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Best multi Approximation of Unbounded Functions by Using Modulus of Smoothness

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A B S T R A C T

We present an estimate of the degree of best multi approximation of unbounded function on $[-1,1]^d$ by algebraic polynomials in weighted space. The studied of the relation between the best approximation of derivatives functions in weighted space and the best approximation of unbounded functions in the same space.

1. INTRODUCTION

This interest originated from the study of a certain class of integro-differential equations and applications in error estimations for singular integral equations. Following the initial works of Kalandiya [1] and Pr¨ossdorf [2], some problems of approximation in H¨older spaces have been studied by Bloom and Elliott [3], Ioakimidis [4], Nevertheless, most interesting and sharp results have been obtained for approximation of algebraic functions (see, for example, [5]).

Also, the approximation problems of the unbounded functions by algebraic polynomials in the weighted spaces have been investigated by several authors. In particular, some direct and inverse theorems in weighted and nonweighted Lebesgue spaces with variable exponent have been obtained in [6,7, 8, 9, 10, 11,12 and 13].

Let $X=[-1,1]$, $L_{p,\wp}(X)$ be the space of all unbounded functions with one variable such that $\|f\|_{p,\wp}<\infty$, where

$$\|f\|_{p,\wp}=\left(\int_X |f(x)\cdot\wp(x)|^p\cdot dx\right)^{\frac{1}{p}}<\infty \quad , \qquad 1\le p<\infty$$

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For $k=1,2,\ldots\ldots$ the k− modulus of smoothness of the function $f\in L_{p,\wp}(X)$ is defined by

$$\omega_k(f,\delta)_{p,\wp}=\sup_{|h|\le\delta}\left\|\Delta_h^k f(\cdot)\right\|_{p,\wp} \quad ; \qquad \delta>0$$

Where

$$\Delta_h^k f(x)=\sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}f(x+ih)$$

Such that $\Delta_h^k f(x)$ is called the $k-th$ difference of f at point x within quantity h.

Let $\mathbb{p}_n$ $(n=0,1,2,\ldots\ldots)$ be the class of algebraic polynomials

$$\left\|\Delta_h^k f(\cdot)\right\|_{p,\wp}\le C(k)\,\|f\|_{p,\wp}$$

And therefore, $\Delta_h^k f(\cdot)$ is defined almost everywhere on $\mathbb{R}$ and Polynomial of degree less than or equal n and d there $E_n(f)_{p,\wp}$ be the best approximation of $f\in L_{p,\wp}(X)$ by elements of $\mathbb{p}_n$, i.e.

$$E_n(f)_{p,\wp}=\inf\left\{\|f-p_n\|_{p,\wp} \quad ; \quad p_n\in\mathbb{p}_n\right\} .$$

Where $\wp : [-1,1] \longrightarrow \mathbb{R}^+$ the weight function

Let $X^d = [-1,1]^d$, $d \in \mathbb{N}$, space $L_{p,\wp}(X^d)$ of all unbounded functions of several variables, with $f \in L_{p,\wp}(X^d)$ given norm defined by

$$\|f\|_{L_{p,\wp}(X^d)} = \left(\int_{X^d} | f(x) \cdot \wp(x) |^p \, dx \right)^{\frac{1}{p}} < \infty \quad , \quad 1 \le p < \infty$$

We consider the following linear operators : $\mathcal{A}_{\delta,j}(f)$, $B(f^{(r)}, \delta, \cdot)$, $\mathcal{X}_i(f, \cdot)$, and $\mathcal{F}_i(f, \cdot)$, defined on $L_{p,\wp}(X^d)$ such that

$$\mathcal{X}_i(f,\cdot) = \sum_{i=0}^{n} G_i(f,\cdot) \qquad , \qquad n = 0,1,2, \dots$$

$$\mathcal{F}_n(f,\cdot) = \frac{1}{n+1} \sum_{i=0}^{n} \mathcal{X}_i(f,\cdot) = \sum_{i=0}^{n} \left(1 - \frac{i}{n+1}\right) G_i(f,\cdot) \qquad , \quad n \in \mathbb{N}$$

$$B(f,\delta,\cdot) = \sum_{i=1}^{\infty} \delta^i \, G_i(f,\cdot) \qquad , \qquad 0 \le \delta < 1$$

$$\mathcal{A}_{\delta,j}(f) = \mathcal{X}_i(f,\cdot) + \sum_{i=j}^{\infty} \beta_{i,j} G_i(f,\cdot)$$

Where

$$\beta_{i,j} \coloneqq \beta_{i,j}(\delta) = \sum_{k=0}^{j-1} \binom{i}{k} (1-\delta)^k \delta^{i-k}$$

$$= \sum_{k=0}^{j-1} \frac{1}{k!} (1-\delta)^k \frac{\partial^k}{\partial \delta^k} \delta^i \qquad , \qquad j \in \mathbb{N}, 0 \le \delta < 1$$

In the terms of Poisson integrals, one can give the following interpretation of the derivative $f^{(r)}$:

Assume that $0 \le \delta < 1$, then

$$B(f^{(r)}, \delta, \cdot) = \delta^r \frac{\partial^r}{\partial \delta^r} B(f, \delta, \cdot) \qquad (⁎)$$

The purpose of this paper is to investigate the operators $\mathcal{A}_{\delta,j}(f)$ and $B(f,\delta,\cdot)$ as the linear methods of approximation of functions in the weighted spaces. In this case, our attention is drawn to the relationship of the approximative properties of the sums $\mathcal{A}_{\delta,j}(f)$ and $B(f,\delta,\cdot)$ with the differential properties of the function f, In addition, we study the relationship between the derivatives of an algebraic polynomial of best approximation and the best approximation of unbounded functions of several variables in weighted space.

2. RESULTS

2. 1. AUXILIARY RESULTS

We begin with the following lemma which needs it is in our main results

Lemma 2.1. [11]. For $r \in \mathbb{R}^+$ let

I. $a_1 + a_2 + \dots + a_n + \dots$ and
II. $a_1 + 2^r a_2 + \dots + n^r a_n + \dots$

Be two series in a Banach space $(B, \|\cdot\|)$

Let

$$R_n^{(r)} \coloneqq \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1}\right)^r\right) a_k$$

And

$$R_n^{(r)*} \coloneqq \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1}\right)^r\right) k^r a_k \qquad , n = 1,2, \dots$$

Then $\|R_n^{(r)*}\| \le c \qquad , n = 1,2, \dots$

For some $c > o$ if and only if there exists $R \in B$ such that

$$\|R_n^{(r)} - R\| \le \frac{C}{n^r} \quad ,$$

Where c and C are constants that depend only on one another.

Lemma 2.2. [11]. Let $f \in L_{p,\wp}(X^d)$, $1 \le p < \infty$ then there are constants $c, C > o$ such that

$$\|\tilde{f}\|_{L_{p,\wp}(X^d)} \le c \|f\|_{L_{p,\wp}(X^d)} \qquad \text{and}$$

$$\|S_n(\cdot, f)\|_{L_{p,\wp}(X^d)} \le C \|f\|_{L_{p,\wp}(X^d)} \qquad \text{for } n = 1,2, \dots$$

Lemma 2.3. [11]. If $k = 1,2,3, \dots$ and $f \in {L_{p,\wp}}^{2k}(X^d)$, $1 \le p < \infty$, then

$\omega_k(f,\delta)_{L_{p,\wp}(X^d)} \le c\delta^{2k}\|f^{2k}\|_{L_{p,\wp}(X^d)} \quad , \ \delta > 0.$

With some constant $c > 0$.

Lemma 2.4. Let $p_n \in \mathbb{p}_n$, and let $r \in \mathbb{R}^+$ then there exists a constant $c > 0$ independent of n and such that

$$\|p_n{}^{(r)}\|_{L_{p,\wp}(X^d)} \le cn^r\|p_n\|_{L_{p,\wp}(X^d)} \quad .$$

Proof: we can assume that $\|p_n\|_{L_{p,\wp}(X^d)} = 1$

Since
$p_n = \sum_{k=0}^{n}\big(A_k(x) + A_{-k}(x)\big)$

We get

$$\frac{\tilde{p}}{n^r} = \sum_{k=0}^{n}\left[\big(A_k(x) - A_{-k}(x)\big)/n^r\right]$$

And

$$\frac{p_n^{(r)}}{n^r} = i^r\sum_{k=1}^{n} k^r\left[\big(A_k(x) - A_{-k}(x)\big)/n^r\right]$$

From lemma 2.2 we have

$$\|R_n{}^{(\propto)}(f,x)\|_{L_{p,\wp}(X^d)} \le c\|f\|_{L_{p,\wp}(X^d)} \quad ,n = 1,2,3,\dots \ ,x \in X^d \ ,f \in L_{p,\wp}(X^d)$$

and $\|\tilde{f}\|_{L_{p,\wp}(X^d)} \le c\|f\|_{L_{p,\wp}(X^d)}$.

Constructed on that, we have

$$\left\|R_n{}^{\langle r\rangle}\left(\frac{\tilde{p}_n}{n^r}\right)\right\|_{L_{p,\wp}(X^d)} \le \frac{c}{n^r}\|\tilde{p}_n\|_{L_{p,\wp}(X^d)} \le \frac{c}{n^r}\|p_n\|_{L_{p,\wp}(X^d)} = \frac{c}{n^r}$$

By using lemma 2.1 (with = 0) to series

$$\sum_{k=1}^{n}\left[\big(A_k(x) - A_{-k}(x)\big)/n^r\right] + 0 + 0 + \cdots + 0 + \cdots$$

$$\sum_{k=1}^{n} k^r\left[\big(A_k(x) - A_{-k}(x)\big)/n^r\right] + 0 + 0 + \cdots + 0 + \cdots$$

We obtain

$$\left\|\sum_{k=1}^{n}\left(1 - \left(\frac{k}{n+1}\right)^r\right)k^r\left[\big(A_k(x) - A_{-k}(x)\big)/n^r\right]\right\|_{L_{p,\wp}(X^d)} \le c$$

Namely

$$\left\|R_n^{\langle r\rangle}\left(\frac{p_n^r}{n^r}\right)\right\|_{L_{p,\wp}(X^d)} = \left\|i^r\sum_{k=1}^{n}\left(1 - \left(\frac{k}{n+1}\right)^r\right)k^r\left[\big(A_k(x) - A_{-k}(x)\big)/n^r\right]\right\|_{L_{p,\wp}(X^d)}$$

$$= \left\|\sum_{k=1}^{n}\left(1 - \left(\frac{k}{n+1}\right)^r\right)k^r\left[\big(A_k(x) - A_{-k}(x)\big)/n^r\right]\right\|_{L_{p,\wp}(X^d)} \le c_*$$

Since $R_n^{\langle r\rangle}(cf) = cR_n^{\langle r\rangle}(f)$ for every real c it follows from relation (10) in [13] and the last inequality that

$$\|p_n^{(r)}\|_{L_{p,\wp}(X^d)} = \|\circledast_n^{\langle r\rangle}(p_n^{(r)})\|_{L_{p,\wp}(X^d)} = n^r\left\|\frac{1}{n^r}\circledast_n^{\langle r\rangle}(p_n^{(r)})\right\|_{L_{p,\wp}(X^d)}$$

$$= n^r\left\|\frac{1}{n^r}\circledast_n^{\langle r\rangle}\left(\frac{p_n^{(r)}}{n^r}\right)\right\|_{L_{p,\wp}(X^d)} \le c_*n^r = c_*n^r\|p_n\|_{L_{p,\wp}(X^d)}$$

Hence

$$\|p_n^{(r)}\|_{L_{p,\wp}(X^d)} \le cn^r\|p_n\|_{L_{p,\wp}(X^d)} \quad .$$

Lemma 2.5. Let $p_n \in \mathbb{p}_n$ be the best approximation of, $\propto \in \mathbb{R}^+$ and $n = 0,1,2,\dots$ then

$$\omega_k\left(p_n, \frac{1}{n+1}\right)_{L_{p,\wp}(X^d)} \le \frac{c}{(n+1)^{\propto}}\|p_n^{(\propto)}\|_{L_{p,\wp}(X^d)}$$

Where the constant $c > 0$ depends only on α .

Proof: First, we prove that if $0 < \alpha < \beta$, $\propto, \beta \in \mathbb{R}^+$ then

$$\omega_\beta(f,\cdot)_{L_{p,\wp}(X^d)} \le c\omega_\propto(f,\cdot)_{L_{p,\wp}(X^d)} \qquad (1)$$

It is easily seen that if $\alpha \le \beta$, $\alpha, \beta \in \mathbb{Z}^+$ then

$$\omega_\beta(f,\cdot)_{L_{p,\wp}(X^d)} \le c(\propto, \beta, p)\, \omega_\propto(f,\cdot)_{L_{p,\wp}(X^d)} \qquad (2)$$

We now assume that $0 < \alpha < \beta < 1$. in this case setting

$\Phi(x) := \sigma_h^\propto f(x)$, we get

$$\sigma_h^{\beta-\propto}\Phi(x) = \sum_{j=0}^{\infty}(-1)^j \binom{\beta-\alpha}{j}\frac{1}{h^j}\int_{\frac{-h}{2}}^{\frac{h}{2}}\dots\int_{\frac{-h}{2}}^{\frac{h}{2}}\Phi(x+u_1+\cdots+u_j)du_1\dots du_j$$

$$= \sum_{j=0}^{\infty}(-1)^j \binom{\beta-\alpha}{j}\frac{1}{h^j}\int_{\frac{-h}{2}}^{\frac{h}{2}}\dots\int_{\frac{-h}{2}}^{\frac{h}{2}}\left[\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\frac{1}{h^k}\int_{\frac{-h}{2}}^{\frac{h}{2}}\dots\int_{\frac{-h}{2}}^{\frac{h}{2}} f(x+u_1+\cdots+u_j+u_{j+1}+\cdots+u_{j+k})du_1\dots du_j du_{j+1}\dots du_{j+k}\right]$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{\beta-\alpha}{j}\binom{\alpha}{k}\left[\frac{1}{h^{j+k}}\int_{\frac{-h}{2}}^{\frac{h}{2}}\dots\int_{\frac{-h}{2}}^{\frac{h}{2}} f(x+u_1+\cdots+u_{j+k})du_1\dots du_{j+k}\right]$$

$$= \sum_{v=0}^{\infty}(-1)^v\binom{\beta}{v}\frac{1}{h^v}\int_{\frac{-h}{2}}^{\frac{h}{2}}\dots\int_{\frac{-h}{2}}^{\frac{h}{2}} f(x+u_1+\cdots+u_v)du_1\dots du_v = \sigma_h^\beta f(x)$$

Then

$$\|\sigma_h^\beta f(x)\|_{L_{p,\wp}(X^d)} = \|\sigma_h^{\beta-\alpha}\Phi(x)\|_{L_{p,\wp}(X^d)} \le c\|\sigma_h^\alpha f(x)\|_{L_{p,\wp}(X^d)}$$

And

$$\omega_\beta(f,\cdot)_{L_{p,\wp}(X^d)} \le c\,\omega_\alpha(f,\cdot)_{L_{p,\wp}(X^d)} \qquad (3)$$

Note that if $r_1, r_2 \in \mathbb{Z}^+$ and $\alpha_1, \beta_1 \in (0,1)$, then, taking $\alpha := r_1 + \propto_1$ and $\beta = r_2 + \beta_1$ for the remaining cases $r_1 = r_2$, $\alpha_1 < \beta_1$, or $r_1 < r_2$, $\alpha_1 = \beta_1$, or $r_1 < r_2$, $\alpha_1 < \beta_1$ and using (2) and (3), one can easily verify that the required inequality (1) is true.

Using relation (1), lemma 2.3, and lemma 2.4, we get:

$$\omega_\alpha\left(p_n, \frac{1}{n+1}\right)_{L_{p,\wp}(X^d)} \le c\,\omega_{[\propto]}\left(p_n, \frac{1}{n+1}\right)_{L_{p,\wp}(X^d)}$$

$$\le c\left(\frac{1}{n+1}\right)^{2[\alpha]}\left\|p_n^{(2[\alpha])}\right\|_{L_{p,\wp}(X^d)}$$

$$\le \frac{c}{(n+1)^{2[\alpha]}}(n+1)^{[\alpha]-(\alpha-[\alpha])}\left\|p_n^{(\alpha)}\right\|_{L_{p,\wp}(X^d)} = \frac{c}{(n+1)^\alpha}\left\|p_n^{(\alpha)}\right\|_{L_{p,\wp}(X^d)} .$$

2.2. MAIN RESULTS

This section of our work can be formulated as follows:

Theorem 3.1.

Let $f \in L_{p,\wp}(X^d)$, $1 \le \mathrm{p} < \infty$, $p_n \in \mathbb{p}_n$ be the best approximation of f and $r, k \in \mathbb{R}^+$ then

1. $\left\|p_n^{(r)}(f)\right\|_{L_{p,\wp}(X^d)} = O(n^{r-k})$, $o < k < r$

$\Leftrightarrow$

2. $E_n(f)_{L_{p,\wp}(X^d)} = O(n^{-k})$

Proof. We suppose that

$$E_n(f)_{L_{p,\wp}(X^d)} = \|f - p_n(f)\|_{L_{p,\wp}(X^d)} = O(n^{-k}) \quad ,k > o \qquad (4)$$

The identity

$$p_n^{(r)}(x) = p_0^{(r)}(x) + \sum_{v=0}^{n-1}\left\{p_{v+1}^{(r)}(x) - p_v^{(r)}(x)\right\} \quad . \qquad (5)$$

From lemma 2.4, we get

$$\left\|p_n^{(r)}(f)\right\|_{L_{p,\wp}(X^d)} \le c_1 n^r \|p_n(f)\|_{L_{p,\wp}(X^d)}$$

Now combining (4), (5), and the last relation, we obtain

$$\left\|p_n^{(r)}(f)\right\|_{L_{p,\wp}(X^d)} \le c_2 c_3 n^r n^{-k} \le c_4 n^{r-k}$$

Now we suppose that

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$$\|p_n^{(r)}(f)\|_{L_{p,\wp}(X^d)} = O(n^{r-k}) \quad (6)$$

Considering [11, lemma 2], direct theorem, and (6) we get

$$\|p_{2n}(f) - p_n(p_{2n}(f))\|_{L_{p,\wp}(X^d)}$$

$$\le \|f - p_{2n}(f)\|_{L_{p,\wp}(X^d)} + \|f - p_n(p_{2n}(f))\|_{L_{p,\wp}(X^d)}$$

$$\le c_5 n^{-r} \|p_n^{(r)}(f)\|_{L_{p,\wp}(X^d)}$$

$$\le c_6 n^{-k} \quad (7)$$

Note that on the other hand, since $p_n(p_{2n}(f))$ is a polynomial of order n, the following inequality holds:

$$\|p_{2n}(f) - p_n(p_{2n}(f))\|_{L_{p,\wp}(X^d)} = \|f - p_n(p_{2n}(f)) - (f - p_{2n}(f))\|_{L_{p,\wp}(X^d)}$$

$$\ge \|f - p_n(p_{2n}(f))\|_{L_{p,\wp}(X^d)} - \|f - p_{2n}(f)\|_{L_{p,\wp}(X^d)}$$

$$\ge E_n(f)_{L_{p,\wp}(X^d)} - E_{2n}(f)_{L_{p,\wp}(X^d)}$$

$$\ge 0 \quad (8)$$

The use of (7) and (8) gives us

$$0 < E_n(f)_{L_{p,\wp}(X^d)} - E_{2n}(f)_{L_{p,\wp}(X^d)} \le c_7 n^{-k} \quad (9)$$

Since $E_n(f)_{L_{p,\wp}(X^d)} \longrightarrow 0$ from the inequality (9) we get to conclude that

$$\sum_{\ell=n_0}^{\infty} \left\{E_{2^\ell}(f)_{L_{p,\wp}(X^d)} - E_{2^{\ell+1}}(f)_{L_{p,\wp}(X^d)}\right\} \le c_8 \sum_{\ell=n_0}^{\infty} 2^{-\ell k}$$

Then from the last inequality, we conclude that

$$E_{2^{n_0}}(f)_{L_{p,\wp}(X^d)} \le c_9 2^{-n_0 k} \quad (10)$$

It is clear that inequality (10) is equivalent to

$$E_n(f)_{L_{p,\wp}(X^d)} \le c_8(n^{-k}) .$$

Theorem 3.2. Let $f \in L_{p,\wp}(X^d)$, $1 \le p < \infty$, $d \in \mathbb{N}$. then the following statements are equivalent :

1. $\|\mathcal{X}_n(f,\cdot)\|_{L_{p,\wp}(X^d)} = O\left(n\omega\left(\frac{1}{n}\right)\right)_{L_{p,\wp}(X^d)}$ as $n \to \infty$
2. $\|f - \mathcal{F}_n(f,\cdot)\|_{L_{p,\wp}(X^d)} = O\left(\omega\left(f, \frac{1}{n}\right)\right)_{L_{p,\wp}(X^d)}$.

Proof:

$$\|f - \mathcal{F}_n(f,\cdot)\|_{L_{p,\wp}(X^d)}$$

$$= \frac{1}{(n+1)^p}\sum_{i=1}^{n} \|iG_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)} + \sum_{i=u+1}^{\infty} \frac{1}{i^p}\left(i\|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right) \quad (11)$$

For a fixed integer $N > n$, we have

$$\frac{1}{(n+1)^p}\sum_{i=1}^{n} \|iG_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)} + \sum_{i=u+1}^{N} \frac{1}{i^p}\left(i\|G(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right)$$

$$= \frac{1}{(n+1)^p}\sum_{i=1}^{n} \|iG_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)} - \frac{1}{n^p}\sum_{i=1}^{n}\left(i\|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right)$$

$$+ \frac{1}{N^p}\sum_{j=1}^{N}\left(j\|G_j(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right) + \sum_{i=u+1}^{N}\left(\frac{1}{(i-1)^p} - \frac{1}{i^p}\right)\sum_{j=1}^{i-1}\left(i\|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right)$$

$$\le p\sum_{i=n}^{N} \frac{1}{i^{p+1}}\|\mathcal{X}_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)} + \frac{1}{N^p}\sum_{j=1}^{N}\left(j\|G_j(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right) \quad (12)$$

So

$$\frac{1}{N^p}\sum_{j=1}^{N}\left(j\|G_j(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right) \longrightarrow 0 \quad as\ N \longrightarrow \infty$$

We obtain, for any N

$$\sum_{i=n}^{N} \frac{1}{i^{p+1}}\|\mathcal{X}_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)} \le O(1)\sum_{i=n}^{\infty}\frac{1}{i}\omega^p\left(\frac{1}{i}\right) = O\left(\omega^p\left(\frac{1}{i}\right)\right) \text{ , as } n \longrightarrow \infty \quad (13)$$

Implies $1 \Longrightarrow 2$.

We need to show $2 \Longrightarrow 1$

$$\|\mathcal{X}_i(f,\cdot)\|_{L_{p,\wp}(X^d)} = \sum_{i=1}^{n}\left(i^p\|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right)$$

$$\leq (n+1)^p \left\| f - \sum_{i=0}^{n} \left(1 - \frac{i}{n+1}\right) G(f,\cdot) \right\|^p_{L_{p,\wp}(X^d)}$$

$$= O\left(n\omega\left(\frac{1}{n}\right)\right) \quad , \text{ as } n \longrightarrow \infty$$

We obtain $2 \Longrightarrow 1$

The proof is complete.

Theorem 3.3 :

Let $f \in L_{p,\wp}(X^d)$, $1 \leq p < \infty$, $d \in \mathbb{N}$. Then

1. $\|f - \mathcal{K}(f)\|_{L_{p,\wp}(X^d)} = O(\delta)^{k-1}\omega(f,\delta)$, $\delta \longrightarrow 0$
2. $\left\|\mathcal{M}(f^{(k)},\delta,\cdot)\right\|_{L_{p,\wp}(X^d)} = O\left(\frac{1}{\delta}\omega(f,\delta)\right)$, $\delta \longrightarrow 0$

Proof: $1 \Longrightarrow 2$

We have

$$\|f\|_{L_{p,\wp}(X^d)} = \left(\sum_{i=1}^{\infty} \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}\right)^{\frac{1}{p}} .$$

Since

$$\sum_{k=0}^{i} \binom{i}{k} (1-\delta)^k \delta^{i-k} = \left((1-\delta)+\delta\right)^i = 1 \quad , i = 0,1,2,\ldots \quad (14)$$

$$(1-\delta)^{jp} \sum_{i=j}^{\infty} \binom{i}{j}^p \delta^{(i-j)} \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}$$

$$\leq \sum_{i=j}^{\infty} \left(\sum_{k=j}^{i} \binom{i}{k} (1-\delta)^k \delta^{i-j} \right)^p \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}$$

$$= \sum_{i=j}^{\infty} \left|1 - \beta_{i,j}(\delta)\right|^p \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}$$

$$= \left\|f - \mathcal{A}_{\delta,j}(f)\right\|^p_{L_{p,\wp}(X^d)} .$$

On the other hand

$$\sum_{i=j}^{\infty} \binom{i}{j}^p \delta^{(i-j)p} \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}$$

$$= \frac{1}{(j!)^p} \left\| \frac{\partial^j}{\partial \delta^j} B(f,\delta,\cdot) \right\|_{L_{p,\wp}(X^d)}$$

From these relations and (⁎), we obtain

$$\left\|B(f^{(r)},\delta,\cdot)\right\|_{L_{p,\wp}(X^d)} \leq j!\,(1-\delta)^{-j} \left\|f - \mathcal{A}_{\delta,j}(f)\right\|_{L_{p,\wp}(X^d)}$$

$$= O\left(\frac{1}{\delta}\omega(\delta)\right)$$

For any $m > j$ and $0 \leq \delta < 1$, then

$$\delta^{(m-j)p} \frac{1}{(j!)^p} \left\|\mathcal{K}(f^{(r)},\cdot)\right\|^p_{L_{p,\wp}(X^d)}$$

$$= \delta^{(m-j)p} \sum_{i=j}^{m} \binom{i}{j}^p \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}$$

$$\leq \sum_{i=j}^{\infty} \binom{i}{j}^p \delta^{(i-j)p} \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}$$

$$= \frac{1}{(j!)^p} \left\| \frac{\partial^j}{\partial \delta^j} B(f,\delta,\cdot) \right\|^p_{L_{p,\wp}(X^d)}$$

We set $\delta = 1 - \frac{1}{m}$ and from the above relation, we obtain

$$\left\|\mathcal{K}(f^{(r)},\cdot)\right\|_{L_{p,\wp}(X^d)} \leq O(1)\left(1 - \frac{1}{m}\right)^{-m} \frac{1}{m}\omega\left(\frac{1}{m}\right)$$

$$= O\left(m\omega\left(\frac{1}{m}\right)\right) \quad , \text{as } m \longrightarrow \infty \quad (15)$$

By using theorem 3.2 , we can conclude that

$$\left\|f^{(r-1)} - \mathcal{K}(f^{(r-1)},\cdot)\right\|_{L_{p,\wp}(X^d)} = O\left(\omega\left(f,\frac{1}{m}\right)\right) \quad , m \longrightarrow \infty \quad (16)$$

Now, to prove that $2 \Longrightarrow 1$.

From the relation (14) and for any $0 \leq \delta \leq 1$

$$\sum_{k=j}^{i} \binom{i}{k} (1-\delta)^k \delta^{i-k} \leq 1, \quad i \geq j$$

We have

$$\left\|f - \mathcal{A}_{\delta,j}(f)\right\|^p_{L_{p,\wp}(X^d)} = \sum_{i=j}^{\infty} \left(\sum_{k=j}^{i} \binom{i}{k} (1-\delta)^k \delta^{i-k} \right)^p \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)}$$

$$\leq \|f\|^p_{L_{p,\wp}(X^d)} < \infty$$

For any $\epsilon > 0$, $\exists\, m_0 \ \ni \ \forall\, m > m_0$ and $0 \leq \delta \leq 1$,

$$\left\|f - \mathcal{A}_{\delta,j}(f)\right\|^p_{L_{p,\wp}(X^d)} = \sum_{i=j}^{m} \left(\sum_{k=j}^{i} \binom{i}{k} (1-\delta)^k \delta^{i-k} \right)^p \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)} + \epsilon \quad (17)$$

Now, let us show that for any $i \geq j$, then

$$\sum_{k=j}^{i} \binom{i}{k} (1-\delta)^k \delta^{i-k} \leq \binom{i}{j} (1-\delta)^j \quad \text{for all } 0 \leq \delta \leq 1 \quad (18)$$

We set $\ell = i - j$ and

$$a_k = \frac{\binom{i}{k+j}}{\binom{i}{j}} \quad , \quad k = 0,1,2,\ldots,\ell$$

The inequality (18) is true if and only if

$$\sum_{k=0}^{\ell} a_k (1-\delta)^k \delta^{\ell-k} \leq 1 \quad \text{for all} \quad 0 \leq \delta \leq 1$$

Since

$$a_k = \frac{i!}{(k+j)!\,(i-k-j)!} \cdot \frac{j!\,(i-j)!}{i!} \leq \frac{(i-j)!}{k!\,(i-j-k)!} = \binom{\ell}{k}$$

By using (17) and (18), we obtain

$$\|f - \mathcal{A}_{\delta,j}(f)\|^p_{L_{p,\wp}(X^d)} \leq (1-\delta)^{jp} \sum_{i=j}^{m} \binom{i}{k}^p \|G_i(f,\cdot)\|^p_{L_{p,\wp}(X^d)} + \epsilon$$

$$= \frac{(1-\delta)^{jp}}{(j!)^p} \|\mathcal{K}(f^{(r)},\cdot)\|^p_{L_{p,\wp}(X^d)} + \epsilon$$

$$\leq \frac{(1-\delta)^{jp}(n+1)}{(j!)^p} \|f^{(r-1)} - \mathcal{K}(f^{(r-1)},\cdot)\|^p_{L_{p,\wp}(X^d)} + \epsilon$$

From (16) , we obtain

$$\|f - \mathcal{A}_{\delta,j}(f)\|^p_{L_{p,\wp}(X^d)} = O(1)(\delta)^{jp} \|f^{(r-1)} - \mathcal{K}(f^{(r-1)},\cdot)\|^p_{L_{p,\wp}(X^d)} + \epsilon$$

$$= O(1)(\delta)^{(jp-1)p}\omega(f,\delta) + \epsilon \quad \text{as} \quad \delta \longrightarrow 0$$

We have $2 \Longrightarrow 1$

3. CONCLUSIONS

We understand from this functional mixture between the two topics, the importance of finding the best multi-approximation of unbounded function by algebraic polynomials in weighted space and the derivative of these functions. From now on, more research exploration continues For more useful to finding best multi- approximate of unbounded functions by some positive linear operators.

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أفضل تقريب متعدد للدوال الغير مقيدة بواسطة مقياس النعومة

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الخلاصة:

سنعرض تخمين درجة افضل تقريب متعدد للدوال الغير مقيدة على الفترة $[-1,1]^d$ بواسطة متعددات الحدود الجبرية في فضاء الوزن وكذلك تم دراسة العلاقة بين افضل تقريب لمشتقات الدوال في فضاء الوزن وافضل تقريب للدوال الفيرة المقيدة في نفس الفضاء.