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**Department of Mathematics**

**On Injective Modules and Related Topics**

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**By**

**Fawzi Noori Hammad**

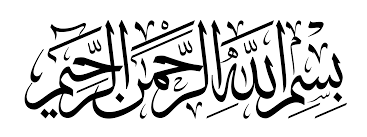
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﴿**فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ ۗ وَلَا تَعْجَلْ بِالْقُرْآنِ مِن قَبْلِ أَن يُقْضَىٰ إِلَيْكَ وَحْيُهُ ۖ وَقُل رَّبِّ زِدْنِي عِلْمًا**﴾

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الاية 114

سورة طه

**اهداء**

**إلى أبي الرجل المثالي أطال الله في عمره ليظل عونًا لي وقدوتي، ومثلي الأعلى في الحياة؛ فهو من علَّمني كيف أعيش بكرامة وشموخ.**

**الى أمي التي فارقتنا بجسدها، ولكن روحها ما زالت تُرفرف في سماء حياتي.**

**إلى إخوتي.... سندي وعضدي ومشاطري أفراحي وأحزاني.**

**إلى زوجتي.... أسمى رموز الإخلاص والوفاء ورفيقة الدرب**

**إلى ابنائي..... فلذات الأكباد.**

**إلى جميع الأخلاء؛ إلى جميع الباحثين، وطلبة العلم.**

***شكر***

**لابد لنا ونحن نخطو خطواتنا الأخيرة في الحياة الجامعية من وقفة نعود إلى أعوام قضيناها في رحاب الجامعة مع أساتذتنا الكرام الذين قدموا لنا الكثير باذلين بذلك جهودا كبيرة في بناء جيل الغد لتبعث الأمة من جديد ...  
وقبل أن نمضي تقدم أسمى آيات الشكر والامتنان والتقدير والمحبة إلى الذين حملوا أقدس رسالة في الحياة ...  
إلى الذين مهدوا لنا طريق العلم والمعرفة ...  
إلى جميع أساتذتي الأفاضل.......**

**كن عالما .. فإن لم تستطع فكن متعلما ، فإن لم تستطع فأحب العلماء ،فإن لم تستطع فلا تبغضهم  
  
وأخص بالتقدير والشكر:  
  
الدكتور ماجد محمد عبد  
الذي نقول له بشراك قول رسول الله صلى الله عليه وسلم:  
إن الحوت في البحر ، والطير في السماء ، ليصلون على معلم الناس الخير  
كما أنني أتوجه له بخاص الشكر ، إلى من علمنا التفاؤل والمضي إلى الأمام، إلى من رعانا وحافظ علينا، إلى من وقف إلى جانبنا عندما ضللنا الطريق.....**

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**وكذلك نشكر كل من ساعد على إتمام هذا البحث وقدم لنا العون ومد لنا يد المساعدة وزودنا بالمعلومات اللازمة لإتمام هذا البحث :**

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**Fawzi Noori .H**

**Supervisor Certification**

*I certify this thesis entitled* ***On Injective Modules and Related Topics*** *submitted by* ***Fawzi Noori Hammad****, has been prepared under my supervision at the University of Anbar, College of Education for Pure Sciences / Department of Mathematics, as a partial fulfillment of the requirements for the degree of master in Mathematics.*

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**List of Publications**

1. A New Results of Injective Module with Divisible Property. In Journal of Physics: Conference Series (Vol. 1818, (2021, March). No. 1, p. 012168). IOP Publishing.‏

2. Noetherian, Artinian Regular Modules and Injective Property, at the Journal of Al-Qadisiyah for computer science and mathematics, (2021), 13(1), Page-161.

3. Study Injective Module Over Dedekind Domain, at the AIP Conference Proceedings**,** (2021, March)**. (Accepted).**

4. Some Rings Give injective module. (2021), IEEE, Conference Proceedings. **(submitted).**

***List of Symbols***

|  |  |
| --- | --- |
| **Symbol** | **Meaning** |
|  | Submodule |
| ⨁ | direct summand |
|  | Arrow |
|  | The group of all homomorphism from M into N |
|  | Annihilator |
|  | Small submodule |
| C! | Contradiction |
| ≤ess | Essential submodule |
|  | Isomorphic |
|  | Equivalent |
| Rad(M) | Radical of M |
|  | External homomorphism |
|  | The kernel of a homomorphism |
|  | Torsion |
|  | The image of a homomorphism |
| **< >** | Generate |
|  | Polynomial ring |
| D.V.R | Discrete valuation ring |

**Abstract**

**ABSTRACT**

The main objective of this work is to study injective modules and related topics. Due to the relations between injective and divisible modules, a notion of several properties about both modules was studied. We introduce all of the key definitions used in the thesis, as well as some findings about the injective module a pear. Every injective module is divisible, but the inverse requires an additional condition P.I.D. Also, if the ring R is semi simple, and is a semi simple R-module, then is injective. Also, if is a cyclic and the regular module is injective. Also, if is regular with N≤ is a finitely generated submodule, so is injective. Here, we study several relationships between injective modules over the Dedekind domain. We prove that every divisible module over D.V.R is injective. Also, any pseudo injective module with R is a Dedekind leads to is an injective. Several facts about the relationship between the injective module and the Euclidean ring are satisfied here. And we find that every divisible module on the Noetherian valuation ring is injective. Also, there are some connections between (D.V.R) and the injective module.

Finally, we investigate the injective module in relation to other rings, such as the Noetherian ring, the local ring, the D.V.R, and the hereditary ring. We prove that if is an R-module and is a maximal ideal of R with quotient ring of R is a field, then is an injective module. Also, if R is a D.V.R, so any-divisible module over R is injective.

***INTRODUCTION***

***INTRODUCTION***

In 1970, Stevens ̈om introduced the notion of injective modules, and generalized the homological properties from Noetherian rings to coherent rings, and in this process, finitely generated modules were replaced by finitely presented modules. Recently, as extending work of Stenstr ̈om’s viewpoint, Gao and Wang introduced the notion of weak injective modules. This class of modules was also investigated by Bravo, Gillespie, and Hovey independently. In this process, finitely presented modules were replaced by super finitely presented modules. The fact shows that weak injective modules play a crucial role in the process of generalizing homological properties from special rings to arbitrary rings.

Let R be a commutative ring with identity and let be a unitary R-module. is called injective R-module provided that

1- is a submodule of Q such that Q is a module.

2- and s.t is an internal direct. See [15]. Or Any module is injective if the short exact

is split.

In our work we study injective module and some modules and some domains in detail were we investigate their basic definitions.

This thesis consists of three chapters. **The first Chapter** contains all the essential definitions used in the thesis.

**The second chapter** consists of **two section.** **In the first section**; we study an important concept namely divisible module, uniform module, non-singular module, semi simple module, P-injective and Q-injective modules. Some new results and properties have been studied in this notion. Every injective module gives divisible but the converse needs another condition P.I.D. Further more, we prove that if R be a P.I.D and is an injective R-module and then is also injective. We prove for any field K and every divisible on K–module, then is injective module. Also we showed if be a divisible module and is a projective module, then the homomorphism from into is also a divisible module. In addition, we get if R be a P.I.D. If is a uniform and non-singular and P-injective module then is injective module. Any R-module is called semi simple if is direct sum of simple submodule we present a good relationship between semi simple and injective-modules through let R be a ring. If R is semi simple and is semi simple R-module, then is injective module.

**In the second section**; we present new results about injective module depending other concepts, namely Noetherian, Artinian, hollow, semi hollow and regular modules. We showed that every Noetherian or Artinian Regular module over abelian ring R with unity is injective module. Further, we found if R be a (QF) and perfect ring or finite–dimensional ring. If is a regular R-module, then is injective. Any ring is called (quasi-Frobenius) if every projective module is injective; or every injective module is discrete. We get every cyclic regular R-module over QF-ring is injective. Let be regular module over P.I.D. If is acyclic module, then it is a Noetherian and is injective module. Let be a regular R-module. If is semi hollow and Rad() is a Noetherian module, so is injective. If proper f-generated submodule of is a small in (P. f-generated N<<), so is called semi hollow module such that is hollow if P-submodule N is small in . We showed the relationship between Noetherian module and injective module. If N≤ is f-generated, then is a Noetherian. We get if and be an R-modules over (QF)-ring and let : → be an onto homomorphism such that is a regular and is f-generated, so is an injective module.

**The third chapter consists of three sections.** In this chapter we will get injective module through some domains and from Euclidean ring and hereditary rings. **In the first section**; we present several results which give injective module over Dedekind domain. We prove Every finitely many prime ideals in a is P.I.D. And Let be a and it is a UFD. If every element of is divisible, so M is injective module. We explain the relationship between valuation ring and Dedekind domain. We prove every valuation ring is a . Let be an integral domain with no invers. If is a oetherian local P.M.I, then is a . We prove if a valuation ring and UFD and it is a oetherian local ring and =< a > so is divisible module then is injective module. An ideal in the ring is namely fractional if it is a submodule of H, there is 0≠S ∋ S⸦. Also, if is an abelian integral domain with 1 and it is a field of fractions, so we say F(D) is the set of non-zero fraction ideals of D such that D is an integral domain. Let D be an integral domain. hen D satisfies generalized domain G(D) if and only if for all ∈F(D), = is invertible. We get that the module is an injective from definition Dedekind domain and (GD) with another condition on a module. Let ∈F(D). If = Ɐ ∈F(D), with is a divisible D-module, then is an injective module. We showed if D be a completely integrally closed; Ɐ ∈ F(D), = If is adivisible module, then is an injective module. And we use a generalized of Dedekind domain in order to obtain module is an injective. Any integral domain D is called generalized (GD) if Ɐ an ideal in fractional ideals of D equal ; so= is invertible. Also, for binary operation ⁎, we say that ⁎-operation same v-operation. Therefore, the map.→ on the fractional ideals F(D) is v-operation. Also we prove be an –module and Ɐ is divisible element so D is ∋ is invertible and v-ideal.

**In the second section,** we study another ring namely Euclidean ring in order to obtain injective module. But this goal needs some additional conditions with Euclidean ring such as divisible module. By van der Warden, we say R is an Euclidean ring if R is integral domain such that the division Algorithm is true. We can rewrite this definition in another way (if ∃:R{0}→+{0} ∋ δ() greater than and equal and , s=0 or ∈ R; ∈R). We prove every divisible over Euclidean domain is injective module. Also we showed if be divisible R-module and R is (D.V.R) ∋ :R\{0}→N is Euclidian then is injective module.

**In the third section**; we study injective module over some other rings for instance Noetherian ring, local ring, D.V.R and hereditary ring. We prove that over Noetherian ring, if is-divisible, so is an injective module. Also, if is an -module and is a maximal ideal of with quotient ring of is a field, then is an injective module. We prove if ;Ɐ and , then is injective module. Any ring R is called hereditary (semi-hereditary) if each ideal (finitely generated ideal) of R is projective. And we prove if be semi-hereditary ring. Then every quotient module of injective module is injective. Also if is a D.V.R, so any -divisible module over is injective. Finlay let be any module over a semi prime ring . If is a lattice and [b) has a complement in (), , then is an injective module.

**CHAPTER ONE**

**Basic Concepts**

***(1.1).Baisic Conceptes***

In this chapter, we present the basic definitions and some results which have several relationships with the study.

**Definition (1.1.1).[22].** Suppose that R is a ring and 1 is its multiplicative identity. A left R-module consists of an abelian group (, +) and an operation⋅ : × → such that for all in R and in , we have:

1-

2

3

4-

**Examples. (1.1.2).[22].**

1. Any ring R is trivially an R-module over itself.
2. If S is a subring of a ring R any left R-module is also a left S-module with the restricted scalar multiplication.
3. Any matrix ring of a ring R is a R-module under componentwise scalar multiplication.
4. The vector space V is an R-module.

**Definition. (1.1.3).[22].** Let R-module A submodule of is a subset satisfying.

1. N is a subgroup of and
2. For all and all we have .

**Examples. (1.1.4). [22].**

1. Any module is submodule of itself called the improper submodule and the zero submodule consisting only of the additive identity of called the trivial submodule.
2. A left ideal I is a submodule of R viewed as an S-module where S is any (not necessarily proper) subring of R.

**Definition (1.1.5).** A module is injective if: is a homomorphism and is a homomorphism, Then there exists is a homomorphism such that

**Definition (1.1.6). [16].** Let be an R – module. If the next conditions are true, then is called injective module.

1. is a sub-module of Q s.t Q is a module.
2. and s.t is an

internal direct. So Any module is injective if the short exact sequence

is split such that the meaning of exact sequence and split can explain by the following:

(1) A pair of module homomorphism

M1 M2 M3

is called exact sequence at if =

(2) In general, we can say a finite sequence of module homomorphism

is exact if () = () : ∀ i=1,2,3,----n-1

**Definition (1.1.7)**. **[22].** Let be a submodule of an R-module . Then is called direct summand of if there is a submodule of such that and . And write

**Definition (1.1.8)**.**[22].**  Any module is called free if it is has a basis that is generating set consisting a linearly independent element.

**Definition (1.1.9)**. **[14].** Let be an module over integral domain R and is called torsion element if such that the set of all torsion elements in denoted by

T(. if T()= then is called torsion module and if T()=0 then is called torsion free module.

**Definition (1.1.10). [22].**  is called a flat module if for every monomorphism is also a monomorphism.

**Definition (1.1.11). [22].** An R-module is called cyclic if it is generated by a single element

**Definition (1.1.12). [27].** A ring R is coherent ring if and only if every direct product of flat module is flat and P-coherent(P-coh) if every principal ideal (P.I) is f-presented.

**Definition (1.1.13). [38].** (**Baer’s criterion**): Any module satisfy (Baer’s criterion) is injective if every ideal and every morphism

, with f(x)=mx, .

**Definition (1.1.14). [16].** Any module is called divisible if

Or; if every element of is divisible.

**Definition (1.1.15). [38].** R is a left P-coherent if and only if any direct product of torsion-free right R-modules is torsion-free.

**Definition (1.1.16). [19].** An R-module is called uniform if every submodule of is an essential in .

**Definition (1.1.17). [11].** If is a non-empty subset of R, then we denote its annihilator in and define it to be the set of elements such that

**Definition (1.1.18). [22**]. A submodule of is called essential, if whenever , then for each submodule of .

**Definition (1.1.19). [19].** An R-module is called nonsingular if

**Definition (1.1.20). [32].** An R-module is called pseudo-injective(P-injective) if every submodule of , each R-monomorphism can be extended to an R-endomorphism of .

**Definition (1.1.21). [21].**  is called Quasi-injective(Q-injective) if for each submodule of , and each R-homomorphism can extend to an R-endomorphism of .

**Definition (1.1.22). [38].** A module is called simple, if and it has only submodules {0} and .

**Definition (1.1.23). [19].** Any R-module is called semi simple if is a direct sum of simple submodule.

**Definition (1.1.24). [42].** Any module is called regular if Ɐ, g∊(,R), then(mg)m=m.

**Definition (1.1.25). [42].** If R has finite direct sum of ideals, so R is called finite dimensional.

**Definition (1.1.26). [5].** A module is called finitely generated(f-generated) if it has a finite set of generators. In other word; is finitely generated if .

**Definition (1.1.27). [8].** A ring R is called hereditary (semi-hereditary) if each ideal (finitely generated ideal) of R is projective.

**Definition (1.1.28).** **[22].** An R-module is called Artinian if satisfies the descending chain condition (DCC) on submodule of .

**Definition (1.1.29). [18].** The ring R is called domain if R is an integral domain and every f-generated. ideal of R is projective (invertible).

**Definition (1.1.30). [35].** A module is said to be discrete if it is lifting and has the property D2 (If N ≤ , such that is isomorphic to a direct summand of , then N is a direct summand of ).

**Definition (1.1.31). [29].** An R-module is called projective if and only if for any :C→V such that C,V are any R-modules and for any homomorphism a homomorphism such that .

**Definition (1.1.32). [22].** An R-module is called Noetherian if M satisfies the ascending chain condition (ACC) on submodule of M.

**Definition (1.1.33). [36].** An integral domain is Dedekind if 0≠ is a proper ideal factors into prime ideals.

**Definition (1.1.34). [41].** G-Dedekind domain carries same meaning of Dedekind domain; so an integral domain D is (GD) if for each A∈F(D); = is invertible**.**

**Definition (1.1.35). [36].**  field F is an abelian ring such that has trivial prime ideal. Or: is a Dedekind domain if it is:

1. is integrally closed.
2. is a oetherian ring; if 0≠ is maximal ∋ is prime ideal.

**Definition (1.1.36). [2].** Let be a domain. So is a valuation ring if it is not field; so∉.

**Definition (1.1.37). [41].**  n integral domain D satisfies generalized of D or (GD) if Ɐ ∈F(D)→=is invertible such that is namely invertible if ∃ is a fractional ideal and then = D.

**Definition (1.1.38). [40].** n ideal of F(D) is namely v-invertible if: ∃ ∈F(D)∋ =D.

**Definition (1.1.39). [41].** We say D is completely integrally closed (C.I.C) if an ideal ∈F(D) is v-invertible.

**Definition (1.1.40). [13].** For binary operation ⁎, we say that ⁎-operation same v-operation. Therefore, the mapping:→ on the fractional ideals F(D) is v-operation.

**Definition (1.1.41). [21].** Any integral domain D is namely type if it has a collection F={ of prime ideals such that,

1. D=∩.
2. Ɐ ; is a valuation domain.
3. 0≠ and not unit of D belongs to only a finite number of .

**Definition (1.1.42). [6].** D is called Mori domain if any set of the integral is a v-ideals of D and satisfies (ACC), such that any ideal ∈F(D) is v-ideal if =.

**Definition (1.1.43). [25].** Let D be an integral domain and let (⁎) an operation on D. We say D is (⁎)-oetherian domain if D has (ACC)on all ideals of D, (Quasi(⁎)-ideals). Or D is (⁎)-oetherian domain if 0≠ is (⁎)f-finite.

**Definition (1.1.44). [17].** Let D be an integral domain and (⁎) is operation on D. Then D is called -(⁎)multiplication domain.(P.(⁎). M. D) if Dm is a valuation such that m is a in S; ≠S= ideals of D (-(⁎)f –maximal ideal).

**Definition (1.1.45). [3].** Any ring R is called Euclidean if it is an integral domain such that R satisfies division algorithm.

**Definition (1.1.46). [34].** We say the function :K→Z is (D.V.R) if:

1. is onto.
2. ()=+.
3. ≥ min{,.}.

**Definition (1.1.47). [33].** For a module over a ring and for an ideal as a module of ; is called -injective if : is a homomorphism, then there exists an elements and such that the image of is (in other words can be extended to homomorphism of into ).

**Definition (1.1.48). [12].** Any -module is called divisible if = such that and is -divisible if . (-divisible give divisible over commutative ring).

**Definition (1.1.49). [7].** Any ring is called local if has a unique maximal ideal. In other word; every ideal is contained in some maximal ideal.

**Definition (1.1.50). [22].** The principal ideal I=<a> is called projective if and only if there exist such that and

**Definition (1.1.51). [4].** Any ring R is called a multiplication ring if all ideals are multiplication. Such that an ideal A is multiplication if every ideal B⊆A, ∃ an ideal C s.t B=AC.

**Definition (1.1.52). [20].** Any set is called saturated in R if . Also an element F in is completely irreducible if: F= So F= And F is called prime if:

or Therefor is completely prime if: .

**Definition (1.1.53). [9].** We say that an integral domain, R, is a UFD if every nonzero nonunit in R can be factored into irreducible elements, and if we have ··· = ··· with each irreducible in R then; (a) n = m and (b) there is a ∈ such that = for all 1≤ ≤n where each is a unit of R

**CHAPTER TWO**

**SOME MODULES AND INJECTIVE PROPERTY**

**SOME MODULES AND INJECTIVE PROPERTY**

**Introduction.**

This chapter introduces various concept namely divisible module, uniform module, non-singular module, semi simple module, P-injective and Q-injective modules. A module is called a divisible module if Or; if every element of is a divisible. We found that every injective module gives divisible but the converse needs another condition P.I.D. Also, some new results have been studied. Also, we showed if be a divisible module and is a projective module, then the homomorphism from into is also a divisible module. We present a good relationship between semi simple and injective-modules. Also we get if R be a P.I.D. If is a uniform and non-singular and P-injective module then is an injective module. In the last section, we provide several relationships between some concepts namely Noetherian, Artinian, hollow, semi hollow and regular modules and injective module. We prove that every Noetherian or Artinian Regular module over abelian ring R with unity is an injective module.

**§1: A New Results of Injective Module With Divisible Property**

In this section, we will present new results that clarify the relationship between divisible module and Injective module. If the short exact is splits, this means that is injective. Also a divisible module over P.I.D is injective.

Now we present the meaning of injective module in more depth way.

**Definition (2.1.1). [16].** Let be an R–module. If the next conditions are true, then is called an injective module.

is a sub-module of s.t is a module.

1. and s.t is an internal direct.

**Equivalent Definitions:**

**Definition (2.1.2).** Any module is injective if the short exact

is split s.t the meaning of exact sequence and split can be explained by the following:

(2) A pair of module homomorphisms

M1 M2 M3

is called exact at if = In general, we can say that a finite sequence of module homomorphism

is exact if () = () : ∀ i=1,2,3,----n-1

(2) An infinite sequence of module homomorphisms

….. →……..

is exact if () = ();

**Remark (2.1.3).** The exact sequence is homomorphism if and only if f is a module monomorphism

0 →

Also;

→0

is an exact sequence of homo. If and only if is module epi.

**Definition (2.1.4).** An exact sequence

0 0

and if for divisor in the ring R with , then is divisible.

**Lemma (2.1.5). [16]** Let R be a P.I. D. If is a divisible R- module, then is injective.

**Example (2.1.6).** If Z is P.I.D injective Z-module; Z is a divisible module over Z.

**Corollary (2.1.7).** Let R be a P.I.D if is an injective R-module and then is also injective.

**Proof.**

Since is divisible; so is divisible with R P.I.D imply is injective.

**Theorem (2.1.8).** Every divisible module over a filed K is injective module.

**Proof.**

Suppose that an ideal normal of K ( ) and so every ; = (K) . Therefore I = K and hence {0} = (0) and only Ideal's in K. Hence is a P.I.D. But is divisible module. Then is an injective.

**Theorem (2.1.9).** Let be a divisible module. If is a projective module, then the homomorphism from into is also a divisible module.

**Proof.**

Suppose that (,) and ( , ). IF

divisor (non–zero divisor in R). In order to prove that (,) is divisible we need to show that . Suppose that ϫ. be a homomorphism from into and define by:

. But we have is divisible. So ϫ is onto also we have is a projective module. Then is injective. Let ∊ . So

() = (()) = () and hence . Thus (, ) is divisible.

Recall that A right –module M and an element is called singular element of if . The set of all singular elements is denoted by . We say that is singular (resp. nonsingular) if (resp. )

**Corollary (2.1.10).** Let satisfies (ACC) over P.I.D. If Z() subset Z(R), then M is an injective R–module.

**Proof.**

If Z (R) be the zero divisors of R; Z() is a zero divisors of . Suppose that = R\Z (R) and = R\Z (). We know let ∊Since satisfies (DCC), then is an Artinian module So. ….

Hence = ; . Now if , then

=  n+1 ∋∊. Then ( – ) =0. Since ∊Г2 and – =0, then = . Therefore. = , ∊ Thus is divisible. But R is a P.I.D, then is injective.

**Corollary (2.1.11).** Let be a divisible R-module. If has no zero-divisors, then is injective.

**Proof.**

Suppose that a nonzero element in R. So .

Hence and then =0. But has no zero–divisors. So Hence is an injective module ) then is injective. Because if , ab and hence

**Theorem (2.1.12).** Let be invertible element of R in If is torsion–free module, then it is divisible ( is injective).

**Proof.**

Suppose that is a divisible module and Then

. [14]. So in thus is injective**.**

**Remark (2.1.13).** We know that a direct divisible over R is also a divisible module. Therefore it is easy say the following statement is true:

From [27], Recall that a ring R is coherent if is a f-generated ideal of R presented and we call R is P-coherent ring if is a principal ideal of R presented and if we have a direct product of copies of a ring R is a torsion free, then R is a P-coherent ring. And let be invertible. So the direct sum of torsion-free modules is also injective.

**Theorem (2.1.14).** Let a direct of P.I.D R is a torsion if

: → is an isom., then is injective.

**Proof.**

Since a direct of a P.I.D is torsion-free, R is a P-coherent. But is an isom. So, β ∊ ∋ ( β+ im(f) = β g = 1m.

Hence ≈ F. Then is divisible with R is P.I.D implies that is injective.

**Corollary (2.1.15).** **[22].** Let R be a ring with unity if and are two R-modules then there exists a one to one mapping : → .

**Corollary (2.1.16). [22].** Let G1 be a commutative group. If G2 is an abelian group, then there exists a mapping β: G1 → G2 is one to one such that G2 is a divisible.

**Proposition (2.1.17). [22].** An abelian group G embedded in a divisible commutative group. Moreover; If D is a divisible commutative group; is injective.

**Example (2.1.18).** We have Z is a PID injective Z-modules are divisible Z-modules (divisible abelian groups).

**Remark. (2.1.19). [39].** Consider *D* being an finite subset of a vector space *V* s.t *D* is a basis of *V*, so every non trivial, f-generated torsion-free is not divisible and we can find a *T:* *V* →inj(*V* ) *T* is injective and surjective.

**Corollary (2.1.20).** If we have a torsion-free Z-module *V*. Then inj (*V*) is a submodule of divisible module (*V* ).

**Proof.**

Set *E* = inj(*V* ). Set *D* = divisible of the module (*V*). If *a* objectinthe *E* and *a* inthe *D* by [26], mult *E* = mult *D*).

**Definition. (2.1.21).** **[27]**. A left R-module is called D*-*injectiveif for every divisible left R-module G and N is D*-*flatif for every divisible left R-modules N and G.

**Proposition (2.1.22). [39].** By Wakamutsu’s Lemma, any kernel of a D-cover is a D-injective and is D-flat iff is D-injective by

for every divisible left R-module N.

**Definition (2.1.23). [19].** An R-module is called uniform if every submodule of is an essential in .

**Definition (2.1.24).** **[19].** An R-module is called nonsingular if

**Definition (2.1.25).** **[32].** An R-module is called pseudo-injective(P-injective) if every submodule N of , each R-monomorphism

can be extended to an R-endomorphism of .

Recall that any module is called Quasi-injective(Q-injective) if for each submodule N of, and each R-homomorphism can extend to an R-endomorphism of . [21].

**Lemma (2.1.26). [22].** Every Q-injective module over P.I.D is an injective module.

Now depending on the last definition and lemma 2.1.26; we can present the following proposition.

**Proposition (2.1.27).** Let R be a P.I.D. If

1. is a uniform module;
2. is a non-singular module;
3. is a P-injective module.

Then is an injective module.

**Proof.**

Assume that is a non-singular and P-injective R-module. Suppose that K≤M and be a homomorphism. We have is a uniform () with non-singular property ; so ker( or ker(.

Case 1: If ker(, then the mapping can extend to homomorphism:

Case 2: If then is (1-1) and can extend to R-homomorphism from Hence is Q-injective. But every Q-injective is injective over P.I.D(lemma 2.1.26).

Recall that any R-module is called semi simple if is a direct sum of simple submodule in the next theorem, we present a good relationship between semi simple and injective-modules.

**Theorem (2.1.28)**. Let R be a ring. If R is semi simple and is a semi simple R-module, then is an injective module.

**Proof**.

We know every module over R is a homomorphism image of a direct sum of copies. So is a semi simple R-module, because:

Let be an R-module → is epimorphism. Hence is a direct summand of , then =+. So =≈N.Therefore N is semi simple (every submodule of semi simple is simple). Now let be an extension of so is a semi simple module. So is direct summand of , because every module can embed in an injective module ( is injective module).

**§2: Noetherian, Artinian Regular Modules and Injective Property**

In this section, we provide that several relationships between some concepts and injective module exist. We investigate, if is a cyclic and regular module is injective. Also, if is regular with N≤ is finitely generated submodule, so is injective. Finally, some relationships about injective module have been studied in details. We present some new results about the relationship between Noetherian, Artinian and regular rings and injective module. We should start with the following definitions.

We need to introduce a basic preliminary in order to proceed towards the main objective of the current study.

**Definition (2.2.1)**. **[42].** Any module is called regular if Ɐ, g∊(,R), then(mg)m=m.

See the following lemma:

**Lemma (2.2.2).** Every Noetherian or Artinian Regular module over abelian ring R with unity is an injective module.

**Proof.**

Any module fulfills all the conditions in theory, this means that is a finite direct sum of projective module have only two submodule are {0} and . Since R is a commutative ring with identity element, so is a flat module if and only if it is injective and a finite direct sum of injective is also injective.

**Definition (2.2.3). [29].**  An R-module is called projective if and only if for any :C→V such that C,V are any R-modules and for any homomorphism g:→V ∃ a homomorphism h:→C such that fh=g.

**Definition. (2.2.4).** **[42]**. A ring R is called a finite–dimensional if R has finite direct summand of ideals.

**Theorem (2.2.5)**. Let R be a (QF) and perfect ring or finite–dimensional ring. If is a regular R-module, then is injective.

**Proof.**

Let R be a perfect ring. Let T=direct limits of projective-module. Then T projective (see [24]). But is a direct limit of ∋ finite submodules. So is a projective. But from [29], for a QF-ring. every projective module is an injective module. Now if R has no infinite direct sums of ideals. Let be regular and Г={∑ Rmα : mα∊} is a partially ordered and Rmα ≤ Rmβ {mα } ⊆ {mβ }.So = Rmα is a maximal in Г (By Zorn’s Lemma). Then ∩Rm≠0, If , suppose that. Rm≈ ; is an ideal of R. So have no infinite direct sums of . But Rm=Rn1…… Rnt ∋ simple, ≠0 Ɐ. so =(Rn1…… Rnt) ∩N⊇. Hence and then = Therefor is a projective. Thus it is an injective module.

**Lemma (2.2.6)**. **[42]**. Every f-generated regular R-module is a projective.

Recall that for all , ∊ ∋ is a some of several generators of . So we can present a definition of finitely generated module by the following way:

**Definition (2.2.7)**. **[5].** A module is called finitely generated(f-generated) if it has a finite set of generators. In other word; is finitely generated if .

**Example (2.2.8).[5].** Any f-dimensional vector space is an f-generated over a field K.

**Proposition (2.2.9)**. Let be a regular R-module. If ≅, then is an injective module.

**Proof.**

Since ≅ , , so there is a homomorphism.

: Rn→ ≅ ∋ (r1,.....,)→ (r1,.....,)+N.

Take =(0,….,0,1,…,0) ∋ (1 being at the place).Hence generate Rn, 1≤ ≤n. so () generate over R, (1≤ ≤n).Therefor is f-generated module. But is regular module. So is projective (Lemma 2.2.6). Thus is injective module.

**Corollary (2.2.10)**. Every cyclic regular R-module over QF-ring is injective.

**Proof.**

Since is cyclic regular R-module then it is f-generated and by (Lemma 2.2.6), is injective.

**Corollary (2.2.11).** Let and be R-modules over (QF)-ring and let : → be an onto homomorphism such that is a regular and is f-generated, so is injective module.

**Proof.**

Suppose that and are two modules over the ring R. Also suppose that is f-generated module. To prove that is injective. Since is a f-generated. R-module, so has a generating set {m1,….,}. Therefore, we need to show that is generated by the set {(m1),…,(mk)}. Ɐ m2∊ , we have is onto, m1∊ ∋ (m1)=m2. But is a f.g module, m1=r1a1+………+, r1,…..,∊R. So m2=(r1a1)+………+() =(a1)+……k). Hence =<(a1), …….k)>. Then is a f-generated with regular property imply is projective and hence is injective.

**Theorem (2.2.12).** Let be a regular module over P.I.D. If is acyclic module, then it is a Noetherian and is an injective module.

**Proof.**

Since is a cyclic, then it is a f-generated. So have generators m1,…...mk. Hence ,∃ defined by

(b1,……,)=b1m1+……,+,

then , But R is a P.I.D, so R is a Noetherian R-module. We have as a regular module. Thus is injective (Lemma 2.2.2).

**Corollary (2.2.13)**. Let be a regular R-module. If and are Noetherian ∋ N is a submodule of , so is injective module.

**Proof.**

Assume that ≤. So in is f-generated. Hence is also f-generated. Let k1,……∊ generate in and let b1,……,generate . Ɐ , r1k1+……+ , ∊R. So K-∑∊. Then -∑=∑, ∊R. Hence =∑+∑ .(K f. g in ) therefore is a Noetherian module with regular property implies is an injective module (Lemma 2.2.2).

Recall that any R-module satisfies the maximal condition for submodules if ≠ Г of submodules have a maximal (Г⸧ H0 ∋ , number containing H0). Therefore it is easy to present a definition of Noetherian R-module, any module satisfies the maximal condition(ACC) is Noetherian.

The next theorem shows the relationship between Noetherian module and injective module; but before that we need to present the following lemma:

**Lemma (2.2.14)**. If every submodule of an R- module is f-generated, then is a Noetherian.

**Proof.** Suppose that N≤ is a f-generated assume that H1⊆ H2 ⊆H3⊆…. is a submodules of .  Take H=∪, =1,……,∞. So H≤ and hence H is a f-generated. Let H=Rh1+……+. All hi in one of Hi, m ∋ h1,…….hn ∊Hm. But H=, . So is a Noetherian module.

**Theorem (2.2.15)**. Let be an R-module. If is a regular module and has N is a f-generated submodule of , then is injective.

**Proof**.

By Lemma (2.2.14), is Noetherian-module (N≤ is an f-generated). But is a regular module. Then from(Lemma 2.2.2), is injective.

**Theorem (2.2.16).** Let be a regular module and let

0→→→→0 be an exact seq. If and are Noetherian, then is an injective module.

**Proof.**

Suppose ≤, = and assume that andare Noetherian.

Let H1⊆ H2 ⊆…....., ⊆⊆⊆…… of and , m ∋ =Hm∩and +=+, n ≥ m. So =∩(+) = ∩(+)= +(∩) by Modular law, Let H, Y, L ≤ and Y⊆ H. So H∩(Y+L) = Y+(H∩L)).= +(∩) = . So is Noetherian module with regular property, we get is injective.

**Example (2.2.17)**. Any module over a division ring is injective, because a division ring R has only 2 ideals 0 and R itself.

**Proposition (2.2.18).** Let be a regular module. If:

1. S1 is the set of f-generated submodules of is Noetherian,

2. ≠N1 is f-generated and N1 ≤ ∋ N1 has maximal element,

3. N ≤ is an f-generated.

Then is injective

**Proof.**

Let S1≠ be a set of f-generated (S1 ≤ ) If S1 has no maximal element, so any s∊S1 {S2∊S1: S2 ⸧ S1, S2≠S1 } ≠,thus we get (ACC)of submodules which is infinite.

Now let N ≤ , there is a maximal element N1. then N1=N. Now let

H1 ⸦ H2⸦……. Be (ACCof submodules of . So ∪⸦ is a f-generated and all generating elements in .

Thus =Ɐ . So is a Noetherian module But every Noetherian module is Artinian with regular property is an injective module. (Lemma2.2.2).

Recall that if proper f-generated submodule of is a small in (P. f-generated N<<), so is called a semi hollow module such that is hollow if P-submodule N is small in . Therefore we present the following theorem.

**Theorem )2.2.19(.** Let be a regular R-module. If is semi hollow and Rad() is a Noetherian module, so is injective.

**Proof.**

Assume that is semi hollow-module. Let Radical of not equal . there are Max(N) ∋ N≤. This means that is also module. Hence Rad() is a maximal and Rad<<. Hence is a simple module and hence is Noetherian. Since

0→Rad()→→ →0

is a short exact seq. Then is a Noetherian ( is Artinian-module) with regular property implies is an injective module.

It is possible to rely on the previous example to discuss its content in another way, follows:

**Proposition 2.2.20**. Let R be a division ring if:

1. is a regular module over R.
2. is a divisible-module.

Then is injective.

**Proof.**

Assume that R is a division ring and is a divisible-module. Let

N such that K is a basis for N. So a basis k1 ofk1 K. Assume that k2 = K1k and N1 is a span of k2. Then N1⨁N= . Hence is a semi simple ( is Artinian). But is a regular. Thus is injective.

**Example )2.2.21(**. Any regular module of the ring R which has only two ideals {0} and R is Artinian, because R has only two ideals {0} and R implies R is division and hence is Artinian with regular property we get is injective.

**Proposition )2.2.22(**. Let , and are three modules if

1. 0→→ →→0 is short exact of R-modules.
2. and are Artinian modules.
3. is a regular module.

Then is injective.

**Proof.**

Take a chain such that are submodules of . From the projecting to , () is stabilized, so if

f:→ , the from chain submodules of . Hence it is stabilizes. Then is Artinian module with condition (3), we get is injective (Lemma 2.2.2).

**Remark )2.2.23(. [22].** Every homomorphic image of Artinian ring is Artinian.

**Theorem )2.2.24(**. Let R be an Artinian ring. If is a f-generated. R-module and regular, so is an injective module.

**Proof.**

We know that ≅ such that N≤ and n+. But is an Artinian ring, so a direct sum of Artinian modules. Hence is an Artinian module (Remark 2.2.23). But is regular. Thus it is injective module.

**Definition )2.2.25(**. **[17].** The ring R is called domain if R is integral domain and every f-generated. ideal of R is projective (invertible).

In the Theorem(2.2.27), we study some conditions over domain in order to get injective module.

**Remark )2.2.26(.** Any ideal I is injective if II-1=R ∋ I-1={ ⊆ R: and (R) is the field of fractions is the smallest field can be embedded.

Recall that if R2 be a unitary extension ring of R1. We say R2 is a p-extension of R1 if Ɐ r1∊ R2 satisfies R1[X] one whose coefficient is a unit of R1(whose coefficients generate a unit ideal of R1).

**Theorem )2.2.27(**. Let R is a ring. If

1. R is a prufer domain Krull domain 1,
2. is a divisible R-module,
3. is Artinian R-module.

Then is injective.

**Proof.**

Over pruferdomain any module is linearly compact and divisible is injective or, Artinian module is linearly compact with divisible property is injective.

**Proposition )2.2.28(**. Let be an R-module. If:

1. satisfies (DCC),
2. Every element is a divisible,
3. R is integrally closed domain with quotient field K,
4. K is a p-extension of R.

Then is injective.

**Proof.**

Assume that K is P-extension of R. and Let I⋌ be a maximal ideal in the ring R1. Let H=all elements f in R1[X] ∋ =R1. So H is a regular multiplicative in R1[X]. Hence H= R1[X] - ∪ I⋌[X]. So if I1 is an ideal of R1[X]⊆∪ I⋌[X]. then I1 contained in one of I⋌[X]. So {I⋌[X]} is the set of prime ideals of R1[X]. Hence R is a domain. Thus is an injective module because from condition (1), is Artinian and by condition (2), is a divisible module.

**CHAPTER THREE**

**INJECTIVE MODULE OVER SOME DOMAINS**

**INJECTIVE MODULE OVER SOME DOMAINS**

**Introduction.**

In this chapter we introduce how we get injective module through some domains and also from Euclidean ring and hereditary rings. In the first section we present several results which give injective module over . One of these domains is the Dedekind domain. We proved that any element of -module is divisible and let be a Dedekind domain and If is a finitely many prime ideals, so is an injective module. We take another domain which is a unique factorization domain (UFD). We showed if be a and it is a UFD and If every element of is divisible, so is injective module. From Krull domain, Mori domain we will get that as injective. In section two we satisfy several facts about the relationship between injective module and Euclidean ring. We get every divisible module over Noetherian valuation ring is injective. In addition, there are some, connections between (D.V.R) and injective module. In the last section we present a main relationship between some rings (hereditary ring, local ring and D.V.R) and injective module.

**§1: Study Injective module over Dedekind domain**

In this section, we study injective module over Dedekind domain. Some results have been obtained about this relationship. Before getting deeply into the relationship between injective module and Dedekind domain, we need some definitions and lemmas related to the topic.

**Definition (3.1.1). [37].** Any ring R is called Dedekind if it is an integral domain and every 0≠ is a factors into product of prime ideals.

To understand , we need to define some concepts, such as filed, and integral domain. **An ideal I of the ring R is a prime ideal if ∈ R then either or for all . Also, in [14],** any ideal I of Z is a f-generated Z-module is called fractional ideal and is denoted by (FI), if for every maximal ideal, is principal ideal over the ring is invertible.

**Lemma (3.1.2)**. **[1].** Every 0≠f is invertible; f is fractional ideal.

**Examples and remark (3.1.3).**

1 Every P..D is a .

2. is a P.I.D if and only ifevery fractional ideal f is principle.

3. If is Dedekind domain, so is UFD if and only if is P.I.D.

4. A localization of a Dedekind domain of multiplicative set is a .

**Lemma (3.1.4).** Every finitely many prime ideals in a is P.I.D.

**roof**.

Let 0≠ is a prime ideals. If 0≠ is an ideal so, ∃ ∋is a P.I(= <a>) and hence is a relatively prime to But S≠ ∅ ∋ S is the set of prime factors of . Hence = and thus <a>= ==

**Lemma (3.1.5).** Let be a and it is a UFD. If every element of is divisible, so is an injective module.

**Proof**.

We know that any commutative ring is P.I.D, because is a UFD. But P.I.D with divisible module indicates that is injective.

**Theorem (3.1.6).** Let be a with nonzero fractional ideal . If is a divisible R-module such that is an integral domain, then is an injective module.

**roof.**

We know that there is 0≠fractional ideal and =. By defining the fractional ideal, there is 0≠ an element in and is an integral ideal. Assume that =. herefore = and hence is a P.I. in . ence is P.I.D. But is a divisible. Thus is an injective module.

**Corollary (3.1.7).** Let any element of -module be divisible and let be a Dedekind domain. If is a finitely many prime ideals, so is injective module.

**roof**.

Let be all prime ideals. If 0≠is an ideal, then ∃ 0≠ and is P.I, (; is a relatively prime to . The factors of =φ. So =. Hence (== =. Then is a P.I.D, but every element of is divisible (=), , . Hence is a divisible module. is injective.

The next lemma explains the relationship between valuation ring and Dedekind domain. Now we start with a clear definition of valuation ring.

**Definition (3.1.8). [2].** Let be an integral domain. So is a valuation ring if it is not field; so∉.

**Lemma (3.1.9)**. Let be an integral domain and with no invers. If is a oetherian local P.M.I, then is a .

**roof**.

Take the maximal ideal =(S) in . We need to show =< a >. (principal). We have is an f-generated. Then, ∃ n is a maximal ∋ ⸦. Ɐ , so b∉; b= u ∋ u is unit and hence b(But this true for all b and since ⸦, =. Thus is a local and P.I.D. But every ..D is Dedekind domain.

**Lemma (3.1.10).** Let be an integral domain has no invers. If is a local such that 0≠ is invertible a .

**roof**.

Since any invertible ideal is a f-generated, then is a oetherian ring. We must prove that is a maximal where = < > ( i.e. is principal). But from Nakayama̕s lemma; . If and s with (s∉); sand M = (S). is a principal. But from lemma (3.1.9); we have is ..D. is a .

**Theorem (3.1.11).** Let be a ring If:

1. a valuation ring and UFD;
2. is a oetherian local ring;
3. =< a >;
4. is a divisible module.

Then is injective module.

**roof**.

Since is a oetherian valuation an ideal by finitely many elements. ence one of them contains all others and be . hen is a P..D. ( is Dedekind). But is UFD and is a divisible module. So is injective.

**Corollary (3.1.12)**. Every V.R is a .

**roof**.

Clear. Every D.V.R is P.I.D and hence is a Dedekind domain

**Corollary (3.1.13).** Let be a ring. If:

1- is D.V.R;

2- is divisible;

is UFD.

Then is an injective module

**roof.**

Since every D.V.R is a Dedekind domain and every D.V.R is a P.I.D, then from condition (3) and condition (2); we get is injective module.

**Example (3.1.14). [22].**  is a fraction field . So is P.I.D with maximal ideal is D.V.R. Thus Dedekind domain.

Recall that an ideal in the ring is namely fractional if it is a submodule of H, there is 0≠S ∋ S⸦. Also, if is abelian integral domain with 1 and it is a field of fractions, so we say that F(D) is the set of non-zero fraction ideals of D such that D is an integral domain.

**Definition (3.1.15).**n integral domain D satisfies generalized of D or (GD) if Ɐ ∈F(D)→=is invertible such that is namely invertible if ∃ is (FI) and then = D.

Now If each fractional ideal in R , so is a . Therefore; from the definition the Dedekind domain and (GD) with another condition on a module, we can get is injective. However before that we need to present the next definition.

**Definition (3.1.16).** Let D be an integral domain. hen D satisfies generalized domain (GD) if and only if for all ∈F(D), = is invertible.

**Theorem (3.1.17).** Let ∈F(D). If = Ɐ ∈F(D), with is a divisible D-module, then is an injective module.

**Proof**.

Ɐ ∈F(D), = , so ()- 1=, because

=(()v)-1 = (()v)-1 =()-1. Hence is invertible in F(D). Therefore is (GD) and so D is a Dedekind with is divisible imply is injective module.

**Proposition (3.1.18)**. Let D be a completely integrally closed;

Ɐ ∈ F(D), = If is a divisible module, then is injective.

Before start with proof of proposition 3.1.18, we need to define some concepts.

**Definition (3.1.19). [40].** n ideal of F(D) is namely v-invertible if: ∃ ∈F(D)∋ (=D.

**Definition (3.1.20). [41].** We say that D is completely integrally closed (C.I.C) if an ideal ∈F(D) is v-invertible.

Now we start with proof of proposition (3.1.18).

**Proof of proposition 3.1.18**

Let ∈F(D). So D is (C.I.C). (by definition). Hence ()v=D ([10]). But if D is a C.I.C, then (= Ɐ ∈F(D). hen (=()v = Ɐ ∈F(D). So is invertible. Therefore D is Dedekind domain. Butis divisible, so is an injective module.

Now we use a generalized Dedekind domain in order to obtain that module is injective. From definition of invertible concept, we can present the following information.

Any integral domain D is called generalized (GD) if Ɐ an ideal in fractional ideals of D equal ; so= is invertible. Also, for binary operation ⁎, we say that ⁎-operation same v-operation. Therefore, the map.→ on the fractional ideals F(D) is v-operation.

**Definition (3.1.21). [23].** Any integral domain D is namely type if it has a collection F={ of prime ideals such that,

1. D=∩,
2. Ɐ ; is a V.R,
3. 0≠ and not unit of D belongs to only a finite number of .

**Remark (3.1.22).[41].** Ɐ ∈F(D), is said to be v-invertible if ∃ ∈F(D) ∋ =D. So Ɐ ∈F(D), we say that is v-invertible and hence D is a completely integrally closed. Note that, D is called a Mori domain if any set of the integral is a v-ideals of D and satisfies (ACC), such that any ideal ∈F(D) is v-ideal if =.

From([31]), an integral domain D is a Mori if and only if ∈F(D),∃ a f-generated. ideal ∈F(D)∋ and =. Therefore

1-Completely integrally closed with domain give(GD).

2-Mori domain and Mori domain with Completely integrally closed give domain

**Example (3.1.23).** (GD) give Completely integrally closed.

**Lemma (3.1.24).** **[41].** Every domain is a completely integrally closed.

**Theorem (3.1.25).** Let be an -module. If:

1. Ɐ is divisible element.
2. D is ∋ is invertible and v-ideal.

hen is injective.

**roof.**

Suppose that D is domain. Hence D is completely integrally closed. Therefore D is a (GD) (proposition 3.1.18). Now if an element such that it is divisible, then is a divisible module over D. But from the definition of (GD); we obtained D is a Dedekind domain. Now is divisible over Dedekind domain; this means is injective module.

**Example (3.1.26).** **[41].** Any divisible D-module over D is the entire function is injective.

Note that from a proof of proposition (3.1.18), we say if D is a completely integrally closed, so D is (GD). Then the converse is true in general (see the following result):

**Corollary (3.1.27)**. Any (GD) is completely integrally closed.

**roof**.

Suppose that we have (GD). So Ɐ ∈F(D),

(=)v= (D=D. Also, every 0≠∈D is v-invertible. Then D is a C.I.C.

**Theorem (3.1.28). [41].** Let D be an integral domain. Then D is completely integrally closed if and only if D is a (GD).

There is another way to prove that D is a domain. This way start with some definitions for example;(⁎)-finite ideal,

G⁎-Noetherian domain, (⁎)-Dedekind domain and P(⁎).M. domain. oetherian ring.

Let D be an integral domain and Let (⁎) be an operation on D. If ∈F(D), so is (⁎)-finite if ∃ ∈F(D)∋⁎=⁎ and is invertible. ([37]).

**Definition (3.1.29).** **[10].** Let D be an integral domain and let (⁎) an operation on D. We say D is (⁎)-oetherian domain if D has (ACC) on all ideals of D. (Quasi(⁎)-ideals). Or D is (⁎)-oetherian domain if 0≠ is (⁎)f-finite.

**Example (3.1.30).** Any oetherian domain is (⁎)-oetherian domain. Also every Mori domain is (⁎)-oetherian domain.

**Definition (3.1.31). [41].** Let D be an integral domain and (⁎) is an operation on D. Then D is called -(⁎)multiplication domain.(P.(⁎). M. D) if Dm is a valuation such that m is a in S; ≠S= ideals of D (-(⁎)f –maximal ideal).

**Definition (3.1.32).[41].** If D is (⁎)-oetherian domain and .(⁎).M.D, so D is called (⁎)-Dedekind.

**Lemma (3.1.33). [10].** Every (⁎)-finite is (⁎)-Noetherian domain.

**Lemma (3.1.34).** Every (⁎)-oetherian domain and P.(⁎).M. domain is (1)-Dedekind domain (Dedekind and v-operation).

**(**2)- (⁎)-Dedekind with v-operation is domain.

**Corollary (3.1.35).** Let D be an integral domain. If:

1. is (⁎)-finite Ɐ ideal of D.
2. Each element is a divisible element ∋ is an D-module.
3. D is P.(⁎).M.D.

hen is injective.

**roof.**

Since is a (⁎)-finite, then Ɐ ∈F(D), ∃ ∈F(D) ∋ ⁎=⁎. Hence is an invertible ideal such that is subset of . Therefore D is (⁎)-oetherian domain. In other words,

0≠ ∈ D, so ∃ ∋ ⁎=⁎ and

is a finitely generated ideal. We have D is a P.(⁎).M.D. Then D is a (⁎)-Dedekind domain

(i.e. D is v-operation). Now D is (⁎)-Dedekind with v-operation; this means D that is domain. Hence D is a completely integrally closed and then it is a (GD). Thus Dedekind with condition (2); implies that is an injective D-module.

Recall that in [32], An R-module is called pseudo-injective(P-injective) if every submodule N of , each R-monomorphism

can be extended to an R-endomorphism of . Also, in [21], Any module is called Quasi-injective (Q-injective) if for each submodule N of , and each R-homomorphism can be extend to an R-endomorphism of .

**Lemma (3.1.36).** **[28].** Every P-injective module over Dedekind domain is Q-injective.

**Proposition (3.1.37).** Let R be a Dedekind domain and P.I.D. If is a P-injective; then it is an injective module.

**Proof.**

Let H≤. Since is a P-injective and every P-injective is Q-injective (see lemma 3.1.36). But R is P.I.D; with Dedekind Property that is an injective module.

**§2: Injective modules and Euclidean Ring**

In this section, we satisfy several facts about the relationship between injective module and Euclidean ring. The main result is that every divisible module over Noetherian valuation ring is injective. Also, there are some, connection between (D.V.R) and injective module. We prove that every divisible module over (UFD) is also injective.

**Definition (3.2.1)**. By van der Warden, we say that R is an Euclidean ring if R is integral domain such that the division Algorithm is true.

**Remark (3.2.2)**. We can rewrite Definition )3.2.1(in another way (if ∃:R/{0}→+{0} ∋ δ() greater than and equal and

, s=0 or ∈ R; ∈R).

**Lemma (3.2.3)**. If is is a divisible module over P.I.D, so is injective module.

**Proof.**

Suppose that is a divisible R-module. Let f:→ be an R-homomorphism, where is a left ideal of R since R is a principal ideal ring, = for some , since is divisible, there exists an m∈ such that =tm define g: → by =. Then g is R-homomorphism Note that every

, Thus is injective.

**Lemma (3.2.4).** Every divisible over Euclidean domain is an injective module.

**Proof.**

Suppose that R is an Euclidean domain. Let ⊲ R. Hence={0}=(0) or

let 0= ∋ d( least, so any ,we obtain and =0 or

d() < d(. But =- ∈ . Since d( is a minimal and and =0, and =(. Hence R is a P.I.D. But is a divisible module. Thus is an injective module.

The following shows that divisible Z-module over the integer numbers is injective module;

Suppose that ⊲Z. if ={0}, so (0) and is a principal ideal. Assume that ≠{0} and is the smallest integer ∋ is a positive in We must prove that =(. Since ∈, so (⊑. If ∈, then for some belong to Z, 0 ≤ r ≤ -1. Hence =; . Since is the smallest positive in , hence r=0. So and ∈(. But is a divisible module Thus is injective.

Recall that any filed is an Euclidean ring and this leads to the following result:

**Theorem (3.2.5).**  Let K be a field. If is a K-divisible module, so is injective.

**Proof.**

The proof of this statement is easy, because; if ⊲K ∋ 0≠∈, so every ∈K and hence: =)∈.

Hence =K. therefore only ideals of K are {0} = (0) and K= (1). Then K is P.I.D (is injective because it is divisible).

**Theorem (3.2.6).**  Let R be an Euclidean ring. If

1. is a divisible R-module.
2. is a maximal ideal of R.

Then is –injective module.

**Proof.**

Suppose that ⊲R ∋ is a maximal ideal of R. Suppose that ∈ and ≠0. To prove that ,∃ ∈ ∋ (=.

Let = <> ∋ is an ideal. So and ().

We have is a maximal ideal. So =R (Definition of maximal ideal). Then . Hence 1= Ɐ y ∈R, . Therefor

=()+==()().

Hence is a field. Then is an Euclidean ring, with is divisible module imply is injective.

**Example (3.2.7).**  Any divisible module over is injective because is a field.

Recall that a domain R is a valuation ring () if R is not field and , ∉R ∋ K is a field. Therefore we can define discrete valuation ring ( on K by the following:

**Definition (3.2.8).[33].**  We say the function :K→Z is (D.V.R) if:

1. is onto.
2. ()=+.
3. ≥ min{,.}.

**Remark (3.2.9). [2].**  An integral domain R which is (V.R) of (D.V) on K is a (D.V.R) and hence every (D.V.R) is an Euclidean domain.

**Proposition (3.2.10).**  Let be an R-module. If:

1. is divisible module;
2. R is (D.V.R) ∋ :R\{0}→N is Euclidian.

Then is an injective module.

**Proof.**

Suppose that ∈ R\{0}. From condition (2) ≤. If ≥, so )≥0 and ∈R. Take the following equation:

a=)b+0 ∋ ≥…………..(⁎)

a= 0b+a ∋ <…………..(⁎⁎)

If (⁎) and (⁎⁎) satisfies Euclidean norm, so ∃ ∈R a=, then

or <. Hence R is an Euclidean domain. But from condition (1) is a divisible module. Thus is injective.

**Corollary (3.2.11). [ 9].** Let R be a ring. If:

1. R is UFD has a unique irreducible element.
2. is a divisible R-module.

Then is an injective module.

**§3: Some Rings Give injective module**

In this section we study injective module over some other rings, such as Noetherian ring, local ring and Hereditary ring.

**Definition (3.3.1). [33].** For a module over a ring and for an ideal as a module of ; is called -injective if : is a homomorphism, then there exists an element and such that the image of is (in other words can be extended to homomorphism of into ).

**Remark (3.3.2).** Every module is injective if it is an -injective such that is a right ideal (in other words any -injective module is injective).

Note that if is a finitely generated, so it is clear that every f-injective module is injective over Noetherian ring ( is f-injective module if it is -injective). Also, any f-injective module over Noetherian ring is injective. By the next lemma, we can start with the main goal of this thesis, which is how to get the injective module.

**Lemma (3.3.3).** Let be a Noetherian ring. If is a f-injective, then it is injective module.

**Proof.**

Suppose that is a Noetherian ring. So every ideal of is a f-generated. Hence every f-injective module is injective.

**Definition (3.3.4).[40].** Any -module is called divisible if such that and is -divisible if . (-divisible gives divisible over commutative ring).

**Theorem (3.3.5).** Let be a Noetherian ring. If is-divisible, so is an injective module.

**Proof.**

From [30], Lemma 3.3.3, every -divisible module is an -injective. But every -injective is f-injective. We have is a Noetherian ring. Hence is a finite generated ideal. Thus is an injective module.

**Corollary (3.3.6).** If Ɐ and , then is injective module.

**Proof.**

Suppose that →M ∋

and =. Take is a restriction of to and respectively. Hence

∃ , ∋= Ɐy∈and . Since coincide with on , so ((= ∈( and ( ∋ =. Assume that

=. Hence . Thus is -injective. Therefore is injective module (Remark 3.3.2).

We need to introduce more results about the relationship between injective module and this goal achieved by studying in the three rings in depth; Noetherian rings, D.V.R and local rings.

**Remark (3.3.7).** If is a finitely generated ideal of so it is normal is a Noetherian and vice versa and more, so when the ring is a P.I.D, we get same result.

**Proposition (3.3.8).** Let be a finitely generated ideal of . If is -divisible, then is injective –module ( is injective over the ring ).

**Proof.**

The main goal of proof is how to prove that is a Noetherian ring. We have is a finitely generated ideal in . So is a Noetherian ring. Suppose that is also an ideal in . We take the following set:

It is clear that is an ideal of . Then is finitely generated and we say:

=(,………).

Assume that can be write by the following

If in , so and hence t can be written as:

+………; Ɐ .

Hence can be write it by +………. So is a Noetherian ring. Thus is injective module (See Theorem 3.3.5).

Note that from (Hilbert Basis Theorem); if is a Noetherian ring (every ideal of is finitely generated), so also is a Noetherian. Therefore we can introduce the following result:

**Corollary (3.3.9). [22].** Every -divisible module over is injective module.

**Definition (3.3.10). [7].** Any ring is called local if has a unique maximal ideal. In other words; every ideal is contained in some maximal ideal.

Now from the definition of Noetherian ring and local ring, we can combine both concepts in order to obtain is injective module. See the following proposition:

**Proposition (3.3.11).** Let be an -module. If.

1. is -divisible.
2. is an irreducible element in .
3. is a unite; .

Then is injective module.

**Proof.**

First, the unique maximal ideal is principal. Suppose that = and . So every element in -{} is a unique and if =, then is also a unit, Hence, therefore and . Thus the uniqueness is true. Hence is local ring. Let ∋ is an ideal. If so ⊆ is maximal ideal. Take n is a maximal element such that and We define the following set by:

.

Note that is an ideal and ). Now if S is a unite otherwise , so Hence any

ideal of is not equal zero and take the from ), . Then all ideals of is finitely generated. Therefore is Noetherian ring. But from condition (1); is -divisible. Thus is injective -module.

**Definition (3.3.12).** **[22].** A ring is called D.V.R if:

; :

1. is a unit.
2. .
3. .

Now from Definition 3.3.12; if is Noetherian and local ring, then is D.V.R. So:

**Corollary (3.3.13).** If is a D.V.R, then any -divisible module over is injective.

**Proof.**

By Definition )3.3.12( and proposition )3.3.11(.

**Example (3.3.14).** If is a ring and is the all non unit, so any I-divisible module is injective, because if non unit ideal of , then is a local ring and we know that is a Noetherian if is a Noetherian ring.

**Example (3.3.15).** If is -divisible over the ring and is a finitely prime generated ideal of , then is injective over where = and is all equivalence classes of fraction ; is the complement of and has no zero divisors.

**Example (3.3.16).** is local ring such that it is a maximal finitely generated ideal (is the localized at the prime ideal (I) and and I not divisible .

**Example (3.3.17).** If is a domain and Then

. So is a local ring and any ideal of is finitely generated, then is a Noetherian ring. Such that = is a multiplicative set ( is called with and ).

Now we study injective module over hereditary (semi hereditary) rings.

**Definition (3.3.18).** **[8].** Any ring is called hereditary(semi-heredity) ring if every quotient module is injective (f-injective).

Recall that any f-injective module is injective implies that is a Noetherian ring. Moreover; any f-injective over Noetherian ring is injective.

**Theorem (3.3.19).** Let be a semi-hereditary ring. Then every quotient module of injective module is injective.

**Proof.**

Suppose that is f-injective and be a homomorphism. Also,  such that is a f-generated right ideal of .

Such that i is inclusion mapping. Then is finitely generated and projective. Hence ∋ h=. Note that is f-injective. So ∃ : → ∋ =. Also , so ==h=. Hence

f-injective. Then the quotation module of injective is also f-injective and hence is injective.

Recall that a ring R is regular ring if for each there exists such that [20].

**Lemma (3.3.20).** Every regular ring is a semi-hereditary self-injective.

**Proof.**

We know that f-generated right ideal of is generated by an idempotent because is a regular ring. Hence is a semi-hereditary. Suppose that an idempotent element. Suppose that . Then =. Also, Hence is f-injective. Thus is injective and is a self f-injective.

**Theorem (3.3.21).** Let an -module. If is a maximal ideal of and the quotient ring of is a field, then is an injective module.

**Proof.**

Let and }. Since is a field, then is not contained in any maximal ideal I containing r. Let , so C is also not contained in any maximal ideal of . We know that. Also since, so is a direct summand of . Now is idempotent of . Also, If =1-. Then is a regular ring. By Lemma 3.3.3 is semi-hereditary self injective and hence is f-injective. Thus is an injective module.

**Remark. (3.3.22).** **[29].** Any ring is called (quasi-Frobenius) if every projective module is injective; or every injective module is discrete.

**Definition (3.3.23). [22].**  Any module is called free if it has a basis that is generating set consisting a linearly independent elements.

**Proposition (3.3.24).**  Let be a ring. If:

1. is a local ring.
2. is a ring.
3. is a flat module.

Then is injective.

**Proof.**

First, we need to prove that is a free module. Assume that is not free. Suppose that is a flat module. Let be a minimum number of all elements such that m generating and t is the minimum of and the sum of Not that is a minimum base of . There is and we assume and

, ).

So and ; ∈R, .

If so Since is a minimal, then is a unite of . If , then

For then as . For ,

so 0 such that . Hence

( ,…,). C!

So is a free module. Therefore is a projective. But is local and ring. Then every projective module is injective.

**Proposition (3.3.25).** Let be any module over a semi prime ring . If is a lattice and [b) has a complement in (), , then is injective module.

**Proof.**

Suppose that is a semi prime ring and [) has a complement in (). So by [ Lemma 4], (Lemma 1 in , =. Now in and = in Hence (). So is a regular ring. Thus is a semi hereditary ring and hence is injective (see Theorem 3.3.21).

**Example (3.3.26).** The integer number is a semi prime ring so it is a regular ring ( is semi prime ring). Hence any module over is injective.

In the next result, we study the relationship between Noetherian ring and semi prime ring.

**Corollary (3.3.27).**  Any semi prime ring is Noetherian and local ring such that () is a Boolean algebra and then every -module over is injective.

**Proof.**

Because is a semi prime ring and is a Boolean algebra. From proposition 3.3.8, is a regular. If P is a prime ideal in , then

is a saturated set. () is a Boolean algebra then by (Lemma 4 in )∋. So . But prime ideal gives principal. Then is Noetherian and local. So every module over is injective.

**Definition (3.3.28). [14].** Let be a module over integral domain R and is called torsion element if such that the set of all torsion elements in denoted by

T(. if T()= then is called torsion module and if T()=0 then is called torsion free module.

**Definition (3.3.29). [4]**.Any ring R is called a multiplication ring if all ideals are multiplication. Such that an ideal A is multiplication if every ideal B⊆A, ∃ an ideal C s.t B=AC.

**Proposition (3.3.30).**  Let R be a semi prime ring, if b is is a non unit in R and any completely irreducible saturated set is completely prime, then R is a Noetherian and a regular ring.

Recall that from [19]; any set is called saturated in R if. Also an element F in is completely irreducible if: F= So F= And F is called prime if or Therefor is completely prime if . Now we return to prove proposition 3.3.30

**Proof.**

Take R is a semi prime ring and is a non unit and every completely irreducible saturated set is completely prime. To prove )is dual semi-complemented. Suppose that . So ∃ b is a non unit. Then It is clear that and (semi-prime property). Hence is dual semi-complemented. Now from the condition (every completely irreducible saturated set is completely prime, Theorem 3 and Lemma 5 in[20]; we obtain that is a Boolean algebra. Also from R is a semi prime and Theorem 2 in [20]; we get R is a Noetherian and regular rings.

**Corollary (3.3.31).**  If any ring R satisfies all conditions of proposition 3.3.30, then every R-module is injective.

**Proof.**

See proposition )3.3.30(and corollary )3.3.27(.

**Proposition (3.3.32).**  Let R be a ring. If:

1. R is multiplication ring or hereditary ring;
2. R is P.I.D;
3. is P-injective module.

Then is an injective module.

**Proof.**

Assume that and P-injective module over multiplication ring or hereditary ring. Suppose that H≤M. We have , so from [27]; any submodule of torsion module is torsion (H is torsion submodule). Also, we have is P-injective, then H is P-injective submodule in . Now H is torsion and P-injective submodule over multiplication or hereditary ring. Hence H is Q-injective ( is injective). But R is P.I.D. Thus is injective module.

**Corollary (3.3.33).** If and P-injective, then any over P.I.D is injective.

Now we introduce another ring which has a good relation to injective module namely P.P-ring. We say that R is a P.P-ring if is P.I. of R is projective module. Also, we know that R is a (QF) ring if every projective module is injective.

**Lemma (3.3.34). [33].**  Let R be a ring. If

1. R is a P.P. ring;
2. is a free module;
3. N≤ is acyclic.

Then is a projective module.

**Theorem )3.3.35(.**  Let R be a ring. If:

1. R is a P.P-ring;
2. R is a QF;
3. N submodule of free module .

Then is an injective module.

**Proof.**

From conditions (1) and (3) is a projective module. But from condition (2) R is a QF. So is an injective module.

From[32]; the following statements are equivalent for any R-module :

1. is a divisible module.
2. is an I-divisible module.
3. is an f-divisible module.

Therefore; the following result is true:

**Corollary )3.3.36(.**  Let R be a ring. If:

1. R is a P.P-ring;
2. R is a P.I.D;
3. is I-divisible module. Then is an injective module.

**Proof.**

Clear. From[33]; we have R is a P.P. ring and is I-divisible module. So is f-divisible. But every f-divisible module is divisible. We have R is P.I.D. Then is an injective module.

Conclusions

Conclusions

In this thesis we worked on how to obtain the injective module through some modules as well as some characteristics such as Including divisible module, Noetherian and regular module, Quasi-injective module, uniform module, non-singular module, semi simple module and semi hollow module. There are some of the main proofs and results that we have proven in chapter two:

1. Let R be a P.I.D if is an injective R-module and then is injective.
2. Let be a divisible module. If is a projective module, then the homomorphism from into is also a divisible module.
3. Let be a regular R-module. If is semi hollow and Rad() is a Noetherian module, so is injective.

Also we got the injective module through some domains, including Dedekind domain and UFD, as well as some rings, including Euclidean ring, semi-hereditary ring and semi prime ring. There are some of the main proofs and results that we have proven in chapter three:

1. Let be a and it is a UFD. If every element of is divisible, so is an injective module.
2. Let be an integral domain and has no invers. If is a oetherian local P.M.I, then is a .
3. Every divisible over Euclidean domain is injective module.
4. Let be a semi-hereditary ring. Then every quotient module of injective module is injective.

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الملخص

الهدف الرئيسي من هذا البحث هو دراسة العلاقة بين المقاس الغامر وبعض المفاهيم ذات العلاقة مثل المقاس القابل للقسمة والمقاسات الأرتيرنية والنيوثرية والمنظمة, قدمنا خواص مهمة حول المقاس الغامر وكذلك اعتمدنا على التعاريف الاساسية في الفصل الاول والتي استخدمت في النتائج. لدينا كل مقاس غامر هو قابل للقسمة ولكن العكس يحتاج الى شرط الحلقة تكون ساحة مثاليات رئيسية (P.I.D). النتيجة المهمة الاخرى هي اذا كان المقاس شبه بسيط على حلقة شبه بسيطة فأن تكون مقاس غامر. كذلك حصلنا على اذا كان المقاس يحمل صفتين المنتظمة والدوار فيكون غامر. وايضا اذا كان منتظم وN مقاس جزئي منه ومنته التولد فيكون ال مقاس غامر.

الجزء المهم الاخر هو دراسة العلاقة بين المقاس الغامر وحلقة ديدكند حيث كل مقاس قابل للقسمة على حلقة (D.V.R) يكون غامر. وكذلك كل مقاس غامر كاذب مع R ديدكيند يحقق ال M مقاس غامر.

الجزء الاخير من الدراسة تمثل بكيفية الحصول على المقاس الغامر اذا كانت الساحات تحمل الصفات التالية:

ساحة اقليدية – ساحة نيوثرية.

ومن هذا الجزء درسنا المقاس I-divisible وعلاقته مع المقاس الغامر.

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**قسم الرياضيات**

حول المقاسات الغامرة والموضوعات ذات الصلة

رسالة مقدمة

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من قبل

**فوزي نوري حماد**

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**بإشراف**

**أ.م.د. ماجد محمد عبد**

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