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Trigonometric Approximation by Modulus of Smoothness in $L_{p,\alpha}(X)$

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ABSTRACT

In this paper, we study the approximate properties of functions by means of trigonometric polynomials in weighted spaces. Relationships between modulus of smoothness of function derivatives and those of the jobs themselves are introduced. In the weighted spaces we also proved of theorems about the relationship between the derivatives of the polynomials for the best approximation and the best approximation of the functions

KEYWORDS: Chalcone; Pyrazoline; Schiff's-Base and antimicrobial.

الخلاصة

في هذا البحث تم دراسة خواص التقريب للدوال بواسطة متعددات الحدود المثلثية في فضاء الوزن ودراسة العلاقة بين مقياس النعومة لمشتقات الدوال مع الدوال نفسها وكذلك تم برهنة العلاقة بين درجة افضل تقريب للدوال مع درجة افضل تقريب لمشتقات متعددات الحدود.

INTRODUCTION

Let $X = [0, 2\pi]$ be a closed interval, for $\alpha \in W$, let $\alpha : X \to [0, \infty)$ be locally integrable and almost everywhere positive where W is weighted space. $L_{p,\alpha}(X)$ be the space of all unbounded functions, $1 \le p < \infty$ with norm of this function defined by:

$$\| f \|_{p,\alpha} = \left(\int_X |f(x).\alpha(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Weighted spaces, introduced by G. Lorentz in the 1950 s . see [2] , [5] and [11] .

Let $k \in Z^+$ and $f \in L_{p,\alpha}(X)$, $x, h \in R$, and we define the symmetric difference by the following:

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + (k-j)h),$$

where $\binom{k}{j} = \frac{k(k-1)\ldots\ldots(k-j+1)}{j!}$, $j > 1$.

Let $1 \le p < \infty$, $f \in L_{p,\alpha}(X)$,we put

$$\psi_\delta^k f(x) = \frac{1}{\delta} \int_0^\delta |\Delta_h^k f(x)| \, dx.$$

If $f \in L_{p,\alpha}(X)$, then according to [4] the Hardy-Littlewood maximal function is unbounded in $L_{p,\alpha}(X)$. Then we have

$\| \psi_\delta^k f \|_{p,\alpha} \le C \, \|f\|_{p,\alpha} < \infty$, where $C > 0$.

Let $f \in L_{p,\alpha}(X)$, $1 \le p < \infty$, $k \in Z^+$ the modulus of smoothness $\omega_k(f, \delta)_{p,\alpha}$ is define by

$$\omega_k(f, \delta)_{p,\alpha} = sup_{|h| \le \delta} \|\Delta_h^k f(x)\|_{p,\alpha} \quad , \quad \delta > 0$$

and have the following properties [3]

$$\omega_k(f_1 + f_2, \delta)_{p,\alpha} \le \omega_k(f_1, \delta)_{p,\alpha} + \omega_k(f_2, \delta)_{p,\alpha}$$

with $\omega_k(f, \delta)_{p,\alpha} = 0$.

For $\in L_{p,\alpha}(X)$, we define the r-th derivative of f as function $f^{(r)} \in L_{p,\alpha}(X)$ satisfying

$$\left\| \frac{\Delta_h^k(f)}{h^k} - f^{(r)} \right\|_{p,\alpha} = 0 \qquad (1.1)$$

Let $\frac{a_0}{2} + \sum_{w=1}^{\infty} \mu_w(f, x)$ (1.2)

$where \ \mu_w(f, x) = a_w(f) \cos\cos wx + b_w(f) \sin\sin wx$ be the Fourier series of the functions in $L_{p,\alpha}(X)$. The n-th partial sums , and de la Vallee-Poussin sum of the series (1) are defined , respectively , as

$$S_n(f) = \frac{a_0}{2} + \sum_{w=1}^{\infty} \mu_w(f, x) \, . \, \alpha(x)$$

$$V_n(f) = \frac{1}{n} \sum_{i=1}^{2n-1} S_i(f)$$

Here denote by $E_n(f)_{p,\alpha}$ $(n = 0,1, \ldots)$ the degree of best approximation of $f \in L_{p,\alpha}(X)$ by

trigonometric polynomials of degree less than n , i.e.,
$E_n(f)_{p,\alpha} = inf\ inf\ \{\|f - t_n\|_{p,\alpha}\ ,\ t_n \in H_n\ \}$, where H_n denotes the class of trigonometric polynomials of degree n

The problems of approximation theory in the weighted and non-weighted space have been investigated in [1] and [6] weight respectively. The approximation problems by trigonometric polynomials in different spaces have been investigated by several authors see, for example, [3],[7],[8] and [9].

In this work we study the approximation problems of unbounded functions by trigonometric polynomials in the weighted space $L_{p,\alpha}(X)$. Relations between modulus of smoothness of the derivatives of a function and those of the function itself are investigated. We also prove a theorem on the relationship between derivatives of a polynomial of best approximation and the best approximation of the function in the weighted space $L_{p,\alpha}(X)$. In addition, in the weighted space $L_{p,\alpha}(X)$ relationship between modulus of smoothness of the function and its de la Vallee-Poussin sums studied.

AUXILIARY RESULTS

In this section we will prove some of the auxiliary results that we needed in proving the main results.

Lemma 1 [9]: Let $1 \le p < \infty$, $f \in L_{p,\alpha}(X)$ and $k = 1,2,\dots$ then there exists a constant $c > 0$ depending on k , p such that

$$E_n(f)_{p,\alpha} \le c\ \omega_k\left(f\ ,\frac{1}{n}\right)_{p,\alpha}$$

holds where $n = 0\ ,1\ ,2\ ,\dots$.

Lemma 2: Let $1 \le p < \infty$, $k \in Z^+$. If $T_n \in H_n\ ,n \ge 1$, then there exists a constant $c > 0$ depending on k , p such that

$$\|T_n^{(k)}\|_{p,\alpha} \le c\ n^k \|T_n\|_{p,\alpha}$$

Proof: since $1 \le p < \infty$,we have in [12]

$$\|S_n(f)\|_{p,\alpha} \le c\|f\|_{p,\alpha}\ ,$$

$$\|\tilde{f}\|_{p,\alpha} \le c\|f\|_{p,\alpha}$$

Now, we obtain the request result

Lemma 3: Let $1 \le p < \infty$, $k \in Z^+$. If $T_n \in H_n\ ,n \ge 1$, then there exists a constant $c > 0$ depending on k , p such that

$$\omega_k(T_n, h)_{p,\alpha} \le c\ h^k \|T_n^{(k)}\|_{p,\alpha}\ , \qquad 0 < h \le \pi$$

Proof: Since

$$\Delta_t^k T_n\left(x - \frac{k^*}{2}t\right) = \sum_{v\in Z_n} \left(2i\ sin\frac{t}{2}v\right)^k c_v e^{ivx},$$

see [8],

$$\Delta_t^{k^*} T_n^{(k-k^*)}\left(x - \frac{k^*}{2}t\right) = \sum_{v\in Z_n} \left(2i\ sin\frac{t}{2}v\right)^{k^*} (iv)^{k-k^*} c_v e^{ivx}$$

with

$Z_n = \{\pm 1\ , \pm 2\ , \pm 3\ , \dots\}\ , k^* \equiv integer\ part\ of\ k$, putting

$$\varphi(z) = \left(2i\ sin\frac{t}{2}v\right)^{k^*} (iz)^{k-k^*}\ , g(z) = \left(\frac{2}{z}\ sin\frac{t}{2}v\right)^{k-k^*},$$

$-n \le z \le n\ , g(0) = t^{k-k^*}$

We get

$$\Delta_t^{k^*} T_n^{(k-k^*)}\left(x - \frac{k^*}{2}t\right) = \sum_{v\in Z_n} \varphi(v) c_v e^{ivx},$$

$$\Delta_t^k T_n\left(x - \frac{k^*}{2}t\right) = \sum_{v\in Z_n} \varphi(v) g(v) c_v e^{ivx}$$

Taking into account that

$$g(z) = \sum_{r=-\infty}^{\infty} d_r e^{\frac{ir\pi z}{n}}$$

Uniformly in $[-n\ ,n\ ]$, with $d_0 > 0\ , (-1)^{r+1} d_r \ge 0\ , d_{-r} = d_r (r = 1,2,\dots)$ we have

$$\Delta_t^k T_n(x) = \sum_{r=-\infty}^{\infty} d_r \Delta_t^{k^*} T_n^{(k-k^*)}\left(x + \frac{r\pi}{n} + \frac{k-k^*}{2}t\right)$$

Consequently, we get $\|\frac{1}{\delta}\int_0^\delta \left|\Delta_t^k T_n(.)\right| dt\|_{p,\alpha} =$

$$\|\frac{1}{\delta}\int_0^\delta \left|\sum_{r=-\infty}^{\infty} d_r \Delta_t^{[k]} T_n^{(k-k^*)}\left(. + \frac{r\pi}{n} + \frac{k-k^*}{2}t\right)\right| dt\|_{p,\alpha}$$

$$\le \sum_{r=-\infty}^{\infty} |d_r| \|\frac{1}{\delta}\int_0^\delta \left|\Delta_t^{k^*} T_n^{(k-k^*)}\left(. + \frac{r\pi}{n} + \frac{k-k^*}{2}t\right)\right| dt\|_{p,\alpha}$$

Since , $\Delta_t^{k^*} T_n^{(k-k^*)}(x) = \int_0^t T_n^{(k)}(x+t)dt$

We find,

$$\omega_k(T_n,h)_{p,\alpha} \le \sum_{r=-\infty}^{\infty} |d_r| \| \frac{1}{\delta}\int_0^\delta \left|\Delta_t^{k^*} T_n^{(k-k^*)}\left(.+\frac{r\pi}{n} + \frac{k-k^*}{2}t\right)\right| dt\|_{p,\alpha}$$

$$= sup_{|\delta|\le h} \sum_{r=-\infty}^{\infty} |d_r| \| \frac{1}{\delta}\int_0^\delta \left|\int_0^t T_n^{(k)}\left(.+\frac{r\pi}{n} + \frac{k-k^*}{2}t\right)\right| dt\|_{p,\alpha}$$

$$\le h^{k^*} sup_{|\delta|\le h} \sum_{r=-\infty}^{\infty} |d_r| \| \frac{1}{\delta}\int_0^\delta \left|T_n^{(k)}\left(.+\frac{r\pi}{n} + \frac{k-k^*}{2}t\right)\right| dt\|_{p,\alpha}$$

$$\le ch^{k^*} sup_{|\delta|\le h} \sum_{r=-\infty}^{\infty} |d_r| \| \frac{1}{\delta}\int_0^\delta \left|T_n^{(k)}\left(.+\frac{r\pi}{n} + \frac{k-k^*}{2}t\right)\right| dt\|_{p,\alpha}$$

$$\le ch^{k^*} sup_{|\delta|\le h} \sum_{r=-\infty}^{\infty} |d_r| \| \frac{1}{\frac{k-k^*}{2}\delta}\int_{.+\frac{r\pi}{n}}^{.+\frac{r\pi}{n}+\frac{k-k^*}{2}\delta} \left|T_n^{(k)}(u)\right| du\|_{p,\alpha}$$

On the other hand

$\sum_{r=-\infty}^{\infty} |d_r| < 2g(0) = 2t^{k-k^*}$, $0 < t \le \pi$

And for $0 < t < \delta < h \le \pi$,we have

$$\sum_{r=-\infty}^{\infty} |d_r| < 2g(0) = 2h^{k-k^*}$$

Therefore, $\omega_k(T_n,h)_{p,\alpha} \le c\, h^k \|T_n^{(k)}\|_{p,\alpha}$

MAIN RESULTS

In this section we will prove some of the main results.

Theorem 1: Let $1 \le p < \infty$, $f \in L_{p,\alpha}(X)$ and T_n be a trigonometric polynomial of degree n satisfying the following conditions:

$\| f - t_n\|_{p,\alpha} = o\left(\frac{1}{n}\right)$ and $\| g - t_n'\|_{p,\alpha} = o\left(\frac{1}{n}\right)$, $n \to \infty$, then we obtain $f' = g$.

Proof: Take > 0 . We choose a natural number $n_0 = n_0(\epsilon)$ such that for $n \ge n_0$

$$\| f - t_n\|_{p,\alpha} \le \epsilon.\left(\frac{1}{n}\right) \quad , \quad \| f' - t_n'\|_{p,\alpha} \le \epsilon \qquad (3.1)$$

Taking account of (3.1) for h satisfying the condition $\frac{\sqrt{\epsilon}}{n} \le h \le \frac{1}{n}$ we obtain $\| \frac{f(.+h)-f(.)}{h} - \frac{t_n(.+h)-t_n(.)}{h}\|_{p,\alpha} \le 2^{\frac{p}{2}}$ (3.2)

Thus, we have

$$\Delta_h^r\, t_n(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i\, t_n\left(x + \left(\frac{r}{2} - i\right)h\right)$$

$$= \sum_{j=r}^{\infty} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left(\frac{r}{2} - i\right)^j \frac{h^j}{j!}\, t_n^{(j)}(x)$$

$$= h^r t_n^{(r)}(x) + \sum_{j=r+1}^{\infty} \eta\,(r,j)^{j-r}\, t_n^{(j)}(x), \qquad (3.3)$$

where $\frac{-r}{2} < \eta(r,j) < \frac{r}{2}$ and $\eta(r,j) = 0$ if $j-r$ is odd . Then using (3.3) and lemma 2 for $\frac{\sqrt{\epsilon}}{n} \le h \le \frac{2\sqrt{\epsilon}}{n}$ we find that

$$\| \frac{t_n(.+h)-t_n(.)}{h} - t_n'(.)\|_{p,\alpha} \le \sum_{m=2}^{\infty} \left(\frac{h^{m-1}}{m!}\right)^p \|t_n^{(m)}\|_{p,\alpha}$$

$$\le \sum_{m=2}^{\infty} (hn)^{(m-1)p} \|t_n\|_{p,\alpha}$$

$$\le 4\left(\frac{\epsilon}{1-2^p\epsilon^{\frac{p}{2}}}\right)\|t_n\|_{p,\alpha}$$

$$\le c\epsilon^p\, \|t_n\|_{p,\alpha} \qquad (3.4)$$

Using (3.2), (3.4) and (3.1) for $\frac{\sqrt{\epsilon}}{n} \le h \le \frac{2\sqrt{\epsilon}}{n}$ we reach

$$\| \frac{f(.+h)-f(.)}{h} - f'\|_{p,\alpha} \le \| \frac{f(.+h)-f(.)}{h} - \frac{t_n(.+h)-t_n(.)}{h}\|_{p,\alpha} + \| \frac{t_n(.+h)-t_n(.)}{h} - t_n'(.)\|_{p,\alpha} + \| t_n' - f'\|_{p,\alpha}$$

$$\le c\left(\epsilon^{\frac{p}{2}} + \epsilon^p \| f\|_{p,\alpha} + \epsilon^p\right)$$

Then the proof of theorem is complete.

Theorem 2: Let $1 \le p < \infty$. Then for a given $f \in L_{p,\alpha}(X)$ and integer k , r satisfying $k > r$ we have

$$\omega_{k-r}\left(f^{(r)},t\right)_{p,\alpha} \le c\left\{\int_0^t \frac{\omega_k(f,u)_{p,\alpha}^s}{u^{sr+1}}\,du\right\}^{\frac{1}{s}},$$

where $c > 0$.

Proof: The function $\omega_m(F,t)_{p,\alpha}$ non-decreasing and according to reference [10] the following inequality holds

$$\omega_k(F,2t)_{p,\alpha} \le c\omega_k(F,2t)_{p,\alpha} \quad (3.5)$$

It is sufficient to prove theorem for $= 2^{-n}$, then using of (3.5) we obtain

$$\left\{\int_0^{2^{-n}} \frac{\omega_k(f,u)_{p,\alpha}^s}{u^{sr+1}}\,du\right\}^{\frac{1}{s}} = \left\{\sum_{v=n}^{\infty} 2^{vsr}\omega_k(f,2^{-v})_{p,\alpha}^s\right\}^{\frac{1}{s}}$$

Therefore, for all n it is sufficient to prove the following inequality:

$$\omega_{k-r}(f^{(r)},2^{-n})_{p,\alpha} \le \left\{\sum_{v=n}^{\infty} 2^{vsr}\omega_k(f,2^{-v})_{p,\alpha}^s\right\}^{\frac{1}{s}} \quad (3.6)$$

For any trigonometric polynomial Q_n of degree n and $G \in L_{p,\alpha}(X)$ we obtain

$$\omega_k\left(G,\frac{1}{n}\right)_{p,\alpha} \le c\left(\|G-Q_n\|_{p,\alpha} + n^{-k}\|Q_n^{(k)}\|_{p,\alpha}\right) \quad (3.7)$$

Therefore, we aim to find Q_{2^n} of degree 2^n such that both $\|f^{(r)} - Q_{2^n}\|_{p,\alpha}$ and $2^{-n(k-r)}\|Q_{2^n}^{(k-r)}\|_{p,\alpha}$ are bounded by the right-hand side of inequality (3.6)

Let $t_n \in T_n$, $where\ n = 0,1,2,\ldots.$ be the trigonometric polynomial of best approximation to $f \in L_{p,\alpha}(X)$.

It is known that the set of trigonometric polynomials is dense in $L_{p,\alpha}(X)$,

then we have $\|f - t_{2^v}\|_{p,\alpha} \to 0 \ as\ v \to \infty$

Let $f \in L_{p,\alpha}(X)$ has the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{w=1}^{\infty} (a_k \cos\cos wx + b_w \sin\sin wx\ ) = \sum_{w=0}^{\infty} A_w(f)$$

We define trigonometric polynomial vNf as

$$vNf = \sum_{w=0}^{\infty} v\left(\frac{w}{N}\right) A_w(f)$$

where $v \in C^{\infty}[0,\infty),\ v(x) = 1\ for\ x \le 1\ and\ v(x) = 0\ for\ x \ge 1$

Note that trigonometric polynomial vNf has the following properties:

1) vNf is a trigonometric polynomial of degree smaller than N
2) If g is a trigonometric polynomial of degree $\frac{n}{2}$, then $vNg = g$
3) $\|vNf\|_{p,\alpha} \le c\,\|f\|_{p,\alpha}$

Thus, we have

$$\|vNf - f\|_{p,\alpha} \le c\,E_{\frac{N}{2}}(f)_{p,\alpha}$$

where $E_k(f)_{p,\alpha}$ is the best approximation of $f \in L_{p,\alpha}(X)$ trigonometric polynomials of degree less than k .

We now choose the Q_n of (3.7) for $G = f^{(r)}$ to be $(v_n f)^{(r)}$

It is Cleary that $\|f - v_n f\|_{p,\alpha} = o(1)\ as\ n \to \infty.$

The following identity holds :

$$v_{2^w}f - v_{2^n}f = \sum_{m=n}^{w-1} (v_{2^{m+1}}f - v_{2^m}f) \equiv \sum_{m=n}^{w-1} \lambda_m f$$

then

$$(v_{2^w}f)^{(r)} - (v_{2^n}f)^{(r)} = \sum_{m=n}^{w-1} (\lambda_m f)^{(r)},$$

we have in weighted spaces $L_{p,\alpha}(X)$

$$c\,\|(v_{2^w}f)^{(r)} - (v_{2^n}f)^{(r)}\|_{p,\alpha} \le \left\|\left(\sum_{m=n}^{w-1}\left\{(\lambda_m f)^{(r)}\right\}^2\right)^{\frac{1}{2}}\right\|_{p,\alpha} \le c\|(v_{2^w}f)^{(r)} - (v_{2^n}f)^{(r)}\|_{p,\alpha} \quad , \quad (3.8)$$

we get

$$\left\|\left(\sum_{m=n}^{w-1}\left\{(\lambda_m f)^{(r)}\right\}^2\right)^{\frac{1}{2}}\right\|_{p,\alpha} \le \left(\sum_{m=n}^{w-1}\|(\lambda_m f)^{(r)}\|_{p,\alpha}^2\right)^{\frac{1}{2}} \quad (3.9)$$

note that $v_n f$ is the near best approximation to f in $L_{p,\alpha}(X)$, we reach the following equivalence

$$\omega_k\left(f,\frac{1}{n}\right)_{p,\alpha} = \|f - v_n f\|_{p,\alpha} + n^{-k}\,\|(v_n f)^{(k)}\|_{p,\alpha} \quad (3.10)$$

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From (3.8), (3.10) and lemma 2 we conclude that

$$\| (v_{2^w} f)^{(r)} - (v_{2^n} f)^{(r)} \|_{p,\alpha} \le c \left(\sum_{i=n}^{w-1} 2^{mrs} \| (\lambda_m f) \|_{p,\alpha}^{s} \right)^{\frac{1}{s}} \le c \left(\sum_{m=n}^{w-1} 2^{mrs} \omega_k (f, 2^{-m})_{p,\alpha}^{s} \right)^{\frac{1}{s}}$$

where c independent of m , k and f.

Use of $Q_{2^n} = v_{2^n} f$ and (3.10) gives us

$$2^{-n(k-r)} \| \left((v_{2^n} f)^{(r)} \right)^{(k-r)} \|_{p,\alpha} = 2^{-n(k-r)} \| (v_{2^n} f)^{(k)} \|_{p,\alpha} \le 2^{nr} \omega_k (f, 2^{-n})_{p,\alpha} \le c \left(\sum_{m=n}^{\infty} 2^{mrs} \omega_k (f, 2^{-m})_{p,\alpha}^{s} \right)^{\frac{1}{s}} .$$

Hence, the proof of theorem 2 is completed.

Theorem 3: Let $1 \le p < \infty$, $f \in L_{p,\alpha}(X)$, $k, r \in Z^+, (k > r > 0)$ and let $t_n(f) \in T_n$ be the polynomial of best approximation to f in $L_{p,\alpha}(X)$ in order that $\| t_n^{(k)}(f) \|_{p,\alpha} = O(n^{k-r})$, it is necessary and sufficient that

$$E_n(f)_{p,\alpha} = O(n^{-r})$$

Proof: Suppose that

$$E_n(f)_{p,\alpha} = \| f - t_n(f) \|_{p,\alpha} = O(n^{-r}), \quad r > 0 \qquad (3.11)$$

Taking into account lemma 2 and the relations (3.11) we obtain

$$\| t_n^{(k)}(f) \|_{p,\alpha} \le c\, n^k \| t_n(f) \|_{p,\alpha} \le n^k \| f - t_n(f) \|_{p,\alpha} + \| t_n(f) \|_{p,\alpha} \le c\, n^{k-r}$$

Now we suppose that

$$\| t_n^{(k)}(f) \|_{p,\alpha} = O(n^{k-r}) \qquad (3.12)$$

using lemma 1 and lemma 3 we get

$$\| t_{2n}(f) - t_n(t_{2n}(f)) \|_{p,\alpha} \le E_n\big(t_{2n}(f)\big)_{p,\alpha} \le c\, \omega_k \left(t_{2n}, \frac{1}{n} \right)_{p,\alpha} \le c\, n^{-k} \| t_{2n}^{(k)} \| \le c\, n^{-k} (n^{k-r}) \le c\, n^{-r} . \qquad (3.13)$$

On the other hand, since $T_n(T_{2n}(f))$ is a polynomial of order n, then the following inequality holds:

$$0 \le E_n(f)_{p,\alpha} - E_{2n}(f)_{p,\alpha} \qquad (3.14)$$
$$\le \| f - t_n\big(t_{2n}(f)\big) \|_{p,\alpha} - \| f - t_{2n}(f) \|_{p,\alpha} \le \| f - t_n\big(t_{2n}(f)\big) - (f - t_{2n}(f)) \|_{p,\alpha} = \| t_{2n}(f) - t_n(t_{2n}(f)) \|_{p,\alpha}$$

use of (3.13) and (3.14) gives us

$$0 \le E_n(f)_{p,\alpha} - E_{2n}(f)_{p,\alpha} \le c\, n^{-r} \qquad (3.15)$$

Since $E_n(f)_{p,\alpha} \to 0$ from the inequality (3.15) we conclude that $\sum_{w=n_0}^{\infty} \{ E_{2^w}(f)_{p,\alpha} - E_{2^{w+1}}(f)_{p,\alpha} \} \le \sum_{w=n_0}^{\infty} 2^{-wr}$

Then from the last inequality we obtain

$$E_{2^{n_0}}(f)_{p,\alpha} \le c\, 2^{-n_0 r} \qquad (3.16)$$

it is clear that inequality (3.16) is equivalent to $E_n(f)_{p,\alpha} \le c\, n^{-r}$

Hence, the proof of theorem 3 is complete.

Theorem 4: Let $1 \le p < \infty$, $f \in L_{p,\alpha}(X)$, $k \in Z^+$, then

$$i)\ c\omega_k \left(f, \frac{1}{n} \right)_{p,\alpha} \le \left(n^{-k} \| V_n^{(k)}(f) \|_{p,\alpha} + \| f - V_n(f) \|_{p,\alpha} \right) \le c\omega_k \left(f, \frac{1}{n} \right)_{p,\alpha}$$

$$ii)\ c\omega_k \left(f, \frac{1}{n} \right)_{p,\alpha} \le \left(n^{-k} \| S_n^{(k)}(f) \|_{p,\alpha} + \| f - S_n(f) \|_{p,\alpha} \right) \le c\omega_k \left(f, \frac{1}{n} \right)_{p,\alpha}$$

Proof: In view of lemma 3 the inequality

$$\omega_k \left(T_n, \frac{1}{n} \right)_{p,\alpha} \le c\, n^{-k} \| T_n^{(k)} \|_{p,\alpha} , \qquad (3.17)$$

holds, where T_n is a trigonometric polynomial of order n. Using the properties of smoothness $\omega_k(f, .)_{p,\alpha}$ and (3.17), we reach

$$\omega_k \left(f, \frac{1}{n} \right)_{p,\alpha} \le \left(\omega_k \left(f - T_n, \frac{1}{n} \right)_{p,\alpha} + \omega_k \left(T_n, \frac{1}{n} \right)_{p,\alpha} \right) \le c \left(\| f - T_n \|_{p,\alpha} + n^{-k} \| T_n^{(k)} \|_{p,\alpha} \right) \qquad (3.18)$$

now, there exists a constant $c > 0$ depending only on r , p such that

$$n^{-k}\|T_n^{(k)}\|_{p,\alpha} \le \omega_k\left(T_n,\frac{1}{n}\right)_{p,\alpha}$$

(3.19)

by virtue of lemma 1

$$E_n(f)_{p,\alpha} \le c\,\omega_k\left(f,\frac{1}{n}\right)_{p,\alpha}$$

(3.20)

now, we get

$$\|f - V_n(f)\|_{p,\alpha} \le c\,E_n(f)_{p,\alpha} \quad ,$$

(3.21)

Thus inequalities (3.19) , (3.21) implies that

$$n^{-k}\|V_n^{(k)}(f)\|_{p,\alpha} + \|f - V_n(f)\|_{p,\alpha}$$
$$\le c\left(\omega_k\left(V_n,\frac{1}{n}\right)_{p,\alpha} + E_n(f)_{p,\alpha}\right)$$
$$\le c\left(\omega_k\left(f,\frac{1}{n}\right)_{p,\alpha} + \omega_k\left(f - V_n,\frac{1}{n}\right)_{p,\alpha} + E_n(f)_{p,\alpha}\right)$$
$$\le c\,\omega_k\left(f,\frac{1}{n}\right)_{p,\alpha}$$

The last inequality and (3.18) imply that (*i*)
There exists a constant *c* such that

$$\|f - S_n(f)\|_{p,\alpha} \le c\,E_n(f)_{p,\alpha} \qquad (3.22)$$

If the inequality (3.22) and the scheme of proof of the estimation (*i*) is used we obtain the estimation (*ii*)
Therefore, Theorem 4 is proved.

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