omega_k(f,\delta)_{p,\alpha} = \omega_k\left(f,\frac{1}{n}\right)_{p,\alpha} = \sup_{|h|\leq\frac{1}{n}}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha}\right\}$$

$$= \sup_{|h|\leq\frac{1}{n}}\left\{\left\|\Delta_h^{k-1}\Delta_h^1 f(.)\right\|_{p,\alpha}\right\}$$

$$= \sup_{|h|\leq\frac{1}{n}}\left\{\left\|\Delta_h^{k-1}\left[f(.)-f(.+\frac{1}{n})\right]\right\|_{p,\alpha}\right\}$$

If $n \to \infty$ then $\frac{1}{n} \to 0$

$$= \sup_{|h|\leq\frac{1}{n}}\left\{\left\|\Delta_h^{k-1}[f(.)-f(.)]\right\|_{p,\alpha}\right\} = \sup_{|h|\leq\frac{1}{n}}\left\{\left\|\Delta_h^{k-1}.[0]\right\|_{p,\alpha}\right\}$$

$$= \sup_{|h|\leq\frac{1}{n}} \|0\|_{p,\alpha} = 0.$$

Lemma 2.5 **:** Let $f, g \in L_{p,\alpha}(X), 1 \leq p < \infty, k \in \mathbb{N}, \delta > 0$, Then $\omega_k(f+g,\delta)_{p,\alpha} \leq \omega_k(f,\delta)_{p,\alpha} + \omega_k(g,\delta)_{p,\alpha}$

Proof : $\omega_k(f+g,\delta)_{p,\alpha} = \sup_{|h|\leq\delta}\left\{\left\|\Delta_h^k(f+g)(.)\right\|_{p,\alpha}\right\}$

$$= \sup_{|h|\leq\delta}\left\{\left\|\Delta_h^k f(.) + \Delta_h^k g(.)\right\|_{p,\alpha}\right\}$$

$$\leq \sup_{|h|\leq\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha} + \left\|\Delta_h^k g(.)\right\|_{p,\alpha}\right\}$$

$$= \sup_{|h|\leq\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha}\right\} + \sup_{|h|\leq\delta}\left\{\left\|\Delta_h^k g(.)\right\|_{p,\alpha}\right\}$$

$$= \omega_k(f,\delta)_{p,\alpha} + \omega_k(g,\delta)_{p,\alpha}$$

Lemma 2.6 **:** Let $f \in L_{p,\alpha}(X), 1 \leq p < \infty, k \in \mathbb{N}, \delta, c > 0$, Then $\omega_k(f,c\delta)_{p,\alpha} \leq c^k\,\omega_k(f,\delta)_{p,\alpha}$.

Proof : $\omega_k(f,c\delta)_{p,\alpha} = \sup_{|h|\leq c\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha}\right\}$

$$\leq \sup_{|h|\leq c\delta}\left\{\left\|\Delta_{c\delta}^k f(.)\right\|_{p,\alpha}\right\}$$

$$= \sup_{|h|\leq c\delta}\left\{\left\|(c\delta)^k D^k f(.)\right\|_{p,\alpha}\right\}$$

$$= c^k \sup_{|h|\leq c\delta}\left\{\left\|\delta^k D^k f(.)\right\|_{p,\alpha}\right\}$$

$$= c^k \sup_{|h|\le c\delta} \left\{ \left\| \Delta_{\delta}^k f(.) \right\|_{p,\alpha} \right\}$$

$$= c^k \omega_k(f, \delta)_{p,\alpha}$$

Lemma 2.7 : Let $f \in L_{p,\alpha}(X), 1 \le p < \infty, k \in \mathbb{N}$, Then $\omega_k(f, \delta_1)_{p,\alpha} \le \omega_k(f, \delta_2)_{p,\alpha}$ for every $\delta_1 \le \delta_2, \delta_1, \delta_2 > 0$

Proof :

$$\omega_k(f, \delta_1)_{p,\alpha} = \sup_{|h|\le \delta_1} \left\{ \left\| \Delta_h^k f(.) \right\|_{p,\alpha} \right\}$$

$$\le \sup_{|h|\le \delta_2} \left\{ \left\| \Delta_h^k f(.) \right\|_{p,\alpha} \right\} \; since \; \delta_1 \le \delta_2$$

$$= \omega_k(f, \delta_2)_{p,\alpha} .$$

Lemma 2.8 : Let $f, \acute{f} \in L_{p,\alpha}(X), 1 \le p < \infty, k \in \mathbb{N}, \delta > 0$, Then $\omega_k(f, \delta)_{p,\alpha} \le \frac{\delta}{2} \omega_{k-1}(\acute{f}, \delta)_{p,\alpha}$ where $\acute{f}$ is the first derivative of a function f

Proof : We have the difference $\Delta_h^k f(x) = \Delta_h^{k-1}(\Delta_h^1 f(x))$

$$= \Delta_h^{k-1} [f(x+H) - f(x-H)]$$

$$\left\| \Delta_h^k f(.) \right\|_{p,\alpha} = \left\| \Delta_h^{k-1} [f(.+H) - f(.-H)] \right\|_{p,\alpha}$$

$$= \left\| \Delta_h^{k-1} [f(.+H) - f(.) + f(.) - f(.-H)] \right\|_{p,\alpha}$$

$$= \left\| \Delta_h^{k-1} [f(.+H) - f(.)] - \Delta_h^{k-1} [f(.-H) - f(.)] \right\|_{p,\alpha}$$

$$= \left\| \Delta_h^{k-1} \int_0^h \acute{f}(.+L) dL - \Delta_h^{k-1} \int_0^h \acute{f}(.-L) dL \right\|_{p,\alpha}$$

$$\le \int_0^h \left\| \Delta_h^{k-1} [\acute{f}(.+L) - \acute{f}(.-L)] \right\|_{p,\alpha} dL$$

$$\le \int_0^h \omega_{k-1}(\acute{f}, \delta)_{p,\alpha} dL \le \frac{\delta}{2} \omega_{k-1}(\acute{f}, \delta)_{p,\alpha}$$

3. Main results

Let X denote the one-dimensional [0 , 1] we denote by
$L_{p,\alpha}(X), 1 \le p < \infty$ the space of all unbounded functions f of one variable on [0 , 1] in each variable and satisfy
$\|f\|_{p,\alpha} < \infty$ where :

$$\|f\|_{p,\alpha} = \begin{cases} \left(\int_x |f(x).\alpha(x)|^p dx \right)^{\frac{1}{p}} , & 1 \le p < \infty \\ ess\, sup|f(x)| , & p = \infty \end{cases}$$

In the following give direct and converse approximation theorems for functions of one variable, which are our main results .

Theorem 3.1 : let $f \in L_{p,\alpha}(T)$, $1 \le p < \infty$ and $0 < \alpha < 1$ then the Jackson type inequality :

$$E_{n,k}(f)_{p,\alpha} \le c\,\omega_k\left(f,\frac{1}{n}\right)_{p,\alpha} \quad , \qquad n = 1,2,\dots\dots$$

Theorem 3.2 **:** let $f \in L_{p,\alpha}(X)$, $1 \le p < \infty$, $n \in \mathbb{N}$, $r \in \mathbb{R}^+$ and $\lambda = max\{2,p\}$. then there exist positive constant c dependent on r and p such that
$\frac{c}{n^{2r}}\left(\sum_{u=1}^{n} u^{2\lambda r-1} E_u^{\lambda}(f)_{p,\alpha}\right)^{\frac{1}{\lambda}} \le \omega_r\left(f,\frac{1}{n}\right)$

Theorem 3.3 **:** let $f \in L_{p,\alpha}(T)$, $1 \le p < \infty$ *and* $0 < \alpha < 1$,
then :
$\omega_k\left(f,\frac{1}{n}\right)_{p,\alpha} \le \frac{c(k)}{n^k}\sum_{i=0}^{n}(i+1)^{k-1}E_i(f)_{p,\alpha}$.

4. Proofs of main results

4.1 Proof of theorem 3.1 :
Let $\delta = \frac{1}{n}$ *and* Let $f \in L_{p,\alpha}(X)$, $1 \le p < \infty$ and the operator of algebraic polynomial defined by :

$$V_n(f,x) = \frac{1}{n+1}\sum_{k=n}^{2n} S_k(f)(x) \quad , \quad n \in \mathbb{N}$$

We see that $V_n \in \mathrm{J}_{2n}$ for $n \in \mathbb{N}$

$$\|f - V_n(f)\|_{p,\alpha} \le cE_n(f)_{p,\alpha}$$

because $E_n(f)_{p,\alpha} = inf\left\{\|f - p_n\|_{p,\alpha}\ , p_n \in \mathcal{P}_n\right\}$

and we have $\|V_n(f)\|_{p,\alpha} \le c\,\|f\|_{p,\alpha}$

$$E_{n,k}(f)_{p,\alpha} = inf\left\{\|f - V_n(f)\|_{p,\alpha}\ \right\}$$

$$\le \|f - V_n(f)\|_{p,\alpha} = \left(\int_X |[f(x) - V_n(f)(x)].\alpha(x)|^p dx\right)^{\frac{1}{p}}$$

$$\le sup\left(\int_X |[f(x) - V_n(f)(x)].\alpha(x)|^p dx\right)^{\frac{1}{p}}$$

$$\le \mathrm{c}\sup_{|h|\le\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha}\ \right\} = c\,\omega_k(f,\delta)_{p,\alpha} = c\,\omega_k\left(f,\frac{1}{n}\right)_{p,\alpha}$$

4.2 Proof of theorem 3.2 :
Since $r \in \mathbb{R}^+$, $1 < \lambda < \infty$, $n \in \mathbb{N}$ we assume that $k \in \mathbb{N}$ such that

$$2^k \le n \le 2^{k+1}$$

By lemma 2.2 , we have

$$\left(\sum_{u=1}^{n}\frac{\mu^{2\lambda r-1}}{n^{2\lambda r}}E_u^{\lambda}(f)_{p,\alpha}\right)^{\frac{1}{\lambda}} \le \left(\sum_{u=1}^{k+1}\sum_{\eta=2^{i-1}}^{2^i-1}\frac{\eta^{2\lambda r-1}}{n^{2\lambda r}}E_\eta^{\lambda}(f)_{p,\alpha}\right)^{\frac{1}{\lambda}}$$

$$\leq \left(\sum_{u=1}^{k+1}\frac{2^{2u\lambda r}}{n^{2\lambda r}}\,E^{\lambda}_{2^{u-1}-1}\,(f)_{p,\alpha}\right)^{\frac{1}{\lambda}} \leq \left(\sum_{u=1}^{k+1}\frac{2^{2u\lambda r}}{n^{2\lambda r}}\left\|\sum_{|\eta|=2^{u-1}}^{2^{u}-1} C_\eta p_n^{i\eta x}\right\|_{p,\alpha}^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$\leq c\left(\sum_{u=1}^{k+1}\frac{2^{2u\lambda r}}{n^{2\lambda r}}\left\|\left(\sum_{\eta=u}^{\infty}|\Delta_u|^2\right)^{\frac{1}{2}}\right\|_{p,\alpha}^{\lambda}\right)^{\frac{1}{\lambda}}$$

Putting $1 < p \leq 2\ ,\ \lambda = 2$

By using Minkowski's inequality , we have

$$\left(\sum_{u=1}^{n}\frac{u^{2\lambda r-1}}{n^{2\lambda r}}\,E^{\lambda}_{u}\,(f)_{p,\alpha}\right)^{\frac{1}{\lambda}} \leq c\left(\sum_{u=1}^{k+1}\frac{2^{u\lambda r}}{n^{ur}}\left\|\left(\sum_{\eta=v}^{\infty}|\Delta_u|^2\right)^{\frac{p}{2}}\right\|_{p,\alpha}^{\frac{2}{p}}\right)$$

$$\leq c\left(\sum_{u=1}^{k+1}\frac{2^{uvr}}{n^{ur}}\left\|\sum_{\eta=u}^{\infty}|\Delta_u|^2\right\|_{p,\alpha}^{\frac{p}{2}}\right)$$

$$\leq c\left(\left\|\left(\sum_{u=1}^{k}\frac{2^{uvr}}{n^{ur}}\ |\Delta_u|^2 + \frac{2^{uvr(k+1)}}{n^{ur}}\sum_{\eta=k+1}^{\infty}\left|\Delta_\eta\right|^2\right)^{\frac{p}{2}}\right\|_{p,\alpha}\right)$$

$$\leq c_0\left\|\left(\sum_{u=1}^{k}\frac{2^{uvr}}{n^{ur}}\ |\Delta_u|^2\right)^{\frac{p}{2}}\right\|_{p,\alpha} + c_1\left\|\left(\sum_{\eta=k+1}^{\infty}\left|\Delta_\eta\right|^2\right)^{\frac{p}{2}}\right\|_{p,\alpha}$$

Using lemma 2.2 we can estimate G_1

$$G_1 = \left\|\left(\sum_{\eta=k+1}^{\infty}\left|\Delta_\eta\right|^2\right)^{\frac{1}{2}}\right\|_{p,\alpha} \leq c\left\|\sum_{\eta=2^k}^{\infty} c_\eta p_n^{inx}\right\|_{p,\alpha}$$

$$\leq c\,E_{2^{k-1}}(f)_{p,\alpha}\ \leq c\ \omega_r(f,\frac{1}{n})_{p,\alpha}$$

On the other hand

$$G_2 = \left\|\left(\sum_{u=1}^{k}\frac{2^{2ur}}{n^{ur}}\ |\Delta_u|^2\right)^{\frac{p}{2}}\right\|_{p,\alpha} \leq \left\|\sum_{u=1}^{k}\frac{2^{2ur}}{n^{2r}}\ |\Delta_u|\right\|_{p,\alpha}$$

$$\leq \left\|\sum_{u=1}^{k}\sum_{|\eta|=2^{u-1}}^{2^{u}-1}\frac{2^{2ur}}{n^{2r}}\left|c_\eta p_n\right|\right\|_{p,\alpha}$$

Now using lemma 2.2 twice , we get

$$G_2 \leq \frac{c\, 2^{2r}}{n^{2r}} \left\| \sum_{|\eta|=1}^{\infty} p_n \left| c_\eta\, p_n^{i\eta x} \right| \right\|_{p,\alpha} \leq \frac{c\, 2^{2r}}{n^{2r}} \left\| \left(I - \sigma_{\frac{1}{n}} \right)^r f \right\|_{p,\alpha}$$

$$= \frac{c\, 2^{2r}}{n^{2r}} \left\| \left(I - \sigma_{\frac{1}{n}} \right)^{\{r\}} .(I-\sigma)^{r-\{r\}} f \right\|_{p,\alpha} \leq c\, \omega_r(f, \frac{1}{n})_{p,\alpha}$$

Therefore, the theorem followed

If $p > 2$, $\lambda = p$, then

right hand $\leq c \left(\sum_{u=1}^{k+1} \frac{2^{upr}}{n^{2upr}} \left\| \left(\sum_{\eta=u}^{\infty} |\Delta_u|^2 \right)^{\frac{1}{2}} \right\|_{p,\alpha}^{p} \right)^{\frac{1}{p}}$

$$\leq c \left(\left\| \sum_{u=1}^{k+1} \frac{2^{upr}}{n^{2upr}} \left(\sum_{\eta=u}^{\infty} |\Delta_u|^2 \right)^{\frac{p}{2}} \right\|_{p,\alpha}^{\frac{1}{p}} \right)$$

$$\leq c \left(\left\| \sum_{u=1}^{k+1} \frac{2^{upr}}{n^{ur}} \left(\sum_{\eta=u}^{\infty} |\Delta_u|^2 \right)^{\frac{p}{2}} \right\| \right)^{\frac{1}{p}} \leq c\, \omega_r(f, \frac{1}{n})_{p,\alpha}$$

4.3 : Proof of theorem 3.3 :

Let $f \in L_{p,\alpha}$(X) for every natural number k there exist a constant c(k) depending on k such that :

$$\omega_k \left(f, \frac{1}{n} \right)_{p,\alpha} \leq \frac{c(k)}{n^k} \sum_{i=0}^{n} (i+1)^{k-1} E_i(f)_{p,\alpha}$$

Let $p_k^* \in \mathcal{P}_n$ be a best algebraic approximation of unbounded function f and b , $b + kh \in$ X

We have $0 \leq \Delta_h^k f(b)\, \alpha(b)$ then :

$$0 \leq \left\| \Delta_h^k f(b) \right\|_{p,\alpha} = \left\| \sum_{i=0}^{k} (i+1)^{k+j} f(b+ih) \right\|_{p,\alpha}$$

$$\leq \left\| \sum_{i=0}^{k} \binom{k}{i} | f(b+ih) - p_k^*(b+ih) | + \sum_{i=0}^{k} \binom{k}{i} | p_k^*(b+ih) | \right\|_{p,\alpha}$$

$$\leq sup \left\| \Delta_h^k\, p_k^*(b+ih) \right\|_{p,\alpha} + \left\| \sum_{i=0}^{k} \binom{k}{i} | f(b+ih) - p_k^*(b+ih) | \right\|_{p,\alpha}$$

$$\leq \omega_k(p_k^*, b, \delta) + \sum_{i=0}^{k} \binom{k}{i} \| f - p_k^* \|_{p,\alpha}$$

$$\leq \frac{c(k)}{n^k} \sum_{i=0}^{k} \binom{k}{i} E_i(f)_{p,\alpha}$$

5. Conclusion

We can be approximated of unbounded functions by using algebraic polynomial in weighted space, also obtain sharp Jackson (direct) inequality of algebraic approximation of unbounded functions in terms modulus of smoothness.

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