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**Applying the duo and essential properties on extending modules**

A Thesis Submitted to the Council of the College of Education for Pure Sciences,

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**By**

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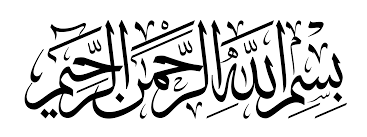
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وَلَقَد اْتَينَا دَاوُود وَسُلَيمَانَ عِلمًا وقَالَا الحَمدُ للهِ الّذِي فَضَّلَنَا عَلَى كَثِيرٍ مِن عِبَادِهِ المُؤمِنينَ (١٥)

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*Abdulsalam F. Talak*

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**Dedication**

*To my kind father: my role model and My idol in Life, he is the one who taught me how to live with dignity and honor.*

*To my compassionate mother: She is the epic of love and the joy of a lifetime, I can’t find words that give her due, an example of dedication and giving.*

*To my brothers: my arms and share my joy, and sorrows.*

*To my university, my teachers and colleagues.*

*I dedicate my scientific research to you*

*Abdulsalam F. Talak*

*2021*

**Supervisor Certification**

*I certify this thesis entitled "****Applying the duo and essential properties on extending modules"*** *submitted by the student "****Abdulsalam Faeq Talak****", has prepared under my supervision at the University of Anbar, College of Education for Pure Sciences / Department of Mathematics, as a partial fulfillment of the requirements for the degree of master in Mathematics.*

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**List of Symbols**

|  |  |
| --- | --- |
|  | Module |
|  | Submodule |
| R | Ring |
|  | Extending module |
|  | Lifting module |
| P.I.D | Principle ideal domain |
| P-(S.P) | Prime and semiprime submodule |
| Q-Injective | Quasi injective |
| P-Injective | Pseudo injective |
|  | essential submodule |
|  | small submodule |
| DCC | descending chain condition |
| ACC | ascending chain condition |
| QF | Quasi Frobenius |

|  |  |
| --- | --- |
|  | Annihilator of a module |
|  | Radical of a module |
|  | Socle of a module |
|  | Heart submodule |
|  | Set of all multiplication modules |
|  | Set notation |
|  | Torsion of element |
|  | is called torsion module |
|  | is called torsion free module |
|  | The residual of *L* by in |
|  | Direct multiplication |

**Thesis Outline**



**ABSTRACT**

A module is called -module (extending) if for all there exists a direct summand *B* is an essential of in . The main goal is to get a module namely -module (extending). Meaning we look for conditions and algebraic structures that lead to obtaining the submodules to be essential, and thus we obtain the -module (extending). The tools that enable us to get this goal are semi simple module, multiplication module and injective module. The second goal is to obtain a generalization of -module using the following tools: duo submodule, prime and semi prime submodules (P-(S.P) submodules) and quasi-injective submodule (Q-injective). Finally; we can say that all the results in this work depended on the concept of submodules of the module and the ring R in this thesis stands for a commutative ring with identity.

**INTRODUCTION**

In (1988); Mahmoud A. Kamal And Bruno J. Muller carried out a study about extending modules over commutative domains where they defined the extending module as follows: a module is extending or satisfies the property if every complement submodule is a direct summand. In (1990); Mohamed ,S.H. and Muller,B.J. introduced the concepts of extending module where they defined the extending module as follows: an R-module is said to be extending if every closed submodule of is a direct summand. In (2009); Fatih Karabacak made a generalization of extending modules where he defined the extending module as follow: a module is called an extending module if every submodule is essential in a direct summand of . In this work we study -module in detail where we look for tools to get an essential submodule, thus to obtain -module. Throughout this thesis every submodule is a direct summand. Also we will introduce a generalization of -module.

A submodule of a module is essential in case for every submodule . This concept was first introduced by F. Kasch.

The main goal of this thesis is to give a comprehensive investigation of the properties, characterization and some examples of -module. So we must get an essential submodule in order to get -module.

This thesis consists of three chapters. Each chapter contains three sections.

In chapter one, we recall some fundamental definitions, remarks and propositions. Section one talks about the subject of -module, where a module is said to be lifting or satisfies (-module), if for any , a direct summand *K* is a coessential submodule of in . Also, a module if satisfies -module so it is -module. Section two talks about the subject of essential submodule and some tools to help me get an essential submodule. Section three explains the subject of duo submodule and some definitions, where a submodule is called fully invariant if is contained in for every *R*-endomorphism *f* of . Also, the right *R*-module is called a duo module if every submodule of is fully invariant. Those definitions are introduced by A.C.zcan, A.Harmanci.

In section one of chapter two, we present some relationship between semi simple module and essential property, where any module is semi simple if it is a sum of simple module. Among the results which we prove in this section are:

1. Let a module be a semisimple indecomposable. If , then .
2. Let a module be an indecomposable over a ring *R*. If is injective, then is essential. Moreover; if is a simple and flat, then .

In section two of chapter two, we study relationship between multiplication module and essential property where any module is multiplication if and I is an ideal of *R*. Among the results which we prove in this section are:

1. If is a multiplication *R*-module and let be one maximal submodule of , then .
2. Let be a module over local ring *R*. If is cyclic module and has only one maximal submodule. Then is -module.

In section three of chapter two, we present the relationship between injective module and essential property. An *R*-module is called injective if for every monomorphism ةخtending) zation of wudyanother property of any submoduleand homomorphism there exists a homomorphism . Among the results which we prove in this section are:

1. Let a module be an injective over a ring *R*. If is indecomposable, then any .
2. If is split and ; is injective R-module.

In section one of chapter three, we study a generalization of -module through the duo module. Among the results which we prove in this section are:

1. Consider as a submodule of over ring R. If and , , then is a duo submodule of .
2. Let be a -module. If has a socle not equal zero, then is duo--module.

In section two of chapter three, we introduce a generalization of -module through the prime and semiprime P-(S.P) submodules. Where any proper submodule is prime (briefly P-submodule) when if , , , then and is said to be semiprime (S.P-submodule) if and whenever , and , then . Among the results which we prove in this section are:



1. Let be a -cyclic R-module If is a prime module and has fully invariant property, then is a P-duo--module.
2. Every semi prime submodule of a multiplication -module is submodules and so is S.P-duo--module.

In section three of chapter three,we present a generalization for -module through quasi-injective submodule, were a module is called Quasi-injective (briefly Q-injective) if , and *R*-homomorphism can be extended to an *R*-homomorphism of , and any *R*-module is said to be pseudo injective (briefly P-injective) if and only if every *R*-isomorphism of each submodule of into can be extended to an *R*-endomorphism of . Also, we have any Q-injective is P-injective. Among the results which we prove in this section are:

1. Let be an R-module over *P.I.D*. If is pseudo-injective module, so it is a Q-injective.
2. If and in Pseudo-injective module , then is Q-injective.

Finally, the ring *R* in this thesis is commutative and identity. Also, a module in chapter two is an indecomposable module.

|  |
| --- |
| **Chapter One**  **Preliminaries** |

**INTRODUCTION**

Let *R* be a commutative ring with identity, and let be a unitary *R*-module. A module is called extending (-module) if for any submodule in , there exists a direct summand *B* in is an essential of [34]. In this chapter, we summarize some definitions, remarks and properties about -module also, some definitions and notes related to -module with some definitions and notes to generalization of -module (Duo submodule, P-(S.P) submodules, Quasi-Injective submodule).

**§1.1: Basic Properties of -Module**

This section contains some known definitions and related concepts about the subject of -Module.

**Definition (1.1.1): [42]**

A module over a ring *R* is an additive commutative group together with mapping

Is called module, for which we have and then:

1. .
2. .
3. .
4. .

**Definition (1.1.2): [42]**

Let be a module over a ring *R*. A subgroup of is called a submodule of if then .

**Remark (1.1.3): [29]**

Every *R*-module has at least two submodules .

**Definition (1.1.4): [42]**

A module is called simple if and has only submodules .

**Example:** A Z-module is a simple module.

**Definition (1.1.5): [15]**

Any module is called semisimple if it is a sum of simple modules.

**Remark (1.1.6): [29]**

1. Every simple module is semisimple.
2. If is semisimple, then every is semi simple.

**Definition (1.1.7): [33]**

Let were be a module. *A* is called a small submodule of (denoted by ) if for any .

**Example:** For any module we have .

**Definition (1.1.8): [42]**

A submodule B of an R-module is called large submodule in if: , we have .

**Definition (1.1.9): [14]**

A module is distributive (D-module) if for any submodules *A, B, C* we have:

.

**Definition (1.1.10): [33]**

A module is said to be hollow if every non-trivial submodule is small in .

**Definition (1.1.11): [34]**

A module is called extending (-module) if a direct summand *B* is an essential of in .

**Definition (1.1.12): [33]**

A module is called lifting or satisfies (-module), if a direct summand *K* is a coessential submodule of in .

**Remark (1.1.13): [34]**

1. Every injective module is a -module.
2. Every projective module is a -module.

**Definition (1.1.14): [17]**

If *X* is a non-empty subset of *R*, then we denote its annihilator in and define it to be the set of elements such that .

.

**Definition (1.1.15): [29]**

A module is called faithful if and only if ⟹.

**Definition (1.1.16): [31]**

A non trivial submodule is called maximal if and only if there is no proper submodule of different from containing .

**Remark (1.1.17) : [31]**

A submodule is maximal in if and only if is simple *R*-module.

**Definition (1.1.18): [30]**

A submodule is semimaximal if and only if is a semisimple *R*-module.

**Remark (1.1.19): [30]**

Every maximal submodule is semimaximal but the converse is not true.

For example: In *Z*-module *Z*, a submodule *6Z* is semi-maximal but not maximal.

**Definition (1.1.20): [29]**

A module is called finitely generatedif and only if there exists a finite generating set.

**Definition (1.1.21): [19]**

A module is indecomposable if and only if and are the only direct summands of .

**§1.2: General View of Essential Submodule**

This section contains some known definitions and related concepts about the subject of essential submodule.

**Definition (1.2.1): [19]**

A submodule is essential in case or if whenever , then .

**Example:** For every *R*-module , we have .

**Definition (1.2.2): [7]**

A submodule of an *R*-module is called a direct summand of in case there is a submodule K of with .

**Definition (1.2.3): [7]**

An *R*-module is called Artinian if satisfies the descending chain condition (DCC) on submodule of . We mean a descending sequence of submodule of . So there exists

.

**Definition (1.2.4): [7]**

An *R*-module is called Noetherian if satisfies the ascending chain condition (ACC) on submodule of . We mean an increasing sequence of submodule of . So there exists

.

**Definition (1.2.5): [29]**

A ring *R* is called semisimple if it is a direct sum of minimal ideals.

**Remark (1.2.6):** If is semisimple. Then is the only small submodule of and is the only large submodule of .

**Definition (1.2.7): [29]**

An *R*-module *R* is called flat module if for every monomorphism

is also monomorphism.

**Definition (1.2.8): [43]**

A submodule of is said to be irreducible if for submodules *A* and *B* of implies that either .

**Definition (1.2.9): [6]**

Let be an *R*-module. Then, the radical of denoted by is defined to be the intersection of the maximal submodules of .

**Definition (1.2.10): [9]**

Any module is called multiplication if , such that and I is an ideal of *R*.

**Definition (1.2.11): [24]**

A module is an -multiplication module if for every submodule .

**Definition (1.2.12): [23]**

An element *x* of a multiplication *R*-module is called nilpotent if , for some positive integer *n*.

**Definition (1.2.13): [36]**

Let be a module and in , is called perfect in if for any index set *A*, the sum is semi perfect in .

**Definition (1.2.14): [13]**

A module over a ring *R* is said to have the extending property if every submodule of is contained as an essential submodule in a direct summand of .

**Definition (1.2.15): [28]**

A submodule of is called complement if has no proper essential extension.

**Definition (1.2.16): [29]**

Let be an R-module and 1, 2 are submodules of . Then we say 2 is an addition complement of 1 in if , and 2 is minimal in the sum of 1 and 2, and equal to .

**Definition (1.2.17): [19]**

An *R*-module is called cyclic if it is generated by single element.

**Example:** 2Z as a Z-module is a cyclic module.

**§1.3: Basic Properties of Duo Submodule**

This section contains some known definitions and related concepts about the subject of (DUO) submodule on -module.

**Definition (1.3.1): [35]**

A submodule is called fully invariant if is contained in for every *R*-endomorphism *f* of .

**Remark (1.3.2): [35]**

It is clear that a submodules and of are fully invariant of .

**Definition (1.3.3): [35]**

An *R*-module is called a duo module if every submodule of is fully invariant.

**Remark (1.3.4): [35]**

If is a simple module, then it is clear that is duo module.

**Definition (1.3.5): [28]**

The heart submodule of an *R*-module denoted by is the intersection of all non-zero submodules of .

**Remark (1.3.6): [28]**

is a minimal submodule contained in every non-zero submodule when is non-zero

.

**Definition (1.3.7): [28]**

Let be a module and is called h-closed if .

**Definition (1.3.8): [17]**

The socle of an R-module is the sum of all the simple submodules of

.

**Remark (1.3.9): [28]**

1. for any R-module .
2. when has a simple socle.

**Definition (1.3.10): [28]**

A submodule of an *R*-module which has no proper essential extension in is called a closed submodule of .

**Definition** **(1.3.11): [28]**

Any submodule of is called closed duo if has no proper essential extension and .

**Definition (1.3.12): [9]**

Let be an *R*-module and . A submodule is pure in if any finite system of equation over which is solvable in is also solvable in .

**Definition (1.3.13): [32]**

A module is called uniform if 1 and 2 are non-zero submodules of ; the intersection of any two non-zero submodules is nonzero, equivalently, is uniform if .

**Definition (1.3.14): [29]**

A module *F* is called a free module which satisfies the conditions:

1. *F* has a basis.
2. .

**Definition (1.3.15): [20]**

If *R* is an integral domain and is an *R*-module, then an element is called torsion element if there exists. So we define:

Note that:

1. If , then a module is called torsion-module.
2. If , then a module is called torsion-free-module.

**Definition (1.3.16): [14]**

A module has square-free-socle if its socle has at most one copy of each simple module.

**Definition (1.3.17): [23]**

Let be a multiplication module and . The residual of *L* by in is

**Definition (1.3.18): [40]**

A submodule is called stable if for each R-homomorphism implies , and an *R*-module is called fully stable in case every submodule of is stable.

**Definition (1.3.19): [23]**

Any proper submodule is prime (briefly P-submodule) when if , , , then and is said to be semiprime (S.P-submodule) if and whenever and for *n* where , then .



Recall that if any submodule is a P-submodule, this means is prime module.

**Definition (1.3.20): [38]**

A Dedekind domain is a commutative domain with property that every non-zero fractional ideal is invertible.

**Definition (1.3.21): [19]**

A ring *R* is said to be local ring if  is a division ring, or a ring *R* is local if it has a unique maximal ideal.

**Definition (1.3.22): [19]**

A ring *R* is called semi-local ring if is semisimple ring.

**Definition (1.3.23): [20]**

An R-module is divisible module if for all .

**Definition (1.3.24): [5]**

An R-module is nonsingular if with implies.

**Remark (1.3.25):**

1. If then is singular.
2. If then is nonsingular.

Such that .

**Definition (1.3.26): [29]**

Let R be a ring and let

Be a sequence of homomorphisms of right R-modules **,** finite or infinite on one or other or both sides, an exact sequence of the form

is called a short exact sequence.

**Definition (1.3.27): [29]**

An exact sequence A is called a split exact sequence if and only if for every subsequence of the form

is a direct summand of Ai.

**Definition (1.3.28): [38]**

An *R*-module P is a projective module if there exists an *R*-module Q such that is a free *R*-module.

**Definition (1.3.29): [38]**

An *R*-module is called injective if for every monomorphism

ةخtending) zation of wudyanother property of any submoduleand homomorphism there exists a homomorphism .

**Remark (1.3.30): [28]**

Every free module over a ring *R* with (identity) is projective.

**Lemma (1.3.31): [ Criterion] [16]**

Let *R* be a ring with (identity). A unitary *R*-module is injective if and only if ideal *L* of *R*, any *R*-module homomorphism may be extend to an *R*-module homomorphism .

**Definition (1.3.32): [27]**

A module is called self-p-injective if satisfy the following condition; every homomorphism from a projection invariant submodule of can be lifted to .

**Definition (1.3.33): [1]**

A ring *R* is (QF) quasi frobenius if R has (DCC) on right ideals and *R* is self-injective.

**Remark (1.3.34): [1]**

Every injective module over a ring *R* is projective and the converse is true if the ring *R* is quasi frobenius.

**Definition (1.3.35): [36]**

A module is called weakly injective in if for every finitely generated submodule of the -injective hull , is contained a submodule *Y* of such that .

**Definition (1.3.36): [11]**

A ring *R* is called hereditary if every ideal *I* of *R* is projective.

**Definition (1.3.37): [8]**

Any ring *R* is called *V*-ring if every simple *R*-module is injective.

**Definition (1.3.38): [22]**

A module is said to be Hopfian if every surjective endomorphism of is an isomorphism.

**Definition (1.3.39): [22]**

A submodule of is said to be non-Hopfian kernel for () if there exists an isomorphism of to .

**Definition (1.3.40): [22]**

A non-simple module is called anti-hopfian if every proper submodule of is a non-hopfian kernel (if there exists an isomorphism ).

**Definition (1.3.41): [26]**

A module is called Quasi-injective (briefly Q-injective) if , and *R*-homomorphism can be extended to an *R*-homomorphism of .

**Definition (1.3.42): [39]**

Any *R*-module is said to be pseudo injective (briefly P-injective) if and only if every *R*-isomorphism of each submodule of into can be extended to an *R*-endomorphism of .

**Remark (1.3.43): [25]**

Any Q-injective is P-injective but the converse is not true.

**Example (1.3.44):**

Let *R* be an algebra over having basis with the following multiplication table:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |
|  |  | 0 | 0 | 0 | 0 |  | 0 |
|  | 0 |  | 0 |  | 0 | 0 |  |
|  | 0 | 0 |  | 0 |  | 0 | 0 |
|  |  | 0 | 0 | 0 | 0 | 0 | 0 |
|  |  | 0 | 0 | 0 | 0 | 0 | 0 |
|  | 0 |  | 0 | 0 | 0 | 0 | 0 |
|  | 0 |  | 0 | 0 | 0 | 0 | 0 |

Then the right *R*-module is pseudo-injective but not quasi-injective.

|  |
| --- |
| **Chapter Two**  **Apply the Essential Property on -module** |

**INTRODUCTION**

This chapter introduces the concept of essential property, where a submodule of is called essential, if whenever , then for each submodule *L* of or a submodule of a module is essential in case for every submodule . Also, we will study some results, characteristics and properties of the essential submodule and the relationship between essential property and some modules, such as semi simple, multiplication and injective modules. Therefore, in this work every submodule is a direct summand. In the first part we will study the relationship between essential property and semi simple module on -module and will prove that if is a submodule of semi simple indecomposable R-module , then . In the next part we will study the relationship between essential property and multiplication module on -module and we will prove that if be one maximal submodule of a multiplication R-module , then . Finally we will study the relationship between essential property and injective module on -module and will prove that every injective indecomposable module over a ring R has .

**§2.1: Semi-Simple Module and Essential Property**

In this section, we present some relationships between semisimple *R*-module and essential property of any submodule of . We investigate every submodule of is a direct summand of , then is essential of and hence is essential -module. Also, we prove any module over a ring *R* is essential -module when *R* is Artinian ring and .But before that, we need some auxiliary results about the topics.

**Lemma (2.1.1):**

Let be an *R*-module such that every submodule of is a direct summand of . Then .

**Proof:**

Suppose that and is a finitely generated So is a maximal submodule. So where \* is a submodule and hence

.

Hence . If *K* is a maximal submodule in , then is simple. Thus is a simple submodule of (this means is simple submodule).

Recall that any module is indecomposable if and only if and are the only direct summands of .

**Lemma (2.1.2):**

Let be a semisimple indecomposable module. If is a submodule of , then .

**Proof:**

Assume that is a semisimple module. Suppose that . Therefore is a direct summand of (). Since is an indecomposable module, so . Hence . Thus .

**Remark (2.1.3):**

By Lemma **(2.1.1)** and Lemma **(2.1.2)**, if and is a semisimple indecomposable R-module, then .

Now we present a clear definition of the concept of semisimple module based on the submodules. See the following:

For a module , if the following conditions are hold, then is called semisimple module:

1. are simple submodules.
2. are simple submodules.
3. .
4. .

Now from the above four conditions, we say that *R* is a semisimple ring if and only if every module of *R* is semisimple module.

**Example (2.1.4):**

The {0} is a semisimple module such that are semisimples. Note that {0} is not simple module.

Recall that The socle of an R-module is the sum of all the simple submodules of .

**Remark (2.1.5):**

If , then is called semisimple module (i.e. a module is a semisimple if is the sum of all its simple submodules).

**Example (2.1.6):**

is a semisimple *Z*-module, because .

**Example (2.1.7):**

is not a semisimple *Z*-module, because

.

**Example (2.1.8):**

*Z* is not semisimple *Z*-module, because (note that there is no simple submodule).

**Theorem (2.1.9):**

Let be a hollow module. If every submodule of is a direct summand of , then is essential of and hence is - module.

**Proof:**

From definition of , we can say is a direct summand of . So

.

We need to prove that . Suppose that . Let and let

.

So *J* is a proper ideal of R and . Since *J* is a proper ideal of *R*, then there exist a maximal ideal *I* of R is a simple R-module. But . So

is a simple R-module. Since , then is a direct summand submodule of . But , then is a direct sum of .

Hence . So . So *L* is a simple module. Therefore .

Then . So and this contradiction. Hence and then . Therefore is a semisimple R-module, since is non zero hollow module so every non zero factor module of is indecomposable. So . Thus is -module.

**Proposition (2.1.10):**

Let be an *R*-module and let . Then is an intersection of essential submodule of if . Thus .

**Proof:**

Suppose that . Let . So , we have that . Let and . We claim that *J* is a direct summand of *K*. Let *B* be a direct of *J* in . . So . Since , then and hence

. Hence *J* is a direct summand of *K*. Since and , then is a direct summand of *K*. Hence . Now we must prove that . Let . Then . So . Hence is a direct summand of *K*. Therefore *F* is a direct summand of *K*. Now , then *F* is a direct summand of *T* (every submodule of *T* is a direct summand of *T*). So *T* is a semisimple submodule of and hence . But , then . Hence and . Thus . Therefore .

**Lemma (2.1.11):**

Let *R* be a semisimple ring and . Then every *R*-module is semisimple module.

**Proof:**

There is a free presentation

.

So is a quotient of free module *F* over *R*. Clear *F* is copy of the ring *R* and *R* is a semisimple ring, this means *F* is a semisimple. Thus is also semisimple module.

**Theorem (2.1.12):**

Let *R* be a ring. If is a proper factor of projective *R*-module, then any submodule of is essential and hence is -module.

**Proof:**

Let be an *R*-module. We have a short exact sequence:

.

Since any module in this ring is projective, then is also projective.

Hence this sequence is splits. So and is submodule and isomorphic to . So *R* is semisimple. From Lemma **(2.1.11)**, is semisimple module, and we have every proper factor module of projective is indecomposable (i.e is indecomposable). So by Lemma **(2.1.2)**. ( is -module).

**Proposition (2.1.13):**

Let be an indecomposable *R*-module. If is addition complement, then .

**Proof:**

From definition **(1.2.9)** of the radical, if this radical does not equal zero. So there exists . Then is addition complement ; is a small in . Since ; *Rr* is a small in , then *Rr* is a small in *Rr* and this contradiction. . Suppose that *B* is addition complement submodule of such that ; subset of . Let . Then and this means is a semisimple module. From our hypothesis is indecomposable. So ( is -module).

**Corollary (2.1.14):**

Let are semisimple uniform *R*-modules. Then the direct summand of is also semisimple and so is essential--module.

Recall that a ring *R* is called hereditary if every ideal *I* of *R* is projective.

**Lemma (2.1.15):**

Every injective module over a hereditary ring *R* has summand sum property.

**Proof:**

Let *R* be a hereditary ring. Then from definition of hereditary we have every of injective module is also injective. is injective and and . Therefore *L* is injective. But . Hence is a direct summand of *K*. Thus has a summand sum property.

**Theorem (2.1.16):**

Let *R* be a ring. If the following are true:

1. *R* is a hereditary ring;
2. ;
3. is a projective module;

then is -module.

**Proof:**

From condition (3) is injective module. But from condition (1) *R* is a hereditary ring, then by Lemma **(2.1.15)**, M has summand sum property. Let . Let *F* be a free module and . So is a projective module and has summand sum property. Let and **.** So is a direct summand of *R*. Hence *R* is a semisimple ring ( is semisimple module). Since ; so is indecomposable. Thus is -module.

**Corollary (2.1.17):**

Let be an *R*-module and . If the following are true:

1. is Projective module;
2. is a perfect module;
3. Every weakly injective module is injective;

then is -module.

**Proof:**

Let satisfy condition (1) and (2). Let such that [] is all modules over *R* whose objects are submodules of , a module is a w-projective. So is injective. Hence is injective ( is semisimple). Since this means M is hollow module so we obtain M is indecomposable. Thus is -module.

**Theorem (2.1.18):**

Let be any indecomposable module over Noetherian ring *R* and any direct sum of modules with summand intersection property has summand sum property. Then is -module.

**Proof:**

Suppose that is injective module. We have *R* is a Noetherian ring. Then is a direct sum of indecomposable modules. But indecomposable modules have summand intersection property. So *R* is a semisimple ring. Hence is a semisimple *R*-module with indecomposable property ( is -module).

There is a good relationship between radical of the ring *R* and semisimplicity property. First; Let *R* be a ring. We can present the meaning of by the following:

.

Moreover;

.

Therefore for any ring *R* we have the following fact is true:

.

Now let us take

.

Also; if *R* is Artinian ring, this means there exists a minimal element:

.

But when *y* is maximal ideal, so . Thus we easly present the following fact:

If *R* is an Artinian ring; the

.

**Proposition (2.1.19):**

Let be a simple module over the ring *R*. Then is -module when *R* is Artinian ring and .

**Proof:**

If *R* is Artinian ring; then we have

(by fact above).

Such that *X* is maximal ideal of *R*. The mapping from R into

But () is a semisimple. Hence *R* is also semisimple. ( is a semisimple *R*-module). Since is simple module so it is indecomposable module. So any submodule of is essential . Thus is a -module.

Recall that for any Artinian ring *R*, the such that an element is nilpotent if .

Therefore we obtain the following result:

**Corollary (2.1.20):**

For any module over an Artinian ring *R*; if *R* has no nilpotent left ideal, then is -module.

**Proof:**

Since is a nilpotent, then (*R* has no non-zero nilpotent). So by proposition **(2.1.19)** is -module.

**Corollary (2.1.21):**

Let be an indecomposable *R*-module. If *R* is a left Artinian, then it has a minimal element ideal and so *R* is semisimple ( is semisimple module and hence it is -module).

**Lemma (2.1.22):**

Let be an indecomposable *R*-module. If is injective, then is essential. Moreover; if is a simple and flat, then .

**Proof:**

Case 1: If is injective module over commutative ring *R*. So *R* is a semisimple. Hence is a semisimple module over *R* with is an indecomposable. Thus , Lemma **(2.1.2)**.

Case 2: Suppose that is semisimple flat module, so is injective. Thus by case (1); .

**Theorem (2.1.23):**

Let be an *R*-module. Then is a direct summand of simple submodule.

**Proof:**

Firstly, From definition of semisimple module we should prove that is a direct sum of simple submodule. If , then and the family of simple submodule of is empty, but the direct sum of an empty family of submodule is also empty. So is a direct sum of simple submodule. If , let *γ=;*  is non-empty independent family of simple submodule of . Since , then contain a simple submodule say . Hence . Therefore by lemma there exists a maximal independent family of simple submodule of M say {}. We claim that .

If not, then there exists a simple submodule of and . Hence . So . Then is an independent family of simple submodule of which is contradiction with maximal of {}. Hence . Thus is a direct summand of simple submodule.

**,𝑠𝑠 r if M is a simple and flate it is Essential-nt ideal and so R is semisimpleTheorem (2.1.24):**

Let be an indecomposable *R*-module. If every submodule of is a direct summand of , then .

**Proof:**

We have is a submodule and direct summand of . So such that is a submodule of . We must prove that . If , let and let

, *I* is a proper ideal of R and

.

Since *I* is a proper ideal of *R*, then there exists a maximal ideal *A* of *R* and

is a simple *R*-submodule. So

is a simple *R*-submodule… (2).

Hence . Therefore is a direct summand of submodule of (by hypothesis). But , So is a direct summand of . Hence . Then and so *B* is a simple submodule. Therefore and . contradiction. Thus and then . But is a direct summand of simple submodule Proposition **(2.1.10)**. Hence is also a direct summand of simple submodule. Thus is a semi-simple module (by definition of semisimple), and we have is indecomposable module. So by lemma **(2.1.2)**, .

**Example (2.1.26):**

Let *Z* be a *Z*-module. Then is a simple submodule of *Z* such that *p* is a prime number. So for each is a semisimple *Z*-module. So contains a submodule which is essential .

**Example (2.1.27):**

The *Z*-module is semisimple small submodule of the *Z*-module. So is essential in *Z*-module.

**Remark (2.1.28): [19]**

If is semimaximal, then is a semisimple module .

**Theorem (2.1.29): [19]**

Let be a proper submodule of . So is a semi-maximal submodule if and only if there exists , *A* semisimple and and semimaximal of *B*.

**Theorem (2.1.30):**

If is an indecomposable P-module such that every submodule of is a closed, then is -module.

**Proof:**

Since is a *P*-module, so is pseudo-injective. But is a closed submodule of . Hence is a direct summand of . This means is a semisimple module, also we have is indecomposable module. Thus is an essential in .

Recall that An R-module is anti-hopfian if is non simple and all nonzero factor modules of are isomorphic to ; that is for all

**[21].**

**Corollary (2.1.31):**

Let be anti-hopfian indecomposable *R*-module, if is closed, then is -module.

**Proof:**

Suppose that is anti-hopfian *R*-module and . So

. Hence is also anti-hopfian. Therefore is a p-module with closed submodule and is indecomposable module imply is essential in .

**Corollary (2.1.32):** Let *R* be a *P.I.D*. If is a cyclic and not isomorphic to *RR*, is closed, then is an essential in .

**Proof:**

Suppose that *R* is a *P.I.D* and is cyclic R-module. So is P-module with closed submodule of imply that is essential in .

**Example (2.1.33):** Let be an indecomposable right R-module; and is a simple. Then .

**§2.2: Multiplication Module and Essential Property**

In this section, we study the relationship between multiplication *R*-module and essential property of any submodule of . We try to obtain as an essential in . Before that we need to introduce some definitions and lemmas about the subject**.**

**Definition (2.2.1): [3]**

Let be an *R*-module. Any submodule *K* of is called prime if implies.

**Definition (2.2.2):** **[3**]

Let be a submodule of a module . Any submodule of is essential if there exists such that , then .

**Lemma (2.2.3):**

If is an multiplication *R*-module and let be only one maximal submodule of , then .

**Proof:**

Suppose that and

.

Assume that . So . Hence . But . Therefore *K* proper in . We have and it is multiplication module. Then *K* is a maximal submodule of  **[17]**. But has only one maximal submodule . Hence . So . Thus .

**Theorem (2.2.4):**

Let *R* be an integral domain. If is divisible simple *R*-module has one maximal submodule, then is essential in .

**Proof:**

Since is a simple module, then it has only two submodules and . If , then is an ideal. So is multiplication module, from lemma **(2.2.3)**, because has one maximal submodule. Thus is essential in -module.

**Theorem (2.2.5):**

Let be a module over local ring *R*. If is:

1. Cyclic module;
2. has only one maximal submodule; then is -module.

**Proof:**

Assume that *R* is a local ring and it has one maximal ideal *I*. If is a cyclic *R*-module, then is multiplication module. So every submodule of has essential property in . Thus is essential in .

**Lemma (2.2.6): [18]**

Let *R* be a ring and let be a finitely generated distributive (*D*-module) *R*-module with . Then is a multiplication R-module.

**Theorem (2.2.7):**

Let be an *R*-module. If:

1. is a cyclic module;
2. has square free socle such that ;
3. is a faithful module;

then is essential submodule of .

**Proof:**

From condition (1), we have is a cyclic module. So is generated by one element . Hence is a finitely generated.

Now condition (2) means:

.

We know that every quotient and submodule of a *D*-module is also *D*-module. If such that is a submodule and S has only two submodule are and S (S is simple). So is *D*-module. So from lemma **(2.2.6)**, is a multiplication module. Hence submodule of is essential .

**Remark (2.2.8):**

*R* is called *D*-domain if any module over R is a *D*-module.

**Lemma (2.2.9):**

Let *R* be a Noetherian domain with quotient division ring *B*. If for all and every ideal *I* of *R* is invertible, then *R* is a *D*-domain.

**Proof:**

We know that

if and only ifthen.

We also have

thenfor some.

Then

But

Then

Also by same way

So if:

two ideals of

Then

Now setting, thenis invariant.

So every ideal is invariant, so *R* is also invariant. Thus by proposition (5) and theorem (6) from **[14]** *R* is a *D*-domain. Then is a *D*-module.

**Corollary (2.2.10):**

Any module have the following properties is -module:

1. is finitely generated;
2. is a faithful module;
3. All conditions of lemma **(2.2.9)** are hold;

then .

**Proof:**

From condition (3), is *D*-module and by lemma **(2.2.6),** is a multiplication module. So any submodule of is essential.

**Corollary (2.2.11):**

Let be an *R*-module. If satisfy the following:

1. is local module;
2. is simple module;
3. ;
4. and *P* prime (maximal) ideal of is an irreducible sub of is proper;

then and is -module.

**Proof:**

From condition (3), let *P* be a prime ideal in *R*. Assume that *A, B* are proper subs of . From **[37]**

andand

But is an irreducible sub of ⟶ the proper subs of are linearly ordered. So is *D*-module [Lemma (2.4), [37]]. From condition (2), is a simple module. So has only two submodules are and it self . Thus we have:

or

is not generated . Hence is a cyclic. But every cyclic module is finitely generated with imply, that is a multiplication module. So . Now from condition (1), we have is local Module. By definition of local module, each proper submodule of imply . But is a small of . Then is a small of . So is a hollow module. But every hollow module is lifting module [Corollary (2.8), [38]]. So M is a -module. Hence is -module.

**§2.3: Injective Module and C1-Module**

In this section, we will connect between Injective R-module and essential property of any submodule of . It is very useful to present the following lemma in order to provide a starting point for this section.

**Lemma (2.3.1):** Every semisimple indecomposable module has .

**Proof:**

Assume that a module is a semi-simple. Suppose that . Therefore (). But is indecomposable R-module, means that . Hence . Thus .

**Lemma (2.3.2):**

Let a module be an injective over a ring *R*. If is indecomposable, then any .

**Proof:**

Since is injective, then *R* is a semi-simple ring. So is a semisimple module over *R* with indecomposable property. Thus is essential.

Recall that *A* is divisible if any ;

**Theorem (2.3.3):**

Suppose that *A* is a divisible group. Then over Z.

**Proof:**

We know that there are cyclic groups as a left ideal. If *A* is divisible and is a homomorphism such that ; .

Let and is homomorphism that extend g. So *A* is injective Z-module. Thus any submodule of *A* is essential in , where -module same *A*.

**Example (2.3.4):**

*Q* with additive as a group has submodule ,, because *Q* is divisible and so injective Z-module.

**Example (2.3.5):**

Let *R* be a not *P.I.D*. Then the quotient of injective *R*-module is not injective.

**Example (2.3.6):**

is a divisible. So .

**Example (2.3.7):**

*Z* is not divisible. So .

**Definition (2.3.8): [29]**

If R is a ring and the

Sequence of homomorphisms of right R-modules **,** is an exact-seq.

*,*

so it is a short exact.

**Definition (2.3.9): [29]**

*A* is called a split exact seq. if and only if for every subsequence

is a direct summand of *Ai*.

**Lemma (2.3.10):**

If

is split and ; then is injective R-module.

**Proof:**

Since

is split exact,

.

Now we have

is split. So is injective. So is injective.

**Example (2.3.11):**

The {0} is injective module.

**Example (2.3.12):**

2Z and Z are injective.

**Remark (2.3.13):**

1. If , then is injective.
2. From **[34]**, every injective module is projective such that a module is called projective over a ring R if we have any diagram of R-module homomorphism

Such that is an exact and g is epimorphism (onto), so homomorphism .



Before entering it to the explanation of the relationship between the projective module and the essential property, we need to clarify the concept of the free module.

If the following statements are hold, then the module is a free.

1- has a non-empty basis.

2- is cyclic module.

3- copies.

4- There exists and any module *K* and unique is a homomorphism;

.

Now we return to studying the relationship between the projective and the injective modules, using the free module.

**Theorem (2.3.14):**

Every free R-module has .

**Proof:**

Let us take a diagram of a unitary R-module:



Such that g is an epi. is a free on a basis *X*

.

But g is onto, then there exists

.

Since is a free, then given by and

.

. So

.

From Remark **(2.3.13),** is injective. Thus every . So is -module.

Throughout, R is an abelian ring with 1 in R. All R-module are assumed to be unitary . A module is injective if for all R-module homomorphisms and such that is injective,

.

**Example (2.3.15): [16]**

If , then and are -modules and there is -module isomorphism . Hence both and are projective -modules.

**Example (2.3.16):**

Given a field *K*, every *K* vector space *W* is an injective *K*-module.

**Example (2.3.17): [16]**

Let R be any ring and I be an indexing set. Then where each is isomorphic to R is an example of free module.

(**Baer’s Criterion):**

Suppose that be an R-module. Then a module is injective if and only if is ideal; extends to .

**Remark (2.3.18):**

Suppose that any two R-modules. There exists another module and maximal of and .

**Definition (2.3.19):**

in last remark is maximal essential extension of in . If is essential and has no proper essential extension, then is a maximal essential of .

The following theorem explains the relationship between essential extension and injective module.

**Theorem (2.3.20):**

If has no essential extension, then .

**Proof:**

Assume that embedding in . So is maximal and Hence is embedding and is an essential extension. So it is an isomorphism. But . Since is a direct summand of an injective module, then is injective . Thus is -module.

**Theorem (2.3.21):**

If and a module is faithful, then any and so is -module.

**Proof:**

Let . We need to prove that is torsion-free. Assume that is not torsion-free. So is torsion is torsion element, such that c one of the regular element in *R* (non-zero divisors). , because . So .

Hence and then .

So is unfaithful module and this contradiction. Then is a torsion-free . Hence an is a one to one (so is injective). So . Thus is -module.

**Corollary (2.3.22):**

If is an anti-Hopfian R-module and . Then .

**Proof:**

Assume that is an anti-hopfian. So is integral domain and hence is a commutative ring with 1 **[10]** imply is a pure multiplication (every commutative with 1 is pure multiplication). Thus .

**Corollary (2.3.23):**

Let *R* be a *P.I.D*. If:

1- is a finite generation module;

2- is a torsion-free ;

then every submodule of is essential ().

**Proof:**

Suppose that is generated by any finite set *K*. There is a maximal subset *H* of *K* and *L* is a submodule generated by is free. Take

and.

Since

is torsion.

Let (finite product). So .

Since *L* is free then is a free. Since is torsion-free then has trivial kernel , So . So is a free, then is projective implys is injective. Thus .

|  |
| --- |
| **Chapter Three**  **Some Generalizations of**  **-module** |

**INTRODUCTION**

In this chapter we will introduce three types of generalizations of -module. We will use some tools of generalization, such as duo submodule, prime or semi prime submodule and quasi-injective submodule. In first part we will study duo submodule on -module and prove if , were as a submodule of over ring R and , there exist , then is a duo submodule of . In next part we will study -module through the submodule of which have two properties, namely prime and semi prime (P-(S.P)) submodule and prove that if any submodule of multiplication -module is a fully invariant in , subset of , subset of for all are submodules of . Then is S.P-duo--module. Finally we will introduce a new generalization of -module. The main method adopted in this generalization is how to obtain a submodule of a module having the characteristic Quasi-injective and we prove if is pseudo-injective R-module over *P.I.D*, so it is a Q-injective. Were any *R*-module is said to be pseudo injective (briefly P-injective) if and only if every *R*-isomorphism of each submodule of into can be extended to an *R*-endomorphism of .

**§3.1: Duo Submodule and -module**

In this section, we will give high priority to some important results about the duality property of submodule. The main reason for choosing this property is that duo is one of the important applications of extending modules. Note that any module will be chosen we will deal with it as a submodule in itself. Before giving the first result of this section we need to present the following definitions.

**Definition (3.1.1): [19]**

Let be an R-module and let . If , then is called fully invariant (FI) such that .

Note that if , this means that is also a fully invariant as a submodule. Moreover; and {0} are called duo submodules.

**Definition (1.3.2): [35]**

The right R-module is called a duo module if every submodule of is fully invariant.

**Examples (3.1.3):**

1- Simple module is Duo module.

2- Multiplication module with projective module is Duo module.

**Lemma (3.1.4):**

Consider as a submodule of over ring R. If and , there exist , then is a duo submodule of .

**Proof:**

Note that . Thus is a duo submodule and so is duo--module.

**Theorem (3.1.5):**

Let be a -module. Consider as a submodule of . If has (ACC) property on cyclic submodule, then is a duo submodule and so is duo--module.

**Proof:**

Assume that has (ACC) property on cyclic submodule. Let

and let be a homomorphism. If , then and so , . Hence ; n is positive integer. So

,

By hypothesis, integer.

There exists such that

*.*

*.*

If , then and so . ….C!

Therefore and hence ..C!. So . Thus is a duo submodule ( is duo--module).

**Remark (3.1.6):**

We can show that some submodules are not duo and hence is not duo--module, for example:

If subring of , then any right *R*1-module *R*2 is not a duo module because if . So defined by:

is an -homomorphism. We have , then is not fully invariant submodule of *R1*-module *R2*.

**Theorem (3.1.7):**

Let be a -module over R be a commutative domain and integrally closed. If is a finitely generated torsion free uniform as a submodule over R, then is a duo--module.

**Proof:**

Let be a subring of *K*-vector space. Suppose that *R* is integrally closed and *U* be any finitely generated torsion-free uniform module as a submodule. Let . Since , *k* integral over *R*, then . So . Thus *U* is a duo submodule and then is duo--module.

**Definition (3.1.8): [28]**

Let be a submodule of an R-module . Then is called an essential extension of If or , then .

**Definition (3.1.9): [28]**

A submodule of an *R*-module which has no proper essential extension in is called a closed submodule of .

**Definition** (**3.1.10): [28]**

Any submodule of is called closed duo if has no proper essential extension and , .

Now we study another property of submodule namely heart submodule , defined by the intersection of all non-zero submodules of .

is a minimal submodule contained in every non-zero submodule when is non-zero.

Recall that if M is any R-module, then the socle of M can be defined by

**Remarks (3.1.11):**

1. for any *R*-module .
2. if has simple Socle.

**Theorem** **(3.1.12):**

Let be a -*R*-module. Consider be a heart submodule of . Then is fully invariant and so is duo ( is duo--module).

**Proof:**

From the definition of we get is a submodule of . Take any homomorphism , . To prove

.

If , then is already an invariant submodule. Let . Therefore is simple and hence

. So

.

Then is fully invariant. Thus is a duo--module.

**Theorem (3.1.13):**

Let be a -module. If is intersection of all submodules of every submodule is fully invariant, then is duo--module.

**Proof:**

Assume that , then has fully invariant submodule ( is duo) and is closed. Thus is closed duo--module.

Now suppose that . Then

.

Then

.

So is h-closed ( is closed duo--module).

**Remark (3.1.14): [28]**

Let be an R-module and . We called h-closed submodule of if .

**Corollary (3.1.15):**

Let be a -module and .

If and , then (h-closed) and so is duo--module.

**Proof:**

Assume that be a h-closed submodule of and

. But . So

(By def. of ) .

Hence is a h closed submodule of . Therefore,

is fully invariant. Thus is duo--module.

**Example (3.1.16):**

Let *F* be a field and a vector space over *F* such that . Consider *R* subring of ;

**.**

There are only three submodules of R:

.

We have and , therefore 0 and *R* are h-closed submodule and then is fully invariant. Thus is duo--module.

**Remark (3.1.17):**

Note that and are not imply is duo--module, because , and

.

Also .

**Example** (**3.1.18):**

Let be *R*-submodule of .Clear that not complement submodule of , therefore it is not h-closed and so not fully invariant.Then is not duo--module.

**Example** (**3.1.19):**

Let and let be a submodule of . So

and then it is h-closed .

Thus is duo--module**.**

The next theorem explains that a direct summand of h-closed submodule gives duo--module.



**Theorem (3.1.20):**

Let be a -module. Then direct summand of h-closed submodules is fully invariant and so is duo--module.



**Proof:**

Suppose that is a direct summand of h-closed. So

**.**

Assume that it is h-closed of Then

Because . Hence is h-closed submodule,

is fully invariant and so is duo--module.

On the other hand, the socle of is the largest submodule of generated by simple modules. Or it is the largest semi simple submodule of .

**Corollary** (**3.1.21):**

Let be a -module. If has a socle not equal zero, then is duo--module.

**Proof:**

Suppose that . Since is a simple, then

. but is a fully invariant, then is duo--module.

**Corollary** (**3.1.22):**

Let be a multiplication -module. If for every submodule of ; R-monomorphism can be extended to an R-endomorphism of , then is duo--module.

**Proof:**

Assume that has a submodule such that R-monomorphism can be extended to an R-endomorphism of . Let be a multiplication module over R. So such that I is an ideal of R. Let be an R endomorphism. If , then . Hence . Then is a fully invariant submodule of . Thus is duo--module.



Recall that a submodule of a module is essential in case for every submodule .

**Corollary (3.1.23):**

Let be a -module. If R monomorphism from into can be extended to an R-endomorphism of , then is duo--module.

**Proof:**

Let be a monomorphism and . Then

. Since is pseudo-injective, then there exists an R-homomorphism extends f. Also, we have is pseudo-injective, then there exists an R-homomorphism extends *g*. Let us claim that . Assume that -. But . So

.

Hence

.

So

Therefore

Hence. Then ; which is contradicts with assumption, then . Hence

.

But

.

So

.

Hence is a fully invariant. Thus is duo--module.

From **[23]**, any module is called -multiplication module if for every submodule of , , where is all multiplication modules.

**Theorem** **(3.1.24):**

Let be -module and . If is an -multiplication module, then is duo--module.

**Proof:**

Since is -module, then has a decomposition such that and where ( is a lifting). So is -module. Assume that . Since is an -multiplication module, then there exists and (). . Then if

, then *L* is fully invariant of . Then is duo--module.

**Example (3.1.25): [23]**

Let . Let (proper), then is a duo submodule.

**§3.2: P-(S.P) Submodule and (Extending) Module**

In this section, we study *C1*-module through the submodule of which have two properties namely prime and semi-prime. Indeed these properties with another concept like fully invariant of any submodules explain the main objective of this study. Also, heart submodule of a module and the socle are studied with fully invariant property to obtain the same objective. Several concepts have been used in this section for the purpose of reaching the main objective of the thesis, for example, heart submodule of and socle of . Also, we will use the fully invariant property to achieve the same goal. Note that P ideal of a ring *R* is called maximal ideal if and J also is ideal of a ring *R* then . So .

Assume that *K* is a proper submodule of multiplication module . So for every , if , then or .

From definition (1.1.10); a module is called lifting or satisfies (-module), if for every submodule of there exists a direct summand *K* of such that *K* is a coessential submodule of in .

From definition (1.3.15); any proper submodule is prime (briefly P-submodule) when if , , , then and is said to be semiprime (briefly S.P-submodule) if and whenever and for *n* where , then .



**Theorem (3.2.1):**



Let be a -module over an integral domain *R*. If is simple (cyclic) and and satisfy , then is prime-duo--module.

**Proof:**

Suppose that is simple (cyclic) and generated by one element.

Or .

So is a multiplication *R*-module. So any is a prime submodule with fully invariant property ( is duo submodule) and *S* is prime ideal of a ring *R* and imply that is P-duo--module.

Recall that , this is called the residual of *L* by in such that is a multiplication R-module and . Also, is an annihilator of in .

**Proposition (3.2.2):**



Let *R* be a semi-local ring If is cyclic *D1*-module such that for any are a fully invariant, and , then is P-duo--module.

**Proof:**



Since *R* is a semi-local ring, then R having just finitely many maximal ideals (*I* is a maximal ideal of a ring R if there are no other ideals existing between I and R). So from {lemma (3), [12]}, is a cyclic *R*-module and hence is a multiplication *R*-module. But and has fully invariant. So is a P-submodule of . Thus is a P-duo--module.ة from hen R having only finitely many maximal ideals.

Recall that if any submodule of is a P-submodule, this means is prime module. Therefore we can introduce the following result:

**Corollary (3.2.3):**



Let be a *D1*-cyclic R-module If is a prime module and has fully invariant property, then is a P-duo--module.

**Proof:**

The proof is very easy, because prime module gives every sub module of is prime with same way in with fully property, we obtain is P-duo--module.

Now we need to introduce two concepts namely heart submodules of and the socle of the module . means intersection of all nonzero submodules of and is a minimal submodule contained in non-zero submodule when .

Recall that if M is any *R*-module, then the socle of can defined by

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On the other hand, the socle of is the largest submodule of generated by simple modules, or it is the largest semisimple submodule of . Also, if *R*be a module and , then is called h-closed submodule of provided that .

**Theorem (3.2.4):**

Let be a -module. If has h-closed submodule and

, invertible element and ; . Then is a P-duo--module.

**Proof:**

Suppose that and are maximal ideal of . Hence

**.**

Since is not invertible element and hence is invertible element. This implies that is invertible element and this contradiction. Since implies that is not invertible element. Then *R* is local ring (*R* semi-local ring). Hence is cyclic *R*-module and then is multiplication *R*-module. Since have h-closed submodule, then is fully invariant. We have and , therefore is a P-submodule.Thus is P-duo--module.

**Proposition (3.2.5):**

Let *R* be a *P.I.D*. Let be a -*R*-module. If has h-closed submodule and *I* is prime ideal of *R*; then is a P-duo--module.

**Proof:**

Clear. *I* prime ideal of *R* gives *I* maximal ideal. Same proof of Theorem **(3.2.4)**, we obtain the required.



**Theorem (3.2.6):**



Let be multiplication *D1*-*R*-module. If any submodule of is a fully invariant in , subset of , subset of for all are submodules of . So is S.P-duo--module.

**Proof:**

Suppose that for some ideal *B* of a ring *R*; . So . But and . Then

. Hence is a semiprime submodule of ( is a semi prime).We have has fully invariant in ; where be an endomorphism .So is a duo submodule. But is -module. Thus is S.P-duo--module.

**Corollary (3.2.7):**

Every semiprime submodules of a multiplication -module is

submodules and so is S.P-duo--module.

**Proof:**

We know that h-closed submodules means intersection of all submodules. But h-closed is fully invariant in with prim property imply is S.P-duo--module.

**Corollary (3.2.8):**



Let be a -module. If is a fully invariant and over *R* is a field, then is a P-duo--module.

**Proof:**

First, we claim *R* is not field. So is a prime. Hence *R* is a domain. So ( is a torsion free *R*-module) over *R*. Note that when *R* is not field this is means ( is not simple module); such that .

If and is not an invertible element of *R*. So *Rxm* is a prime and ; contradiction (because ) or , then and this contradiction. Hence is a prime submodule of . We have is a fully invariant (duo submodule). Thus is a P-duo--module.

**Corollary (3.2.9):**



Let be a *D1*-module. If the following statements are true:

1. ; I proper ideal of *R* and ;
2. ;
3. ;
4. is duo submodule;

then is P-duo-*C1*- module.

**Lemma (3.2.10):**

is a fully invariant submodule.

**Proof:**

From definition of , we get is a submodule of a module . Take any homomorphism *g* in , . We need to prove that

.

If , then is invariant submodule.

If . Therefore is simple and hence

. So

**,**

then is also fully invariant.

**Theorem (3.2.11):**



Let be a *D1*-module. If satisfy the following statements:

1- is a multiplicatively closed such that ;

2- ;

3- ;

then is P-duo--module.

**Proof:**

From condition (2), is a multiplication module. Let . From condition (1), (see [6]). So is a proper prime submodule of . We have is fully invariant submodule with condition (3), is also fully invariant submodule ( is a duo submodule). Since is a *D1*-module, then it is -module. Thus is P-duo--module.



**Theorem (3.2.12):**



Let be a *D1*-module and satisfy the following conditions:

1- is semisimple module;

2- is cyclic module;

1. Let . For every , if subset of P, then or ;
2. is closed-duo-module;

then is P-duo--module.

**Proof:**

Assume that be any submodule of a semisimple module . By assuming that is a direct summand of ; hence, it is a closed submodule. We have is a closed-duo-module, then is fully invariant. Since is cyclic module, then is a multiplication module with condition (3), we obtain is prime submodule, and we obtained the result.

**§3.3: On Q-Injective, Duo Submodules of -Modules**

This section investigates modules having a submodules which is a duo and quasi-injective. We introduce a new generalization of *C1*-module. The main method adopted in this generalization is how to obtain a submodule of a module having the characteristic Quasi-injective. Also, we study duo property of a submodule in . We investigate the relationship between pseudo-injective module and Quasi-injective property of *C1*-module. Finally, we introduce a new relationship between Quasi-injective and anti-hopfian module. Now we start with Pseudo-injective and Quasi-injective submodule. We study two important properties of namely Quasi-injective and P-injective. Via this submodule, we obtain a new characterization of *C1*-module. Moreover; we should provide another property namely fully invariant of this submodule. Note that Q-injective is a self injective.

**Lemma (3.3.1): [40]**

Let be an R-module over *P.I.D*. If is pseudo-injective module, so it is a Q-injective.

Now we need to find such that is Q-injective with fully invariant property.

From **[39]**, any pseudo-injective module over *P.I.D* is a Q-injective (i.e. if is a module on *P.I.D*, then on *P.I.D*, but is pseudo injective is pseudo-injective and hence is Q-injective).

**Theorem (3.3.2):**

Let *R* be a *P.I.D*. If is a pseudo-injective *C1*-module over *R*, then any submodule is a Q-injective and ; so is Q-injective-duo- -module.

**Proof:**

Suppose that a module is a pseudo-injective. Let us take . We have any module on *P.I.D*. So also on *P.I.D*. But is pseudo-injective, then is pseudo-injective over *P.I.D*. Hence N is Q-injective with imply is Quasi-injective and fully invariant (duo) submodule of .

Now we introduce another method to obtain any submodule of *C1*-module and be Q-injective. This way depends on a new domain namely Dedekind domain (*R* is a Dedekind domain if it is integrally closed, Noetherian and if is a maximal; p is prime ideal). So if *R* is a Dedekind domain, then it is a *UFD* iff *R* is *P.I.D*. See the next lemma:

**Lemma (3.3.3): [41]**

Let be any *R*-module over Dedekind domain. Then is Q-injective and so is also Q-injective submodule.

**Theorem (3.3.4):**

Let be a Pseudo-injective-*C1*-module over Dedekind domain. If is stable, then is Q-injective-duo-*C1*-module.

**Proof**:

Assume that is Pseudo-injective and *R* is a Dedekind domain. From lemma **(3.3.3),**  is a Q-injective. So is also Q-injective. But is stable, so is a fully invariant. Therefore is a duo submodule of -module.

**Lemma (3.3.5): [41]**

Let be an *R*-module. If the following statements are true:

1. *R* is Multiplication ring;
2. and *I* is an ideal of *R*;
3. ;

then is Q-injective and so is also Q-injective.

**Theorem (3.3.6):**

Let be a module. If:

1. *R* is a multiplication ring;
2. ;
3. and stable;
4. is -module and Pseudo-injective;

then is Q-injective-duo--module.

**Proof:**

Assume that and *R* is multiplication ring. Then from **[25]**, (any submodule of torsion module is torsion). Since is P-injective, then is a Q-injective and hence is P-injective and . Hence is a Q-injective. Since and stable, then is a fully invariant. But from condition (4), is - module. Then is a Q-injective-duo-*C1*-module.

**Corollary (3.3.7):**

If is -Pseudo-injective *R*-module, then is Q-injective-duo--module such that and .

**Proof:**

By theorem **(3.3.6)**.

Recall that any *R*-m is called nonsingular if, for all with implies . Or

a right an ideal *I* of *R* such that and **[5]**.

**Lemma (3.3.8):**

If and in Pseudo-injective module , then is Q-injective.

**Proof:**

Let and . Let be an *R*-homomorphism. So or .

Suppose that , so g can be extended to homomorphism .

Now if , so *g* is a one to one and can be extended to *R*-homomorphism from ( is Pseudo-injective). Hence is Q-injective.

**Corollary (3.3.9):**

Let be a -pseudo-injective R-module. If , and ; then M is Q-injective-duo-C1-module.

**Proof:**

By lemma **(3.3.8)**.

Now we present another way in order to decide that any submodule of *R*-module is a Q-injective. But before that we need to present some important definitions that are closely related to the mentioned way. Firstly, a concept of stable-Q-injective was explained in **[40]**.

Let . Then is called a stable module. So if every is stable this means that is a fully stable module (F-stable) If is stable and can be extended *R*-homomorphism () to an *R*-endomorphism (), then is called stable-Q-injective *R*-module. Also, If *R* is an integral domain and is an *R*-module, then an element is called torsion element if ∋ . **[20]**. So we define:

a torsion element

Note that:

1. If , then a module is called torsion-module.
2. If , then a module is called torsion-free-module.

**Lemma (3.3.10): [40]**

Let be a stable-Q-injective R-module. If is an injective *R*-module, then it is Q-injective.

**Theorem (3.3.11):**

Let be a -module. If is a F-stable and stable-Q-injective; then is Q-injective-duo--module.

**Proof:**

Let and let be an *R*-homomorphism of . So is stable because is a F-stable. But from stable-Q-injective of , there is an

extends *ϕ*. Hence is a Q-injective. Thus is a Q-injective-duo--module.

**Corollary (3.3.12):**

Let be a -module. If and be a homomorphism and is a stable-Q-injective, then is Q-injective-duo--module.

**Proof:**

By theorem **(3.3.11)**.

**Remark (3.3.13):**

From definition of fully invariant submodule and definition of stable, we find that the two meanings are the same.

Recall that a ring *R* is called Quasi-Frobenius (QF-ring) if every projective module is injective; or every injective module is discrete. From **[29]**, every projective module is injective and then every injective module is Q-injective.

**Corollary (3.3.14):**

Let be a -module over QF-ring. If is a projective module and stable in *R*, then is Q-injective-duo--module ( Q-injective submodule).

**Proof:**

Let *R* be a QF-ring. Since is a projective *R*-module, then is an injective module and hence Q-injective. Therefore any submodule of is Q-injective. Note that is stable module; so for any *R*-homomorphism , we get ( is fully invariant). Thus is Q-injective-duo--module.

Recall that a module is called -module if for any , there exists is a coessential sub of . Or if and , then and . So -module is extending.

**Proposition (3.3.15):**

Let be an *R*-module over QF-ring *R*. If:

1. is-module;
2. is stable module;
3. is a free-module;

then is Q-injective-duo-*C1*-module.

**Proof:**

From condition (1), is *C1*-module. From condition (2), there exists an *R*-homomorphism ( is fully invariant). So is a duo submodule. Condition (3), gives is a free module. So if we take *F* is a free *R*-module on a set *S*. Suppose that two modules over the ring *R*. Let is a homomorphism.

we choose

Also,

andonto

Then

Since F is a free-R-module on S, a unique homomorphism

To prove that . Let . So

because F is generated by

Now

h is a homomorphism.

Now

homomorphism

*.*

g homomorphism. So . Then is a projective and hence is injective ( is a Q-injective). Then is Q-injective. Thus is a Q-injective-duo--module.

**Lemma (3.3.16):**

For a ring *R*, we have is a semisimple if and only if *R* is a semisimple and so any module over *R* is a semi simple module.

**Proof:**

We need to prove the following,

(1)- semisimple if and only if *R* is semisimple.

(2)- is a semisimple module over *R*.

From **[29]**, we can get the proof of (1).

Now we need to proof (2):

If is a semisimple and if , then *R* is a semisimple as an epimorphic image of . So as a sum of semisimple module is again semisimple.

**Lemma (3.3.17):**

Let *R* be a semisimple ring and be an *R*-module. Then every submodule of is Q-injective.

**Proof:**

Since *R* is a semisimple ring, then every module over *R* is a semisimple. So is a direct summand. Hence is injective *R*-module. But every injective *R*-module is a Q-injective. Thus is Q-injective.

**Theorem (3.3.18):**

Let *R* be a semisimple ring and is an R-module. If is -module and stable; then it is Q-injective-duo submodules module.

**Proof:**

It is clear that from lemma **(3.3.17)**, is Q-injective. But is a stable. Then . So is a fully invariant and hence is a duo ( is a duo submodule). We have is -module. So it is -module. Thus is Q-injective of .

**Corollary (3.3.19):**

Let be an R-module. If:

1. is projective module;
2. is a simple module;
3. is Q-injective;

then is Q-injective and duo submodule of -module.

**Proof:**

It is clear that projective module means extending or -module. Also, if is a simple module, then is duo module. ( N is fully invariant; and is an R-homomorphism). Now from condition (3), we have is Quasi-projective. So is a Q-injective and hence is a Q-injective of -module.

Recall that any ring *R* is called V-ring if every simple *R*-module is injective **[8].**

**Corollary (3.3.20):**

Let be a -R-module over V-ring. Then is Q-injective-duo--module.

**Proof:**

It is clear by theorem **(3.3.18)**.

The following are some new results about the relationship between Hopfian, Self-injective and Q-injective Submodule. From **[27]**, a module is called self-p-injective if satisfy the following condition: every homomorphism from a projection invariant submodule of to can be lifted to .

**Definition (3.3.21): [29]**

A module is indecomposable if and only if and are the only direct summands of .

**Example** **(2.3.22):**

*Z* is indecomposable *Z*-module but *Z* is not simple *Z*-module (*Z* contains proper submodule *2Z*).

Therefore every simple module is an indecomposable but the converse is not true.

**Theorem (3.3.23):**

Let be an indecomposable self-P-injective *R*-module. Then any - module is Q-injective-duo--module.

**Proof:**

From definition of self-p-injective, there exists *K* submodule of such that *K* is fully invariant. Assume that is an indecomposable module, so every submodule of is projective invariant. Then is Q-injective. Thus is Q-injective-duo-*C1*-module.

Recall that any module is called Hopfian if every surjective f in is isomorphism and a non simple module is called anti-Hopfian if proper submodule of is a non-Hopfian kernel such that a submodule of is non-Hopfian kernel (for ) if there exists an isomorohism to [19]. Or An *R*-module is anti-hopfian if is non simple and all nonzero factor modules of are isomorphic to ; that is for all **[34]**.

**Example (3.3.24):**

Any module of semisimple Artinian ring with finite length is a Hopfian module.

**Lemma (3.3.25): [4]**

Let be an R-module. If is anti-hopfian, then every submodule of is Q-injective.

**Theorem (3.3.26):**

Let be -R-module. If has exactly one non-zero proper submodule and are simple modules, then is a Q-injective of .

**Proof:**

From **[21]**, is anti-hopfian module. Since 1 and 2 are simple modules, then is a simple module and so it is a duo module ( is a duo submodule). From lemma **(3.3.25)**, the proof is completed.

**Corollary (3.3.27):**

Let *R* be a Dedekind domain, and is -module with . If is a non-zero ideal of *R* and is duo submodule of , then is Q-injective in -module.

**Proof:**

From **[21]** and lemma **(3.3.25)**.

**CONCLUSIONS**

In this work we had two objectives, the main goal achieved in the two chapters is got a module namely -module (extending). Meaning we looked for conditions and algebraic structures that lead to obtain the submodules to be essential, and thus we obtained the -module (extending). The tools that helped to achieve this goal are semi simple module, multiplication module and injective module. These are some of the main proofs and results that we have proven in chapter two:

1. Let be a semisimple indecomposable module. If is a submodule of , then .
2. If is an multiplication *R*-module and let be one maximal submodule of , then .
3. Let a module be an injective over a ring *R*. If is indecomposable, then any .

The second goal achieved in the third chapter is got a generalization of -module by using the following tools: duo submodule, prime and semi prime submodules (P-(S.P) submodules) and quasi-injective submodules (Q-injective). These are some of the main proofs and results that we have proven in chapter three:

1. Let be a -module. If has a socle not equal zero, then is a duo--module.
2. Let be a -cyclic R-module. If is a prime module and has fully invariant property, then is a P-duo--module.
3. Every semi prime submodules of a multiplication -module is submodules and so is S.P-duo--module.
4. Let be a Pseudo-injective--module over Dedekind domain. If is stable, then is Q-injective-duo--module.

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**المستخلص**

المقاس يسمى C1 الممتد اذا كان لأي مقاس جزئي في يوجد جمع مباشر لمقاس جزئي آخر B في بحيث B يكون جوهري ل في .

لذلك في هذا العمل كل مقاس جزئي من يكون جمع مباشر.

الهدف الرئيسي الاول في هذا البحث هو الحصول على المقاس الممتد من النوع C1. نركز على الشروط والتراكيب الجبرية التي تقود الى الحصول على المقاسات الجزئية بحيث تكون جوهرية. الادوات المتوفرة لدينا التي تؤدي الى الحصول على هدفنا هي المقاس البسيط، المقاس الضربي و المقاس الغامر. أما الهدف الثاني الرئيسي هو أجراء تعميم للمقاس C1 باستخدام الادوات التالية:

المقاس الجزئي الثانيوي، المقاس الجزئي الأولي وشبه الأولي واخيرا المقاس الجزئي شبه الغامر.

وفي النهاية نقول كل النتائج في هذا العمل اعتمدت على مفاهيم اساسية في المقاسات الجزئية.

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**قسم الرياضيات**

**تطبيق خاصيتي الثنائية و الجوهرية على الموديول الموسع**

رسالة مقدمة

إلى مجلس كلية التربية للعلوم الصرفة - جامعة الانبار

وهي جزء من متطلبات نيل شهادة الماجستير في الرياضيات

من قبل

**عبدالسلام فائق طلك**

بكالوريوس رياضيات- كلية التربية للعلوم الصرفة - جامعة الأنبار 2018 -

**بإشراف**

**أ.م.د. ماجد محمد عبد**

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