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**On Injective Modules and Related Topics**

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**By**

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﴿**فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ ۗ وَلَا تَعْجَلْ بِالْقُرْآنِ مِن قَبْلِ أَن يُقْضَىٰ إِلَيْكَ وَحْيُهُ ۖ وَقُل رَّبِّ زِدْنِي عِلْمًا**﴾

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الاية 114

سورة طه

**اهداء**

**إلى أبي الرجل المثالي أطال الله في عمره ليظل عونًا لي وقدوتي، ومثلي الأعلى في الحياة؛ فهو من علَّمني كيف أعيش بكرامة وشموخ.**

**الى أمي التي فارقتنا بجسدها، ولكن روحها ما زالت تُرفرف في سماء حياتي.**

**إلى إخوتي.... سندي وعضدي ومشاطري أفراحي وأحزاني.**

**إلى زوجتي.... أسمى رموز الإخلاص والوفاء ورفيقة الدرب**

**إلى ابنائي..... فلذات الأكباد.**

**إلى جميع الأخلاء؛ إلى جميع الباحثين، وطلبة العلم.**

***شكر***

**لابد لنا ونحن نخطو خطواتنا الأخيرة في الحياة الجامعية من وقفة نعود إلى أعوام قضيناها في رحاب الجامعة مع أساتذتنا الكرام الذين قدموا لنا الكثير باذلين بذلك جهودا كبيرة في بناء جيل الغد لتبعث الأمة من جديد ...
وقبل أن نمضي تقدم أسمى آيات الشكر والامتنان والتقدير والمحبة إلى الذين حملوا أقدس رسالة في الحياة ...
إلى الذين مهدوا لنا طريق العلم والمعرفة ...
إلى جميع أساتذتي الأفاضل.......**

**كن عالما .. فإن لم تستطع فكن متعلما ، فإن لم تستطع فأحب العلماء ،فإن لم تستطع فلا تبغضهم

وأخص بالتقدير والشكر:

الدكتور ماجد محمد عبد
الذي نقول له بشراك قول رسول الله صلى الله عليه وسلم:
إن الحوت في البحر ، والطير في السماء ، ليصلون على معلم الناس الخير
كما أنني أتوجه له بخاص الشكر ، إلى من علمنا التفاؤل والمضي إلى الأمام، إلى من رعانا وحافظ علينا، إلى من وقف إلى جانبنا عندما ضللنا الطريق.....**

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**Supervisor Certification**

 *I certify this thesis entitled* ***On Injective Modules and Related Topics*** *submitted by* ***Fawzi Noori Hammad****, has been prepared under my supervision at the University of Anbar, College of Education for Pure Sciences / Department of Mathematics, as a partial fulfillment of the requirements for the degree of master in Mathematics.*

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**List of Publications**

1. A New Results of Injective Module with Divisible Property. In Journal of Physics: Conference Series (Vol. 1818, (2021, March). No. 1, p. 012168). IOP Publishing.‏

2. Noetherian, Artinian Regular Modules and Injective Property, at the Journal of Al-Qadisiyah for computer science and mathematics, (2021), 13(1), Page-161.

3. Study Injective Module Over Dedekind Domain, at the AIP Conference Proceedings**,** (2021, March)**. (Accepted).**

4. Some Rings Give injective module. (2021), IEEE, Conference Proceedings. **(submitted).**

***List of Symbols***

|  |  |
| --- | --- |
| **Symbol** | **Meaning** |
| $$\leq $$ | Submodule |
| ⨁ | direct summand |
| $$\rightarrow $$ | Arrow |
| $$Hom\_{R}(M,N)$$ | The group of all homomorphism from M into N |
| $$Ann$$ | Annihilator |
| $$\ll $$ | Small submodule |
| C! | Contradiction |
| ≤ess | Essential submodule |
| $$≅$$ | Isomorphic |
| $$≈$$ | Equivalent |
| Rad(M) | Radical of M |
| $$Ext( N,M )$$ | External homomorphism |
| $Ker(f)$  | The kernel of a homomorphism $f$ |
|  $ Tor(M,N )$ | Torsion |
|  $Img(f)$ | The image of a homomorphism $f$ |
|  **< >** | Generate |
| $R[X]$ | Polynomial ring |
| D.V.R | Discrete valuation ring |

**Abstract**

**ABSTRACT**

 The main objective of this work is to study injective modules and related topics. Due to the relations between injective and divisible modules, a notion of several properties about both modules was studied. We introduce all of the key definitions used in the thesis, as well as some findings about the injective module a pear. Every injective module is divisible, but the inverse requires an additional condition P.I.D. Also, if the ring R is semi simple, and $M$ is a semi simple R-module, then $M$ is injective. Also, if $M$ is a cyclic and the regular module is injective. Also, if $M$ is regular with N≤$ M$ is a finitely generated submodule, so $M$ is injective. Here, we study several relationships between injective modules over the Dedekind domain. We prove that every divisible module $M$ over D.V.R is injective. Also, any pseudo injective module with R is a Dedekind leads to $M$ is an injective. Several facts about the relationship between the injective module and the Euclidean ring are satisfied here. And we find that every divisible module on the Noetherian valuation ring is injective. Also, there are some connections between (D.V.R) and the injective module.

 Finally, we investigate the injective module in relation to other rings, such as the Noetherian ring, the local ring, the D.V.R, and the hereditary ring. We prove that if $M$ is an R-module and $I$ is a maximal ideal of R with quotient ring $R\_{I}$ of R is a field, then $M$ is an injective module. Also, if R is a D.V.R, so any$ I$-divisible module over R is injective.

***INTRODUCTION***

***INTRODUCTION***

 In 1970, Stevens ̈om introduced the notion of injective modules, and generalized the homological properties from Noetherian rings to coherent rings, and in this process, finitely generated modules were replaced by finitely presented modules. Recently, as extending work of Stenstr ̈om’s viewpoint, Gao and Wang introduced the notion of weak injective modules. This class of modules was also investigated by Bravo, Gillespie, and Hovey independently. In this process, finitely presented modules were replaced by super finitely presented modules. The fact shows that weak injective modules play a crucial role in the process of generalizing homological properties from special rings to arbitrary rings.

 Let R be a commutative ring with identity and let $M$ be a unitary R-module. $M$ is called injective R-module provided that

 1- $M$ is a submodule of Q such that Q is a module.

 2- $K \leq M \ni M + K = Q$ and $M∩K = \{ 0 \}$ s.t $K +M$ is an internal direct. See [15]. Or Any module $M$ is injective if the short exact

$$0\rightarrow Q \rightarrow M \rightarrow K \rightarrow 0$$

 is split.

 In our work we study injective module and some modules and some domains in detail were we investigate their basic definitions.

This thesis consists of three chapters. **The first Chapter** contains all the essential definitions used in the thesis.

**The second chapter** consists of **two section.** **In the first section**; we study an important concept namely divisible module, uniform module, non-singular module, semi simple module, P-injective and Q-injective modules. Some new results and properties have been studied in this notion. Every injective module gives divisible but the converse needs another condition P.I.D. Further more, we prove that if R be a P.I.D and $M$ is an injective R-module and $N\leq M,$ then $\frac{M}{N}$ is also injective. We prove for any field K and every divisible $M$ on K–module, then $M$ is injective module. Also we showed if $M\_{1}$ be a divisible module and $M\_{2}$ is a projective module, then the homomorphism from $M\_{2}$into $M\_{1}$is also a divisible module. In addition, we get if R be a P.I.D. If $M$ is a uniform and non-singular and P-injective module then $M$ is injective module. Any R-module $M$ is called semi simple if $M$ is direct sum of simple submodule we present a good relationship between semi simple and injective-modules through let R be a ring. If R is semi simple and $M$ is semi simple R-module, then $M$ is injective module.

**In the second section**; we present new results about injective module depending other concepts, namely Noetherian, Artinian, hollow, semi hollow and regular modules. We showed that every Noetherian or Artinian Regular module over abelian ring R with unity is injective module. Further, we found if R be a (QF) and perfect ring or finite–dimensional ring. If $M$ is a regular R-module, then $M $is injective. Any ring $R$ is called $QF$(quasi-Frobenius) if every projective module is injective; or every injective module is discrete. We get every cyclic regular R-module over QF-ring is injective. Let $M $be regular module over P.I.D. If $M$ is acyclic module, then it is a Noetherian and is injective module. Let $M $be a regular R-module. If $M$ is semi hollow and Rad($M$) is a Noetherian module, so $M$ is injective. If proper f-generated submodule of $M $is a small in $M$ (P. f-generated N<<$M$), so $M$ is called semi hollow module such that $M$ is hollow if P-submodule N is small in $M$. We showed the relationship between Noetherian module and injective module. If N≤$ M$ is f-generated, then $M$ is a Noetherian. We get if $M\_{1}$ and $M\_{2}$ be an R-modules over (QF)-ring and let $δ$:$ M\_{1}$ →$M\_{2}$ be an onto homomorphism such that $M\_{2}$ is a regular and $M\_{1}$ is f-generated, so $M\_{2} $is an injective module.

**The third chapter consists of three sections.** In this chapter we will get injective module through some domains and from Euclidean ring and hereditary rings. **In the first section**; we present several results which give injective module over $R$ Dedekind domain. We prove Every finitely many prime ideals $P\_{i}$ in a $Dedekind domain$ is P.I.D. And Let $R$ be a $Dedekind domain$ and it is a UFD. If every element of $M$ is divisible, so M is injective module. We explain the relationship between valuation ring and Dedekind domain. We prove every valuation ring is a $Dedekind domain$. Let $R$ be an integral domain with no invers. If $R$ is a $Ν$oetherian local $ring with$ P.M.I, then $R$ is a $Dedekind domain$. We prove if $R$ a valuation ring and UFD and it is a $Ν$oetherian local ring and $I$=< a > so $M$ is divisible module then $M$ is injective module. An ideal $Ι$ in the ring is namely fractional if it is a submodule of H, there is 0≠S ∋ S$I$⸦$ R$. Also, if $R$ is an abelian integral domain with 1 and it is a field of fractions, so we say F(D) is the set of non-zero fraction ideals of D such that D is an integral domain. Let D be an integral domain. $Τ$hen D satisfies generalized domain G(D) if and only if for all $Ι$∈F(D), $Ι$= $(Ι^{-1})^{-1}$ is invertible. We get that the module $M$ is an injective from definition Dedekind domain and (GD) with another condition on a module. Let $Ι$∈F(D). If $(ΙJ)^{-1}$= $Ι^{-1}$$J^{-1}$Ɐ $J$∈F(D), with $M$ is a divisible D-module, then $M$ is an injective module. We showed if D be a completely integrally closed; Ɐ $I,J$∈ F(D), $(IJ)\_{v}$=$I\_{v}J\_{v}.$ If $M$ is adivisible module, then $M$ is an injective module. And we use a generalized of Dedekind domain in order to obtain module $M$ is an injective. Any integral domain D is called generalized (GD) if Ɐ $Ι$ an ideal in fractional ideals of D equal $(Ι^{-1})^{-1}$ ; so$Ι$=$(Ι^{-1})^{-1}$ is invertible. Also, for binary operation ⁎, we say that ⁎-operation same v-operation. Therefore, the map.$I$→$I\_{v}$ on the fractional ideals F(D) is v-operation. Also we prove $M$ be an $R$ –module and Ɐ $m\in M$ is divisible element so D is $Krull$ ∋ $I∙J$ is invertible and v-ideal.

 **In the second section,** we study another ring namely Euclidean ring in order to obtain injective module. But this goal needs some additional conditions with Euclidean ring such as divisible module. By van der Warden, we say R is an Euclidean ring if R is integral domain such that the division Algorithm is true. We can rewrite this definition in another way (if ∃$δ$:R{0}→$Z^{+}$+{0} ∋ δ($xy$) greater than and equal $δ(y)$ and $x=yq+s$, s=0 or $δ(s)<δ(y)$ $Ɐ x,y$ ∈ R; $q ,s$ ∈R). We prove every divisible over Euclidean domain is injective module. Also we showed if $M$ be divisible R-module and R is (D.V.R) ∋ $T$:R\{0}→N is Euclidian then $M$ is injective module.

**In the third section**; we study injective module over some other rings for instance Noetherian ring, local ring, D.V.R and hereditary ring. We prove that over Noetherian ring, if $M$ is$ I$-divisible, so $M$ is an injective module. Also, if $M$ is an $R$-module and $I$ is a maximal ideal of $R$ with quotient ring $R\_{I}$ of $R$ is a field, then $M$ is an injective module. We prove if $ann(P)=$ $M\_{p}$ ;Ɐ $p\in R$ and $ann(S∩T)=ann(S)+ann(T)$, then $M$ is injective module. Any ring R is called hereditary (semi-hereditary) if each ideal (finitely generated ideal) of R is projective. And we prove if $R$ be semi-hereditary ring. Then every quotient module of injective module is injective. Also if $R$ is a D.V.R, so any $I$-divisible module $M$ over $R$ is injective. Finlay let $M$ be any module over a semi prime ring $R$. If $T(R)$ is a lattice and [b) has a complement in $T$($R$), $b\in R$, then $M$ is an injective module.

**CHAPTER ONE**

**Basic Concepts**

***(1.1).Baisic Conceptes***

 In this chapter, we present the basic definitions and some results which have several relationships with the study.

 **Definition (1.1.1).[22].** Suppose that R is a ring and 1 is its multiplicative identity. A left R-module $M$ consists of an abelian group ($M$, +) and an operation⋅ : $R$ × $M$ → $M$ such that for all $r, t$ in R and $m$ in $M$, we have:

1-$ r(m\_{1}+m\_{2})=rm\_{1}+rm\_{2}.$ $ Ɐ m\_{1} , m\_{2} \in M$

2$- (r+t)m=rm+tm.$

3$- (rt)m=r(tm).$

4-$ 1.m=m=m.1.$

**Examples. (1.1.2).[22].**

1. Any ring R is trivially an R-module over itself.
2. If S is a subring of a ring R any left R-module is also a left S-module with the restricted scalar multiplication.
3. Any matrix ring of a ring R is a R-module under componentwise scalar multiplication.
4. The vector space V is an R-module.

**Definition. (1.1.3).[22].** Let $M$ R-module A submodule of $M$ is a subset $N⊆M$ satisfying.

1. N is a subgroup of $M$ and
2. For all $r\in R$ and all $n\in N$ we have $rn\in N$.

**Examples. (1.1.4). [22].**

1. Any module $M$ is submodule of itself called the improper submodule and the zero submodule consisting only of the additive identity of $M$ called the trivial submodule.
2. A left ideal I is a submodule of R viewed as an S-module where S is any (not necessarily proper) subring of R.

**Definition (1.1.5).** A module $M$ is injective if: $f:X \rightarrow Y$ is a homomorphism and $g:X \rightarrow M $is a homomorphism, Then there exists $h: Y \rightarrow M$ is a homomorphism such that $g=h∘f.$

**Definition (1.1.6). [16].** Let $M$ be an R – module. If the next conditions are true, then $M$ is called injective module.

1. $M$ is a sub-module of Q s.t Q is a module.
2. $K \leq M \ni M + K = Q$ and $M∩K = \{ 0 \}$ s.t $K +M$ is an

$ $internal direct. So Any module $M$ is injective if the short exact sequence

$$0\rightarrow Q \rightarrow M \rightarrow K \rightarrow 0$$

is split such that the meaning of exact sequence and split can explain by the following:

 (1) A pair of module homomorphism

 M1$→$ M2 $→$M3

is called exact sequence at $M\_{2}$ if $ker(g)$ = $Img(f).$

(2) In general, we can say a finite sequence of module homomorphism

 $M\_{0}→$ $M\_{1}→$ $M\_{2}$ $→M\_{3}$ $→M\_{n-1}→M\_{n}$

is exact if $ker$($f\_{i+1}$) = $Img$($f\_{i}$) : ∀ i=1,2,3,----n-1

**Definition (1.1.7)**. **[22].** Let$ N$ be a submodule of an R-module $M$. Then $N$ is called direct summand of $M$ if there is a submodule$ K$ of $M$ such that$ K+N=M$ and $K∩N=0$. And write $M= K⨁N.$

**Definition (1.1.8)**.**[22].**  Any module $M$ is called free if it is has a basis that is generating set consisting a linearly independent element.

**Definition (1.1.9)**. **[14].** Let $M$ be an module over integral domain R and $m\in M$ is called torsion element if $,∃ 0\ne r\in R$ such that $rm=0$ the set of all torsion elements in $M$ denoted by

T($M)=\{m\in M; rm=0 for som r\in R\}⊆M$. if T($M$)=$M$ then $M$ is called torsion module and if T($M$)=0 then $M$ is called torsion free module.

**Definition (1.1.10). [22].** $M$ is called a flat module if for every monomorphism $α:A\_{R}\rightarrow B\_{R}$ $. α⊗1\_{m}$ is also a monomorphism.

**Definition (1.1.11). [22].** An R-module $M$ is called cyclic if it is generated by a single element

**Definition (1.1.12). [27].** A ring R is coherent ring if and only if every direct product of flat module is flat and P-coherent(P-coh) if every principal ideal (P.I) is f-presented.

**Definition (1.1.13). [38].** (**Baer’s criterion**): Any module $M$ satisfy (Baer’s criterion) is injective if every ideal $I$ and every morphism

$f: I\rightarrow M$, $∃ m∊M$ with f(x)=mx, $x∊I$.

**Definition (1.1.14). [16].** Any module $M$ is called divisible if

 $r.m=m,$ $0\ne r∊R.$ Or; if every element of $M$ is divisible.

**Definition (1.1.15). [38].** R is a left P-coherent if and only if any direct product of torsion-free right R-modules is torsion-free.

**Definition (1.1.16). [19].** An R-module $M$ is called uniform if every submodule of $M$ is an essential in $M$.

 **Definition (1.1.17). [11].** If $X$ is a non-empty subset of R, then we denote its annihilator in $M$ $(ann(x))$ and define it to be the set of elements $m\in M$ such that $xm=0 ∀x\in X.$

**Definition (1.1.18). [22**]. A submodule $N$ of $M$ is called essential, if whenever $N ∩ L \ne (0)$, then $L\ne (0)$ for each submodule $L$ of $M$.

**Definition (1.1.19). [19].** An R-module $M$ is called nonsingular if $Z(M)=0$ $\ni Z(M)=\{m\in M :ann(m)\leq \_{ess}R\}.$

**Definition (1.1.20). [32].** An R-module $M$ is called pseudo-injective(P-injective) if every submodule $N$ of $M$, each R-monomorphism $f: N\rightarrow M$ can be extended to an R-endomorphism of $M$.

**Definition (1.1.21). [21].** $M$ is called Quasi-injective(Q-injective) if for each submodule $N$ of $M$, and each R-homomorphism $f: N\rightarrow M$ can extend to an R-endomorphism of $M$.

 **Definition (1.1.22). [38].** A module $M$ is called simple, if $M \ne 0$ and it has only submodules {0} and $M$.

**Definition (1.1.23). [19].** Any R-module $M$ is called semi simple if $M$ is a direct sum of simple submodule.

**Definition (1.1.24). [42].** Any module $M$ is called regular if Ɐ$m∊M$,$ ∃$ g∊$Hom$($M$,R), then(mg)m=m.

**Definition (1.1.25). [42].** If R has finite direct sum of ideals, so R is called finite dimensional.

**Definition (1.1.26). [5].** A module $M$ is called finitely generated(f-generated) if it has a finite set of generators. In other word; $M$ is finitely generated if $M =∑x\_{i}r\_{i} ; Ɐ x\in M, r\in R$.

**Definition (1.1.27). [8].** A ring R is called hereditary (semi-hereditary) if each ideal (finitely generated ideal) of R is projective.

**Definition (1.1.28).** **[22].** An R-module $M$ is called Artinian if $M$ satisfies the descending chain condition (DCC) on submodule of $M$.

**Definition (1.1.29). [18].** The ring R is called $prufer$ domain if R is an integral domain and every f-generated. ideal of R is projective (invertible).

**Definition (1.1.30). [35].** A module $M$ is said to be discrete if it is lifting and has the property D2 (If N ≤ $M$, such that $\frac{M}{N}$ is isomorphic to a direct summand of $M$, then N is a direct summand of $M$).

**Definition (1.1.31). [29].** An R-module $M$ is called projective if and only if for any $epimorphism$ $f$:C→V such that C,V are any R-modules and for any homomorphism $g:M\rightarrow V ∃$ a homomorphism $h:M\rightarrow C$ such that $f∘h=g$.

**Definition (1.1.32). [22].** An R-module $M$ is called Noetherian if M satisfies the ascending chain condition (ACC) on submodule of M.

**Definition (1.1.33). [36].** An integral domain $R$ is Dedekind if 0≠$I$ is a proper ideal factors into prime ideals.

**Definition (1.1.34). [41].** G-Dedekind domain carries same meaning of Dedekind domain; so an integral domain D is (GD) if for each A∈F(D); $A\_{v}$=$(A^{-1})^{-1}$ is invertible**.**

**Definition (1.1.35). [36].** $Α$ field F is an abelian ring such that has trivial prime ideal. Or: $R$ is a Dedekind domain if it is:

1. $R$ is integrally closed.
2. $R$ is a $Ν$oetherian ring; if 0≠$Ρ$ is maximal ∋ $Ρ$ is prime ideal.

 **Definition (1.1.36). [2].** Let $R$ be a domain. So $R$ is a valuation ring if it is not field; $aϵR, $so$ a^{-1}$∉$ R$.

**Definition (1.1.37). [41].**  $Α$n integral domain D satisfies generalized of D or (GD) if Ɐ $Ι$∈F(D)→$Ι$=$(Ι^{-1})^{-1}$is invertible such that $an ideal$ $Ι$ is namely invertible if ∃ $J$ is a fractional ideal and then $ΙJ$= D.

**Definition (1.1.38). [40].** $Α$n ideal $Ι$ of F(D) is namely v-invertible if: ∃ $J$∈F(D)∋ $IJ\_{v}$=D.

 **Definition (1.1.39). [41].** We say D is completely integrally closed (C.I.C) if an ideal $I$∈F(D) is v-invertible.

**Definition (1.1.40). [13].** For binary operation ⁎, we say that ⁎-operation same v-operation. Therefore, the mapping:$I$→$I\_{v}$ on the fractional ideals F(D) is v-operation.

**Definition (1.1.41). [21].** Any integral domain D is namely $Krull$ type if it has a collection F={$p\_{i}\}\_{i\in I}$ of prime ideals such that,

1. D=∩$D\_{p\_{i}}$.
2. Ɐ $i\in I$; $D\_{p\_{i}}$ is a valuation domain.
3. 0≠$I$ and $I$ not unit of D belongs to only a finite number of $p\_{i}$.

 **Definition (1.1.42). [6].** D is called Mori domain if any set of the integral is a v-ideals of D and satisfies (ACC), such that any ideal $Ι$∈F(D) is v-ideal if $I$=$I\_{v}$.

**Definition (1.1.43). [25].** Let D be an integral domain and let (⁎) an operation on D. We say D is (⁎)-$Ν$oetherian domain if D has (ACC)on all ideals of D, $i.e$ (Quasi(⁎)-ideals). Or D is (⁎)-$Ν$oetherian domain if 0≠$I$ is (⁎)f-finite.

**Definition (1.1.44). [17].** Let D be an integral domain and (⁎) is operation on D. Then D is called $prufer$-(⁎)multiplication domain.(P.(⁎). M. D) if Dm is a valuation such that m is a $maximal element$ in S; $φ$≠S= $set$ ideals of D ($Quasi$-(⁎)f –maximal ideal).

**Definition (1.1.45). [3].** Any ring R is called Euclidean if it is an integral domain such that R satisfies division algorithm.

**Definition (1.1.46). [34].** We say the function $T$:K→Z is (D.V.R) if:

1. $T$ is onto.
2. $T$ ($ab$)=$ T(a)$+$ T(b)$.
3. $T(a+b)$≥ min{$ T(a)$,$T(b)$.}.

 **Definition (1.1.47). [33].** For a module $M$ over a ring $R$ and for an ideal as a module of $R$; $M$ is called $I$-injective if $δ$: $I\rightarrow M$ is a homomorphism, then there exists an elements $m\in M$ and $x\in I$ such that the image of $x$ is $mx$ (in other words $δ$ can be extended to homomorphism of $R$ into $M$).

**Definition (1.1.48). [12].** Any $R$-module $M$ is called divisible if $M$ =$Ma$ such that $0\ne a\in R$ and $M$ is $I$-divisible if $Ma=$ $ann\_{M}(r(a)); a\in R$. ($I$-divisible give divisible over commutative ring).

**Definition (1.1.49). [7].** Any ring $R$ is called local if $R$ has a unique maximal ideal. In other word; every ideal is contained in some maximal ideal.

**Definition (1.1.50). [22].** The principal ideal I=<a> is called projective if and only if there exist $b\in R$ such that $a=ab$ and $ann(a)=ann(b)$

**Definition (1.1.51). [4].** Any ring R is called a multiplication ring if all ideals are multiplication. Such that an ideal A is multiplication if every ideal B⊆A, ∃ an ideal C s.t B=AC.

**Definition (1.1.52). [20].** Any set $φ\ne F⊆R$ is called saturated in R $( T(R) )$ if $Ɐ a,b \in R, ab\in F⟺a,b\in F$. Also an element F in $T(R)$ is completely irreducible if: F=$⋂F\_{i}\ni F\_{i}\in T\left(R\right).$ So F=$F\_{i}.$ And F is called prime if:

 $F\_{1}⋂F\_{2}⊆F⟹F\_{1}⊆F$ or $F\_{2}⊆F.$ Therefor $F\in T(R)$ is completely prime if: $⋂F\_{i}⊆F(\{F\_{1}\}⊆T(R) ⟹F\_{i}⊆F$.

**Definition (1.1.53). [9].** We say that an integral domain, R, is a UFD if every nonzero nonunit in R can be factored into irreducible elements, and if we have $α\_{1}α\_{2}$ ···$α\_{n}$ = $β\_{1}β\_{2}$ ···$β\_{m}$ with each $α\_{i} ,β\_{i} $irreducible in R then; (a) n = m and (b) there is a $σ$ ∈ $S\_{n}$ such that $α\_{i}$ = $u\_{i}β\_{σ(i)}$for all 1≤ $i$ ≤n where each $u\_{i}$ is a unit of R

**CHAPTER TWO**

**SOME MODULES AND INJECTIVE PROPERTY**

**SOME MODULES AND INJECTIVE PROPERTY**

**Introduction.**

This chapter introduces various concept namely divisible module, uniform module, non-singular module, semi simple module, P-injective and Q-injective modules. A module $M$ is called a divisible module if $r.m=m,$ $0\ne r∊R.$ Or; if every element of $M$ is a divisible. We found that every injective module gives divisible but the converse needs another condition P.I.D. Also, some new results have been studied. Also, we showed if $M\_{1}$ be a divisible module and $M\_{2}$ is a projective module, then the homomorphism from $M\_{2}$into $M\_{1}$is also a divisible module. We present a good relationship between semi simple and injective-modules. Also we get if R be a P.I.D. If $M$ is a uniform and non-singular and P-injective module then $M$ is an injective module. In the last section, we provide several relationships between some concepts namely Noetherian, Artinian, hollow, semi hollow and regular modules and injective module. We prove that every Noetherian or Artinian Regular module over abelian ring R with unity is an injective module.

**§1: A New Results of Injective Module With Divisible Property**

In this section, we will present new results that clarify the relationship between divisible module and Injective module. If the short exact is splits, this means that $M$ is injective. Also a divisible module over P.I.D is injective.

Now we present the meaning of injective module in more depth way.

**Definition (2.1.1). [16].** Let $M$ be an R–module. If the next conditions are true, then $M$ is called an injective module.

$1- M$ is a sub-module of $Q$ s.t $Q$ is a module.

1. $K \leq M \ni M + K = Q$ and$ M∩K = \{ 0 \}$ s.t $K +M$ is an internal direct.

**Equivalent Definitions:**

**Definition (2.1.2).** Any module $M$ is injective if the short exact

$$0\rightarrow Q \rightarrow M \rightarrow K \rightarrow 0$$

is split s.t the meaning of exact sequence and split can be explained by the following:

(2) A pair of module homomorphisms

 M1$→$ M2 $→$M3

is called exact at $M\_{2}$ if $ker(g)$ = $Img(f).$ In general, we can say that a finite sequence of module homomorphism

 $M\_{0}→$ $M\_{1}→$ $M\_{2}$ $→M\_{3}$ $→M\_{n-1}→M\_{n}$

is exact if $ker$($f\_{i+1}$) = $Img$($f\_{i}$) : ∀ i=1,2,3,----n-1

(2) An infinite sequence of module homomorphisms

 …..$→M\_{i-1}→M\_{i}$ $→$ $M\_{i+1}→ $→……..

is exact if $ker$($f\_{i+1}$) = $Img$($f\_{i}$); $∀i∊Z .$

**Remark (2.1.3).** The exact sequence is homomorphism if and only if f is a module monomorphism

 0 $→M\_{1}$ →$M\_{2}$

Also;

 $ M\_{1}$ →$M\_{2}→$0

 is an exact sequence of homo. If and only if $g$ is module epi.

**Definition (2.1.4).** An exact sequence

 0$\rightarrow M\_{1}\rightarrow M\_{2}$ $\rightarrow $ $M\_{3}$ $\rightarrow $ 0

and if for $0\ne M \ni $ divisor in the ring R with $M=mM$ , then $M$ is divisible.

**Lemma (2.1.5). [16]** Let R be a P.I. D. If $M$ is a divisible R- module, then $M$ is injective.

**Example (2.1.6).** If Z is P.I.D injective Z-module; Z is a divisible module over Z.

**Corollary (2.1.7).** Let R be a P.I.D if $M$ is an injective R-module and $N\leq M,$ then $\frac{M}{N}$ is also injective.

**Proof.**

Since $M$ is divisible; so $\frac{M}{N}$ is divisible with R P.I.D imply $\frac{M}{N}$ is injective.

**Theorem (2.1.8).** Every divisible module $M$ over a filed K is injective module.

**Proof.**

 Suppose that an ideal normal of K ( $I⊲ K$ ) and $0\ne i \in I$ so every $k\in K$; $K$ = (K$i^{-1}$) $i \in I$. Therefore I = K and hence {0} = (0) and $K=(1)$ only Ideal's in K. Hence $K$ is a P.I.D. But $M$ is divisible module. Then $M$ is an injective.

**Theorem (2.1.9).** Let $M\_{1}$ be a divisible module. If$ M\_{2}$ is a projective module, then the homomorphism from $M\_{2}$into $M\_{1}$ is also a divisible module.

**Proof.**

 Suppose that $α\in Hom\_{R}$($M\_{2}$,$ M\_{1}$) and $β∊Hom\_{R}$($M\_{2}$ , $M\_{1}$). IF

 $0 \ne r ∊ R ∃ r$ divisor (non–zero divisor in R). In order to prove that $Hom\_{R}$($M\_{2}$,$ M\_{1}$) is divisible we need to show that $β=r α$. Suppose that ϫ. be a homomorphism from $M\_{1}$ into$ M\_{1}$ and define by:

 $ϫ(m)=rm$. But we have $M\_{1}$is divisible. So ϫ is onto also we have $M\_{2}$ is a projective module. Then $M\_{2}$ is injective. Let $m\_{2}$∊ $M\_{2}$. So

 $β$($m\_{2}$) = $ϫ$($α$($m\_{2}$)) = $rα$ ($m\_{2}$) and hence $β=rα$ . Thus $Hom\_{R}$ ($M\_{2}$, $M\_{1}$) is divisible.

Recall that A right –module M and an element $m∊M$ is called singular element of $M$ if$:ann(m)\leq \_{ess}R$ . The set of all singular elements is denoted by $Z(M)$ . We say that $M$ is singular (resp. nonsingular) if $Z\left(M\right)=M$ (resp.$ Z\left(M\right)=0.$ )

**Corollary (2.1.10).** Let $M$ satisfies (ACC) over P.I.D. If Z($M$) subset Z(R), then M is an injective R–module.

**Proof.**

If Z (R) be the zero divisors of R; Z($M$) is a zero divisors of $M$. Suppose that $Г\_{1}$= R\Z (R) and $Г\_{2 }$= R\Z ($M$). We know $Г\_{1}⊆$$Г\_{2 }$let $α$∊$Г\_{1}⊆$$Г\_{2 }.$Since $M$ satisfies (DCC), then $M$ is an Artinian module So. $α$ $M⊇ α^{2}M⊇$….

Hence $α^{n}$$M$= $α^{n+1}$$M$; $n ∊Z^{+}$. Now if $m\_{1}∊M$, then

 $α^{n}m\_{1}$ = $α$ n+1 $m\_{2}$∋$m\_{2}$∊$M$. Then $α^{n}$($m\_{1}$ – $αm\_{2}$) =0. Since $α^{n}$∊Г2 and $m\_{1}$ –$ α$ $m\_{2}$=0, then $m\_{1}$=$ α$ $m\_{2}$. Therefore. $M$ = $α$ $M$ ,$∀$ $α$ ∊$Г\_{1}.$ Thus $M$ is divisible. But R is a P.I.D, then $M$ is injective.

**Corollary (2.1.11).** Let $M$ be a divisible R-module. If $M$ has no zero-divisors, then $M$ is injective.

**Proof.**

 Suppose that a nonzero element in R. So $∃ m\in M \ni am =1$.

Hence $mam = m$ and then $(ma-1)m$=0. But $M$ has no zero–divisors. So $ma =1. $Hence $M$ is an injective module $( 0 \ne a ∊ R ∃ a invertible$) then $M$ is injective. Because if $0 \ne b∊I(m)$ , $∃a∊R$ $∃$ ab $\in $ $M$ and hence $b= a ^{-1}(a b) ∊M .$

**Theorem (2.1.12).** Let $0\ne a$ be invertible element of R in $M$ If $M$ is torsion–free module, then it is divisible ( $M$ is injective).

**Proof.**

Suppose that $M$ is a divisible module and $0 \ne b ∊ I \left(M\right).$ Then $∃$

$0\ne a ∊ R \ni ab∊M$. [14]. So $b= a^{-1}$ $(a b)$ in $M$ thus $M$ is injective**.**

**Remark (2.1.13).** We know that a direct divisible over R is also a divisible module. Therefore it is easy say the following statement is true:

From [27], Recall that a ring R is coherent if $I$ is a f-generated ideal of R presented and we call R is P-coherent ring if $I$ is a principal ideal of R presented and if we have a direct product of copies of a ring R is a torsion free, then R is a P-coherent ring. And let $0 \ne a∊R$ be invertible. So the direct sum of torsion-free modules is also injective.

**Theorem (2.1.14).** Let a direct of P.I.D R is a torsion if

$α$: $Ext(M, M)$ → $Hom(M,M)$ is an isom., then $M$ is injective.

**Proof.**

 Since a direct of a P.I.D is torsion-free, R is a P-coherent. But $α$ is an isom. So, $∃$β ∊$Hom(D, M)$ ∋$ α$ ( β+ im(f) = β g = 1m.

Hence $M$ ≈ $⨁$ F. Then $M$ is divisible with R is P.I.D implies that $M$ is injective.

**Corollary (2.1.15).** **[22].** Let R be a ring with unity if $M\_{1}$ and $M\_{2}$ are two R-modules then there exists a one to one mapping $α$: $M\_{1}$→ $M\_{2}$.

**Corollary (2.1.16). [22].** Let G1 be a commutative group. If G2 is an abelian group, then there exists a mapping β: G1 → G2 is one to one such that G2 is a divisible.

 **Proposition (2.1.17). [22].** An abelian group G embedded in a divisible commutative group. Moreover; If D is a divisible commutative group; $Hom\_{z}(R, D)$ is injective.

**Example (2.1.18).** We have Z is a PID injective Z-modules are divisible Z-modules (divisible abelian groups).

**Remark. (2.1.19). [39].** Consider *D* being an finite subset of a vector space *V* s.t *D* is a basis of *V*, so every non trivial, f-generated torsion-free is not divisible and we can find a *T:* *V* →inj(*V* ) $\ni $ *T* is injective and surjective.

**Corollary (2.1.20).** If we have a torsion-free Z-module *V*. Then inj (*V*) is a submodule of divisible module (*V* ).

**Proof.**

Set *E* = inj(*V* ). Set *D* = divisible of the module (*V*). If *a* objectinthe *E* and *a* inthe *D* by [26], mult *E* = mult *D*).

**Definition. (2.1.21).** **[27]**. A left R-module $M$ is called D*-*injectiveif $Ext(G, M) = 0 $for every divisible left R-module G and N is D*-*flatif $Tor(N, G) = 0$ for every divisible left R-modules N and G.

**Proposition (2.1.22). [39].** By Wakamutsu’s Lemma, any kernel of a D-cover is a D-injective and $M$ is D-flat iff $M^{+}$ is D-injective by

 $Ext(N, M^{+})≅$ $Tor(M, N)^{+}$ for every divisible left R-module N.

**Definition (2.1.23). [19].** An R-module $M$ is called uniform if every submodule of $M$ is an essential in $M$.

**Definition (2.1.24).** **[19].** An R-module $M$ is called nonsingular if $Z(M)=0$ $\ni Z(M)=\{m\in M :ann(m)\leq \_{ess}R\}.$

**Definition (2.1.25).** **[32].** An R-module $M$ is called pseudo-injective(P-injective) if every submodule N of $M$, each R-monomorphism

$f: N\rightarrow M$ can be extended to an R-endomorphism of $M$.

Recall that any module $M$ is called Quasi-injective(Q-injective) if for each submodule N of$ M$, and each R-homomorphism $f: N\rightarrow M$ can extend to an R-endomorphism of $M$. [21].

**Lemma (2.1.26). [22].** Every Q-injective module over P.I.D is an injective module.

Now depending on the last definition and lemma 2.1.26; we can present the following proposition.

**Proposition (2.1.27).** Let R be a P.I.D. If

1. $M$ is a uniform module;
2. $M $is a non-singular module;
3. $M$ is a P-injective module.

Then $M$ is an injective module.

**Proof.**

Assume that $M$ is a non-singular and P-injective R-module. Suppose that K≤M and $φ:K\rightarrow M$ be a homomorphism. We have $M$ is a uniform ($N\leq \_{ess}M$) with non-singular property $(Z(M)=0)$; so ker($φ)=0$ or ker($φ)=K$.

Case 1: If ker($φ)=K$, then the mapping $φ$ can extend to homomorphism:$ φ:M\rightarrow M.$

Case 2: If $ker(φ)=0,$ then $φ$ is (1-1) and can extend to R-homomorphism from $M\rightarrow M.$ Hence $M$ is Q-injective. But every Q-injective is injective over P.I.D(lemma 2.1.26).

Recall that any R-module $M$ is called semi simple if $M$ is a direct sum of simple submodule in the next theorem, we present a good relationship between semi simple and injective-modules.

**Theorem (2.1.28)**. Let R be a ring. If R is semi simple and $M$ is a semi simple R-module, then $M$ is an injective module.

**Proof**.

We know every module over R is a homomorphism image of a direct sum of copies. So $M$ is a semi simple R-module, because:

Let $M^{⸝}$ be an R-module $f: M$→$M^{⸝}$ is epimorphism. Hence $ker(f)$ is a direct summand of $M$, then $∃ N\leq M \ni M$=$ker(f)$+$N$. So $M^{⸝}$=$ \frac{M}{ker⁡(f)}$≈N.Therefore N is semi simple (every submodule of semi simple is simple). Now let $M^{⸝}$ be an extension of $M$ so $M^{⸝}$ is a semi simple module. So $M$ is direct summand of $M^{⸝}$, because every module can embed in an injective module ($M$ is injective module).

**§2: Noetherian, Artinian Regular Modules and Injective Property**

In this section, we provide that several relationships between some concepts and injective module exist. We investigate, if $M$ is a cyclic and regular module is injective. Also, if $M$ is regular with N≤$ M$ is finitely generated submodule, so $M$ is injective. Finally, some relationships about injective module have been studied in details. We present some new results about the relationship between Noetherian, Artinian and regular rings and injective module. We should start with the following definitions.

We need to introduce a basic preliminary in order to proceed towards the main objective of the current study.

**Definition (2.2.1)**. **[42].** Any module $M$ is called regular if Ɐ$m∊M$,$ ∃$ g∊$Hom$($M$,R), then(mg)m=m.

See the following lemma:

**Lemma (2.2.2).** Every Noetherian or Artinian Regular module over abelian ring R with unity is an injective module.

**Proof.**

Any module $M$ fulfills all the conditions in theory, this means that $M$ is a finite direct sum of projective module have only two submodule are {0} and $M$. Since R is a commutative ring with identity element, so $M$ is a flat module if and only if it is injective and a finite direct sum of injective is also injective.

**Definition (2.2.3). [29].**  An R-module $M$ is called projective if and only if for any $epimorphism$ $f$:C→V such that C,V are any R-modules and for any homomorphism g:$M$→V ∃ a homomorphism h:$M$→C such that f$∘$h=g.

**Definition. (2.2.4).** **[42]**. A ring R is called a finite–dimensional if R has finite direct summand of ideals.

**Theorem (2.2.5)**. Let R be a (QF) and perfect ring or finite–dimensional ring. If $M$ is a regular R-module, then $M $is injective.

**Proof.**

Let R be a perfect ring. Let T=direct limits of projective-module. Then T projective (see [24]). But $M$ is a direct limit of $N\_{i}$∋ $N\_{i}$finite submodules. So $M$ is a projective. But from [29], for a QF-ring. every projective module is an injective module. Now if R has no infinite direct sums of ideals. Let $M$ be regular and Г={∑ $⨁$ Rmα : mα∊$M$} is a partially ordered and $∑$ $⨁$ Rmα ≤ $∑$ $⨁$ Rmβ $iff$ {mα } ⊆ {mβ }.So $∃$ $N$=$∑$ $⨁$ Rmα is a maximal in Г (By Zorn’s Lemma). Then $N$∩Rm≠0, $0\ne m∊M. $If $N=M$, suppose that$ m∊M$. Rm≈ $I$; $I$ is an ideal of R. So have no infinite direct sums of $N\_{i}$. But Rm=Rn1$⨁$……$⨁$ Rnt ∋ $Rni $ simple, $Rn\_{i}∩N$ ≠0 Ɐ$i$. so $Rni∩N= Rni.$$ Rm∩N$=(Rn1$⨁$……$⨁$ Rnt) ∩N⊇$∑⨁( Rn\_{i}∩N)= ∑⨁ Rn\_{i}=Rm$. Hence $Rm⊆N$ and then $M$=$N.$ Therefor $M$ is a projective. Thus it is an injective module.

**Lemma (2.2.6)**. **[42]**. Every f-generated regular R-module is a projective.

Recall that for all $m\_{i}∊M$, $i$∊$J$ ∋$ J$ is a some of several generators of $M$. So we can present a definition of finitely generated module $M$ by the following way:

**Definition (2.2.7)**. **[5].** A module $M$ is called finitely generated(f-generated) if it has a finite set of generators. In other word; $M$ is finitely generated if $M =∑x\_{i}r\_{i} ; Ɐ x\in M, r\in R$.

**Example (2.2.8).[5].** Any f-dimensional vector space is an f-generated over a field K.

**Proposition (2.2.9)**. Let $E$ be a regular R-module. If $E$≅$\frac{R^{n}}{N}$, then $E$ is an injective module.

**Proof.**

 Since $E$≅$\frac{R^{n}}{N}$ , $n∊Z^{+}$$, N\leq R^{n}$, so there is a homomorphism.

 $γ$: Rn→$\frac{R^{n}}{N}$ ≅$E$ ∋$γ$ (r1,.....,$r\_{n}$)→ (r1,.....,$ r\_{n}$)+N.

Take $e\_{i}$=(0,….,0,1,…,0) ∋ (1 being at the $i-th$ place).Hence $e\_{i}$generate Rn, 1≤ $i$ ≤n. so $γ$ ($e\_{i}$) generate $E$ over R, (1≤ $i$ ≤n).Therefor $E$ is f-generated module. But $E$ is regular module. So $E$ is projective (Lemma 2.2.6). Thus $E$ is injective module.

**Corollary (2.2.10)**. Every cyclic regular R-module over QF-ring is injective.

**Proof.**

 Since $M$ is cyclic regular R-module then it is f-generated and by (Lemma 2.2.6), $M$ is injective.

**Corollary (2.2.11).** Let $M\_{1}$ and $M\_{2}$ be R-modules over (QF)-ring and let $δ$:$ M\_{1}$ →$M\_{2}$ be an onto homomorphism such that $M\_{2}$ is a regular and $M\_{1}$ is f-generated, so $M\_{2} $is injective module.

**Proof.**

 Suppose that$ M\_{1}$ and $M\_{2 }$are two modules over the ring R. Also suppose that $M\_{1 }$is f-generated module. To prove that $M\_{2 }$ is injective. Since$ M\_{1}$ is a f-generated. R-module, so $M $has a generating set {m1,….,$m\_{k}$}. Therefore, we need to show that $M\_{1}$is generated by the set {$δ$(m1),…,$ δ$(mk)}. Ɐ m2∊$M\_{2 }$ , we have $δ$ is onto, $∃$ m1∊$M\_{1}$ ∋ $δ$(m1)=m2. But $M\_{1}$ is a f.g module, m1=r1a1+………+$r\_{k }a\_{k}$, r1,…..,$ r\_{k }$∊R. So m2=$ δ$(r1a1)+………+$ δ$($r\_{k }a\_{k}$) =$ r δ$(a1)+……$r\_{k } δ(a$k). Hence $M\_{2 }$=<$ δ$(a1), …….$, δ(a$k)>. Then $M\_{2 }$is a f-generated with regular property imply $M\_{2 }$ is projective and hence is injective.

**Theorem (2.2.12).** Let $M $be a regular module over P.I.D. If $M$ is acyclic module, then it is a Noetherian and is an injective module.

**Proof.**

Since $M$ is a cyclic, then it is a f-generated. So $M$ have generators m1,…...mk. Hence ,∃ $φ:R^{k}\rightarrow M$ defined by

$φ$(b1,……,$b\_{k}$)=b1m1+……,+$b\_{k}m\_{k}$,

then $M≅ \frac{R^{k}}{N}$ , But R is a P.I.D, so R is a Noetherian R-module. We have $M$ as a regular module. Thus $M$ is injective (Lemma 2.2.2).

**Corollary (2.2.13)**. Let $M$ be a regular R-module. If $N$ and $\frac{M}{N}$ are Noetherian ∋ N is a submodule of $M$, so$ M$ is injective module.

**Proof.**

 Assume that $K$≤$ M$. So $Img(K)$ in $\frac{M}{N}$ is f-generated. Hence $K∩N$ is also f-generated. Let k1,……$k\_{n}$∊$K$ generate $Img(K)$ in $\frac{M}{N}$ and let b1,……,$b\_{m}$generate $K∩N$. Ɐ $k∊K$ , $k≡$r1k1+……+$r\_{n}k\_{n}$ , $r\_{i}$∊R. So K-∑$r\_{i}k\_{i }$∊$K∩N$. Then $K$-∑$r\_{i}k\_{i}$=∑$t\_{i}b\_{j}$, $t\_{i}$∊R. Hence $K$=∑$r\_{i}k\_{i}$+∑$t\_{i}b\_{j}$ .(K f. g in $M$) therefore $M$ is a Noetherian module with regular property implies $M$ is an injective module (Lemma 2.2.2).

 Recall that any R-module $M$ satisfies the maximal condition for submodules if $φ$≠ Г of submodules have a maximal (Г⸧ H0 ∋ , $∄$ number containing H0). Therefore it is easy to present a definition of Noetherian R-module, any module $M$ satisfies the maximal condition(ACC) is Noetherian.

 The next theorem shows the relationship between Noetherian module and injective module; but before that we need to present the following lemma:

**Lemma (2.2.14)**. If every submodule of an R- module $M$ is f-generated, then $M$ is a Noetherian.

**Proof.** Suppose that N≤$M$ is a f-generated assume that H1⊆ H2 ⊆H3⊆…. is a submodules of $M$.  Take H=∪$H\_{i}$, $i$=1,……,∞. So H≤$ M$ and hence H is a f-generated. Let H=Rh1+……+$Rh\_{n}$. All hi in one of Hi, $∃$ m ∋ h1,…….hn ∊Hm. But H=$H\_{m}$, $n\geq m$. So $M$ is a Noetherian module.

**Theorem (2.2.15)**. Let $M$ be an R-module. If $M$ is a regular module and has N is a f-generated submodule of $M$, then $M$ is injective.

**Proof**.

By Lemma (2.2.14), $M$ is Noetherian-module (N≤$ M$ is an f-generated). But $M$ is a regular module. Then from(Lemma 2.2.2), $M$ is injective.

**Theorem (2.2.16).** Let $M$ be a regular module and let

 0→$M\_{1}$→$M$→$M\_{2}$→0 be an exact seq. If $M\_{1}$and $M\_{2}$ are Noetherian, then $M$ is an injective module.

**Proof.**

 Suppose $M\_{1}$≤$M$, $M\_{2} $=$ \frac{M}{M\_{1}}$ and assume that $M\_{1}$ and$ \frac{M}{M\_{1}} $are Noetherian.

Let H1⊆ H2 ⊆…....., $\frac{(H\_{1}+M\_{1})}{M\_{1}}$ ⊆$ \frac{(H\_{2}+M\_{1})}{M\_{1}} $⊆$ \frac{(H\_{3}+M\_{1})}{M\_{1}} $⊆…… of $M\_{1}$ and $\frac{M}{M\_{1}}$, $∃$ m ∋ $H\_{n}∩M$ =Hm∩$M\_{1}$and $H\_{n}$+$M\_{1}$=$H\_{m}$+$M\_{1}$, n ≥ m. So $H\_{n}$=$H\_{m}$∩($H\_{n}$+$M\_{1}$) = $H\_{n}$∩($H\_{m}$+$M\_{1}$)= $H\_{m}$+($H\_{n}$∩$M\_{1}$) by Modular law, Let H, Y, L ≤$ M$ and Y⊆ H. So H∩(Y+L) = Y+(H∩L)).= $H\_{m}$+($H\_{n}$∩$M\_{1}$) = $H\_{m}$. So $M $is Noetherian module with regular property, we get $M$ is injective.

**Example (2.2.17)**. Any module over a division ring is injective, because a division ring R has only 2 ideals 0 and R itself.

**Proposition (2.2.18).** Let $M$ be a regular module. If:

1. S1 is the set of f-generated submodules of $M$ is Noetherian,

2.$ φ$ ≠N1 is f-generated and N1 ≤ $M$ ∋ N1 has maximal element,

3. N ≤ $M$ is an f-generated.

 Then $M$ is injective

**Proof.**

 Let S1≠$ φ$ be a set of f-generated (S1 ≤ $M$) If S1 has no maximal element, so any s∊S1 {S2∊S1: S2 ⸧ S1, S2≠S1 } ≠$ φ$,thus we get (ACC)of submodules which is infinite.

 Now let N ≤ $M$, there is a maximal element N1. then N1=N. Now let

H1 ⸦ H2⸦……. Be (ACC$)$of submodules of $M$. So ∪$H\_{i}$⸦ $M$ is a f-generated and all generating elements in $H\_{i}, i∊H$.

Thus $H\_{i}$=$H\_{i+r} $Ɐ $r∊N$. So $M$ is a Noetherian module But every Noetherian module is Artinian with regular property $M$ is an injective module. (Lemma2.2.2).

Recall that if proper f-generated submodule of $M $is a small in $M$ (P. f-generated N<<$M$), so $M$ is called a semi hollow module such that $M$ is hollow if P-submodule N is small in $M$. Therefore we present the following theorem.

**Theorem )2.2.19(.** Let $M $be a regular R-module. If $M$ is semi hollow and Rad($M$) is a Noetherian module, so $M$ is injective.

**Proof.**

Assume that $M$ is semi hollow-module. Let Radical of $M$ not equal $M$. there are Max(N) ∋ N≤$ M$. This means that $M$ is also module. Hence Rad($M$) is a maximal and Rad<<$M$. Hence $\frac{M}{Rad(M)}$ is a simple module and hence is Noetherian. Since

 0→Rad($M$)→$M$→ $\frac{M}{Rad(M)}$→0

is a short exact seq. Then $M$ is a Noetherian ($M$ is Artinian-module) with regular property implies $M$ is an injective module.

It is possible to rely on the previous example to discuss its content in another way, follows:

**Proposition 2.2.20**. Let R be a division ring if:

1. $M$ is a regular module over R.
2. $M$ is a divisible-module.

 Then $M$ is injective.

**Proof.**

Assume that R is a division ring and $M$ is a divisible-module. Let

 N$\leq M$ such that K is a basis for N. So $∃$ a basis k1 of$ M\ni $k1 $⊃$ K. Assume that k2 = K1$∩$k and N1 is a span of k2. Then N1⨁N= $M$. Hence $M$ is a semi simple ($M$ is Artinian). But $M$ is a regular. Thus $M$ is injective.

**Example )2.2.21(**. Any regular module $M$ of the ring R which has only two ideals {0} and R is Artinian, because R has only two ideals {0} and R implies R is division and hence $M$ is Artinian with regular property we get $M$ is injective.

**Proposition )2.2.22(**. Let $M\_{1}$, $M\_{2}$ and $M\_{3}$ are three modules if

1. 0→$M\_{1}$→ $M\_{2}$→$M\_{3}$→0 is short exact of R-modules.
2. $M\_{1}$ and $M\_{3}$ are Artinian modules.
3. $M\_{2}$ is a regular module.

Then $M\_{2}$ is injective.

**Proof.**

Take a chain $M\_{2\_{m}}$ such that $M\_{2\_{m}}$are submodules of $M\_{2}$. From the projecting to $M\_{3}$, $Im$($M\_{2\_{m}}$) is stabilized, so if

f:$M\_{2\_{m}}$→ $M\_{3}$, the $ker(f)$ from chain submodules of $M\_{1}$ . Hence it is stabilizes. Then $M\_{2}$ is Artinian module with condition (3), we get $M\_{2}$ is injective (Lemma 2.2.2).

 **Remark )2.2.23(. [22].** Every homomorphic image of Artinian ring is Artinian.

**Theorem )2.2.24(**. Let R be an Artinian ring. If $M$ is a f-generated. R-module and regular, so $M$ is an injective module.

**Proof.**

We know that $M$≅$\frac{R^{n}}{N}$ such that N≤ $R^{n}$and n+. But $R^{n}$ is an Artinian ring, so a direct sum of Artinian modules. Hence $M$ is an Artinian module (Remark 2.2.23). But $M$ is regular. Thus it is injective module.

**Definition )2.2.25(**. **[17].** The ring R is called $prufer$ domain if R is integral domain and every f-generated. ideal of R is projective (invertible).

In the Theorem(2.2.27), we study some conditions over $prufer$ domain in order to get injective module.

**Remark )2.2.26(.** Any ideal I is injective if I$⋅$I-1=R ∋ I-1={$rI$ ⊆ R: $r∊q(R) $and $q$(R) is the field of fractions is the smallest field can be embedded.

Recall that if R2 be a unitary extension ring of R1. We say R2 is a p-extension of R1 if Ɐ r1∊ R2 satisfies R1[X] one whose coefficient is a unit of R1(whose coefficients generate a unit ideal of R1).

**Theorem )2.2.27(**. Let R is a ring. If

1. R is a prufer domain Krull domain 1,
2. $M$ is a divisible R-module,
3. $M$ is Artinian R-module.

Then $M$ is injective.

**Proof.**

 Over prufer$ $domain any module is linearly compact and divisible is injective or, Artinian module is linearly compact with divisible property $M$ is injective.

**Proposition )2.2.28(**. Let$ M$ be an R-module. If:

1. $M$ satisfies (DCC),
2. Every element $m∊M$ is a divisible,
3. R is integrally closed domain with quotient field K,
4. K is a p-extension of R.

Then $M$ is injective.

**Proof.**

Assume that K is P-extension of R. and Let I⋌ be a maximal ideal in the ring R1. Let H=all elements f in R1[X] ∋ $A\_{f}$=R1. So H is a regular multiplicative in R1[X]. Hence H= R1[X] - ∪ I⋌[X]. So if I1 is an ideal of R1[X]⊆∪ I⋌[X]. then I1 contained in one of I⋌[X]. So {I⋌[X]} is the set of prime ideals of R1[X]. Hence R is a $prufer$ domain. Thus $M$ is an injective module because from condition (1), $M$ is Artinian and by condition (2), $M$ is a divisible module.

**CHAPTER THREE**

**INJECTIVE MODULE OVER SOME DOMAINS**

**INJECTIVE MODULE OVER SOME DOMAINS**

**Introduction.**

In this chapter we introduce how we get injective module through some domains and also from Euclidean ring and hereditary rings. In the first section we present several results which give injective module over $R$. One of these domains is the Dedekind domain. We proved that any element of $R$-module $M$ is divisible and let $R$ be a Dedekind domain and If$ Ι$ is a finitely many prime ideals, so$ M$ $ $is an injective module. We take another domain which is a unique factorization domain (UFD). We showed if $R$ be a $Dedekind domain$ and it is a UFD and If every element of $M$ is divisible, so $M$ is injective module. From Krull domain, Mori domain we will get that $M$ as injective. In section two we satisfy several facts about the relationship between injective module and Euclidean ring. We get every divisible module over Noetherian valuation ring is injective. In addition, there are some, connections between (D.V.R) and injective module. In the last section we present a main relationship between some rings (hereditary ring, local ring and D.V.R) and injective module.

**§1: Study Injective module over Dedekind domain**

In this section, we study injective module over Dedekind domain. Some results have been obtained about this relationship. Before getting deeply into the relationship between injective module and Dedekind domain, we need some definitions and lemmas related to the topic.

**Definition (3.1.1). [37].** Any ring R is called Dedekind if it is an integral domain and every 0≠$I$ is a factors into product of prime ideals.

To understand $Dedekind domain$, we need to define some concepts, such as filed, and integral domain. **An ideal I of the ring R is a prime ideal if** $a,b$ **∈ R then either** $a\in I$ **or** $b\in I$ **for all** $a.b\in I$**. Also, in [14],** any ideal I of Z is a f-generated Z-module is called fractional ideal and is denoted by (FI), if for every maximal ideal, $I\_{i}$ is principal ideal over the ring $R\_{i}$ is invertible.

**Lemma (3.1.2)**. **[1].** Every 0≠f is invertible; f is fractional ideal.

**Examples and remark (3.1.3).**

1 Every P.$Ι$.D is a $Dedekind domain$.

2. $R$ is a P.I.D if and only if$ $every fractional ideal f is principle.

3. If $R$ is Dedekind domain, so $R$ is UFD if and only if $R$ is P.I.D.

4. A localization of a Dedekind domain of multiplicative set is a $ Dedekind domain$.

**Lemma (3.1.4).** Every finitely many prime ideals $P\_{i}$ in a $Dedekind domain$ is P.I.D.

$Ρ$**roof**.

Let 0≠$P\_{i}$ is a prime ideals. If 0≠$I$ is an ideal so, ∃ $I\_{1}$∋$ II\_{1}$is a P.I($II\_{1}$= <a>) and hence $I\_{1}$is a relatively prime to $I$ But S≠ ∅ ∋ S is the set of prime factors of $I\_{1}$. Hence $I\_{1}$= $R$ and thus <a>= $II\_{1}$=$IR$=$I.$

**Lemma (3.1.5).** Let $R$ be a $Dedekind domain$ and it is a UFD. If every element of $M$ is divisible, so $M$ is an injective module.

**Proof**.

We know that any commutative ring is P.I.D, because $R$ is a UFD. But P.I.D with divisible module $M$ indicates that $M$ is injective.

**Theorem (3.1.6).** Let $R$ be a $Dedekind Domain$ with nonzero fractional ideal $I\_{i}$. If $M$ is a divisible R-module such that $I\_{i}$ is an integral domain, then $M$ is an injective module.

$Ρ$**roof.**

We know that there is 0≠$I\_{3}$fractional ideal and $I\_{1}I\_{3}$=$ R$. By defining the fractional ideal, there is 0≠$r$ an element in $R$ and $rI\_{3}$ is an integral ideal. Assume that $I\_{2}$=$rI\_{3}$. $Τ$herefore $I\_{1}I\_{2}$=$ R$ and hence $I\_{1}I\_{2}$is a P.I. in $R$. $ Η$ence $R$ is P.I.D. But $M$ is a divisible. Thus $M$ is an injective module.

**Corollary (3.1.7).** Let any element of $R$-module $M$ be divisible and let $R$ be a Dedekind domain. If$ Ι$ is a finitely many prime ideals, so$ M$ $ $is injective module.

$Ρ$**roof**.

Let $I\_{i}$ be all prime ideals. If 0≠$I\_{1}$is an ideal, then ∃ 0≠$I\_{2}$ and $I\_{1}I\_{2}$ is P.I, ($α)$; $I\_{2}$ is a relatively prime to $I\_{i}$. The factors of $I\_{2}$ =φ. So $I\_{2}$=$ R$. Hence ($α)$=$ I\_{1}I\_{2}$=$I\_{1} R$ =$I\_{1}$. Then $R$ is a P.I.D, but every element of $M $is divisible ($rm$=$m$), $rϵR$, $mϵM$. Hence $M$ is a divisible module. $So$ $M$ is injective.

 The next lemma explains the relationship between valuation ring and Dedekind domain. Now we start with a clear definition of valuation ring.

**Definition (3.1.8). [2].** Let $R$ be an integral domain. So $R$ is a valuation ring if it is not field; $aϵR, $so$ a^{-1}$∉$ R$.

**Lemma (3.1.9)**. Let $R$ be an integral domain and with no invers. If $R$ is a $Ν$oetherian local $ring with$ P.M.I, then $R$ is a $Dedekind domain$.

$Ρ$**roof**.

 Take the maximal ideal $M$=(S) in $R$. We need to show $I$=< a >. (principal). We have $I $is an f-generated. Then, ∃ n is a maximal ∋ $I$ ⸦$M^{n}$. Ɐ $b\in I$, so b∉$M^{n+1}$; b= u$s^{n}$ ∋ u is unit and hence b$\in $($s^{n}). $But this true for all b and since $M^{n+1}$⸦$(s^{n})$, $I$=$(s^{n})$. Thus $R$ is a local and P.I.D. But every $Ρ$.$Ι$.D is Dedekind domain.

**Lemma (3.1.10).** Let $R$ be an integral domain has no invers. If $R$ is a local such that 0≠$I$ is invertible $ideal, then R is$ a $Dedekind domain$.

$Ρ$**roof**.

Since any invertible ideal is a f-generated, then $R$ is a $Ν$oetherian ring. We must prove that $M$ is a maximal where $M$= < $m$ > ( i.e. $M$ is principal). But from Nakayama̕s lemma; $M\ne M^{2}$. If $s\in M-M^{2} $and s$M^{-1}$ $⊄$ $R$ with $sM^{-1}⊄M$ (s∉$M^{2}$); $then$ s$M^{-1}=R $and M = (S). $So$ $M$ is a principal. But from lemma (3.1.9); we have $R$ is $Ρ$.$Ι$.D. $Thus$ $R$ is a $Dedekind domain$.

**Theorem (3.1.11).** Let $R$ be a ring If:

1. $R$ a valuation ring and UFD;
2. $R$ is a $Ν$oetherian local ring;
3. $I$=< a >;
4. $M$ is a divisible module.

Then $M$ is injective module.

$Ρ$**roof**.

Since $R$ is a $Ν$oetherian valuation $ring, so$ an ideal $Ι$ $is generated$ by finitely many elements. $Η$ence one of them contains all others and be $Ι$. $Τ$hen $R$ is a P.$ Ι$.D. ($R$ is Dedekind). But $R$ is UFD and $M$ is a divisible module. So $M$ is injective.

**Corollary (3.1.12)**. Every V.R is a $Dedekind domain$.

$Ρ$**roof**.

Clear. Every D.V.R is P.I.D and hence$ R$ is a Dedekind domain

**Corollary (3.1.13).** Let $R$ be a ring. If:

1-$ R$ is D.V.R;

2-$ M$ is divisible;

$3-R$ is UFD.

Then $M$ is an injective module

$Ρ$**roof.**

Since every D.V.R is a Dedekind domain and every D.V.R is a P.I.D, then from condition (3) and condition (2); we get $M$ is injective module.

**Example (3.1.14). [22].** $Q\_{p}$ is a fraction field $Z\_{p}$. So is P.I.D with maximal ideal is D.V.R. Thus Dedekind domain.

Recall that an ideal $Ι$ in the ring is namely fractional if it is a submodule of H, there is 0≠S ∋ S$I$⸦$ R$. Also, if $R$ is abelian integral domain with 1 and it is a field of fractions, so we say that F(D) is the set of non-zero fraction ideals of D such that D is an integral domain.

**Definition (3.1.15).**$ [41]. Α$n integral domain D satisfies generalized of D or (GD) if Ɐ $Ι$∈F(D)→$Ι$=$(Ι^{-1})^{-1}$is invertible such that $an ideal$ $Ι$ is namely invertible if ∃ $J$ is (FI) and then $ΙJ$= D.

Now If each fractional ideal in R $is invertible$, so $R$ is a $Dedekind domain$. Therefore; from the definition the Dedekind domain and (GD) with another condition on a module, we can get $M$is injective. However before that we need to present the next definition.

**Definition (3.1.16).** Let D be an integral domain. $Τ$hen D satisfies generalized domain (GD) if and only if for all $Ι$∈F(D), $Ι$= $(Ι^{-1})^{-1}$ is invertible.

**Theorem (3.1.17).** Let $Ι$∈F(D). If $(ΙJ)^{-1}$= $Ι^{-1}$$J^{-1}$Ɐ $J$∈F(D), with $M$ is a divisible D-module, then $M$ is an injective module.

**Proof**.

Ɐ $J$∈F(D), $(ΙJ)^{-1}$= $Ι^{-1}$$J^{-1}$, so ($I\_{v}J$)- 1=$I^{-1}$$J^{-1}$, because

 $(IJ)^{-1}$=(($ΙJ$)v)-1 = (($I\_{v}J$)v)-1 =($I\_{v}J$)-1. Hence $I\_{v}$ is invertible in F(D). Therefore $I\_{v }$is (GD) and so D is a Dedekind with $M$ is divisible imply $M$ is injective module.

**Proposition (3.1.18)**. Let D be a completely integrally closed;

Ɐ $I,J$∈ F(D), $(IJ)\_{v}$=$I\_{v}J\_{v}.$ If $M$ is a divisible module, then $M$ is injective.

Before start with proof of proposition 3.1.18, we need to define some concepts.

**Definition (3.1.19). [40].** $Α$n ideal $Ι$ of F(D) is namely v-invertible if: ∃ $J$∈F(D)∋ ($IJ\_{v})$=D.

**Definition (3.1.20). [41].** We say that D is completely integrally closed (C.I.C) if an ideal $I$∈F(D) is v-invertible.

Now we start with proof of proposition (3.1.18).

**Proof of proposition 3.1.18**

 Let $I$∈F(D). So D is (C.I.C). (by definition). Hence ($ΙΙ^{-1}$)v=D ([10]). But if D is a C.I.C, then ($IJ)\_{v}$=$I\_{v}J\_{v}$ Ɐ $I, J$ ∈F(D). $Τ$hen ($II^{-1})\_{v}$=$I\_{v}$($I^{-1}$)v =$ I\_{v}I^{-1}$ Ɐ $I$∈F(D). So $I\_{v}$ is invertible. Therefore D is Dedekind domain. But$ M $is divisible, so $M$ is an injective module.

 Now we use a generalized Dedekind domain in order to obtain that module $M$ is injective. From definition of invertible concept, we can present the following information.

Any integral domain D is called generalized (GD) if Ɐ $Ι$ an ideal in fractional ideals of D equal $(Ι^{-1})^{-1}$ ; so$Ι$=$(Ι^{-1})^{-1}$ is invertible. Also, for binary operation ⁎, we say that ⁎-operation same v-operation. Therefore, the map.$I$→$I\_{v}$ on the fractional ideals F(D) is v-operation.

**Definition (3.1.21). [23].** Any integral domain D is namely $Krull$ type if it has a collection F={$p\_{i}\}\_{i\in I}$ of prime ideals such that,

1. D=∩$D\_{p\_{i}}$,
2. Ɐ $i\in I$; $D\_{p\_{i}}$ is a V.R,
3. 0≠$I$ and $I$ not unit of D belongs to only a finite number of $p\_{i}$.

**Remark (3.1.22).[41].** Ɐ $Ι$∈F(D), is said to be v-invertible if ∃ $J$∈F(D) ∋ $(IJ\_{v})$=D. So Ɐ $I$∈F(D), we say that $Ι$ is v-invertible and hence D is a completely integrally closed. Note that, D is called a Mori domain if any set of the integral is a v-ideals of D and satisfies (ACC), such that any ideal $Ι$∈F(D) is v-ideal if $I$=$I\_{v}$.

 From([31]), an integral domain D is a Mori if and only if $ I$∈F(D),∃ a f-generated. ideal $J$∈F(D)∋$J⊑Ι$ and $I\_{v}$=$J$. Therefore

1-Completely integrally closed with $Krull$ domain give(GD).

2-Mori domain and Mori domain with Completely integrally closed give $Krull$ domain

**Example (3.1.23).** (GD) give Completely integrally closed.

**Lemma (3.1.24).** **[41].** Every $Krull$ domain is a completely integrally closed.

**Theorem (3.1.25).** Let $M$ be an $R$ -module. If:

1. Ɐ $m\in M$ is divisible element.
2. D is $Krull$ ∋ $I∙J$ is invertible and v-ideal.

$Τ$hen $M$ is injective.

$Ρ$**roof.**

Suppose that D is $Krull$ domain. Hence D is completely integrally closed. Therefore D is a (GD) (proposition 3.1.18). Now if an element $m\in M$ such that it is divisible, then $M$ is a divisible module over D. But from the definition of (GD); we obtained D is a Dedekind domain. Now $M$ is divisible over Dedekind domain; this means $M$ is injective module.

**Example (3.1.26).** **[41].** Any divisible D-module over D is the entire function is injective.

 Note that from a proof of proposition (3.1.18), we say if D is a completely integrally closed, so D is (GD). Then the converse is true in general (see the following result):

**Corollary (3.1.27)**. Any (GD) is completely integrally closed.

$Ρ$**roof**.

Suppose that we have (GD). So Ɐ $I$∈F(D),

($ΙΙ^{-1})\_{v}$=$(I\_{v}I^{-1}$)v= (D$)\_{v}$=D. Also, every 0≠$I$∈D is v-invertible. Then D is a C.I.C.

**Theorem (3.1.28). [41].** Let D be an integral domain. Then D is completely integrally closed if and only if D is a (GD).

There is another way to prove that D is a $Krull $domain. This way start with some definitions for example;(⁎)-finite ideal,

G⁎-Noetherian domain, (⁎)-Dedekind domain and P(⁎).M. domain. $Ν$oetherian ring.

Let D be an integral domain and Let (⁎) be an operation on D. If $I$∈F(D), so $I$ is (⁎)-finite if ∃ $J$∈F(D)∋$Ι$⁎=$J$⁎ and $I$ is invertible. ([37]).

**Definition (3.1.29).** **[10].** Let D be an integral domain and let (⁎) an operation on D. We say D is (⁎)-$Ν$oetherian domain if D has (ACC) on all ideals of D. $(i.e$ (Quasi(⁎)-ideals). Or D is (⁎)-$Ν$oetherian domain if 0≠$I$ is (⁎)f-finite.

**Example (3.1.30).** Any $Ν$oetherian domain is (⁎)-$Ν$oetherian domain. Also every Mori domain is (⁎)-$Ν$oetherian domain.

**Definition (3.1.31). [41].** Let D be an integral domain and (⁎) is an operation on D. Then D is called $prufer$-(⁎)multiplication domain.(P.(⁎). M. D) if Dm is a valuation such that m is a $maximal element$ in S; $φ$≠S= $set$ ideals of D ($Quasi$-(⁎)f –maximal ideal).

**Definition (3.1.32).[41].** If D is (⁎)-$Ν$oetherian domain and $Ρ$.(⁎).M.D, so D is called (⁎)-Dedekind.

**Lemma (3.1.33). [10].** Every (⁎)-finite is (⁎)-Noetherian domain.

**Lemma (3.1.34).** Every (⁎)-$Ν$oetherian domain and P.(⁎).M. domain is (1)-Dedekind domain (Dedekind and v-operation).

**(**2)- (⁎)-Dedekind with v-operation is $Krull$ domain.

**Corollary (3.1.35).** Let D be an integral domain. If:

1. $I$ is (⁎)-finite Ɐ $I$ ideal of D.
2. Each element $m\in M$ is a divisible element ∋ $M$ is an D-module.
3. D is P.(⁎).M.D.

 $Τ$hen $M$ is injective.

$Ρ$**roof.**

Since $I $is a (⁎)-finite, then Ɐ $I$∈F(D), ∃ $J$∈F(D) ∋ $I$⁎=$J$⁎. Hence $I$ is an invertible ideal such that $J$ is subset of $ I$. Therefore D is (⁎)-$Ν$oetherian domain. In other words,

 0≠$I$ ∈ D, so ∃ $J⊑I$ ∋ $I$⁎=$J$⁎ and

$J$ is a finitely generated ideal. We have D is a P.(⁎).M.D. Then D is a (⁎)-Dedekind domain

(i.e. D is v-operation). Now D is (⁎)-Dedekind with v-operation; this means D that is $Krull$ domain. Hence D is a completely integrally closed and then it is a (GD). Thus Dedekind with condition (2); implies that $M$ is an injective D-module.

Recall that in [32], An R-module $M$ is called pseudo-injective(P-injective) if every submodule N of $M$, each R-monomorphism

 $f: N\rightarrow M$ can be extended to an R-endomorphism of $M$. Also, in [21], Any module $M$ is called Quasi-injective (Q-injective) if for each submodule N of $M$, and each R-homomorphism $f: N\rightarrow M$ can be extend to an R-endomorphism of $M$.

**Lemma (3.1.36).** **[28].** Every P-injective module over Dedekind domain is Q-injective.

**Proposition (3.1.37).** Let R be a Dedekind domain and P.I.D. If $M$ is a P-injective; then it is an injective module.

**Proof.**

Let H≤$M$. Since $M$ is a P-injective and every P-injective is Q-injective (see lemma 3.1.36). But R is P.I.D; with Dedekind Property that$ M$ is an injective module.

**§2: Injective modules and Euclidean Ring**

 In this section, we satisfy several facts about the relationship between injective module and Euclidean ring. The main result is that every divisible module over Noetherian valuation ring is injective. Also, there are some, connection between (D.V.R) and injective module. We prove that every divisible module over (UFD) is also injective.

**Definition (3.2.1)**. By van der Warden, we say that R is an Euclidean ring if R is integral domain such that the division Algorithm is true.

**Remark (3.2.2)**. We can rewrite Definition )3.2.1(in another way (if ∃$δ$:R/{0}→$Z^{+}$+{0} ∋ δ($xy$) greater than and equal $δ(y)$ and

 $x=yq+s$, s=0 or $δ(s)<δ(y)$ $Ɐ x,y$ ∈ R; $q ,s$ ∈R).

**Lemma (3.2.3)**. If $M$ is is a divisible module over P.I.D, so $M$ is injective module.

**Proof.**

 Suppose that $M$ is a divisible R-module. Let f:$I$→$M$ be an R-homomorphism, where $I $is a left ideal of R since R is a principal ideal ring, $I$=$Rt$ for some $t\in R$, since $M$ is divisible, there exists an m∈$M$ such that $f(t)$=tm define g: $R$→$M$ by $g(r)$=$rm$. Then g is R-homomorphism Note that every

 $st\in Rt$, $g(st)=(st)m=s(tm)=sf(t)=f(st).$ Thus $M$ is injective.

**Lemma (3.2.4).** Every divisible over Euclidean domain is an injective module.

**Proof.**

 Suppose that R is an Euclidean domain. Let $I\_{1}$⊲ R. Hence$ I\_{1}$={0}=(0) or

let 0=$α\in I\_{1}$ ∋ d($α)$ least, so any $β\in I\_{1}$,we obtain $β=qα+r$ and $r$=0 or

 d($r$) < d($α)$. But $r$=$ q$- $βα$ ∈ $I\_{1}$. Since d($α)$ is a minimal and and $r$=0, and $I\_{1}$=($α)$. Hence R is a P.I.D. But $M$ is a divisible module. Thus $M$ is an injective module.

The following shows that divisible Z-module over the integer numbers is injective module;

 Suppose that $I$⊲Z. if $I$={0}, so $I$(0) and $I$ is a principal ideal. Assume that $I$≠{0} and $α$ is the smallest integer ∋ $α$ is a positive in $I$ We must prove that $I$=($ α)$. Since $α$∈$I$, so ($α)$⊑$I$. If $β$∈$I$, then for some $q,r$ belong to Z, 0 ≤ r ≤ $α$-1. Hence $r$=$ β-q α$; $r\in I$. Since $α$ is the smallest positive in $I$, hence r=0. So $β=q α$ and $β$∈($ α)$. But $M$ is a divisible module Thus $M$ is injective.

 Recall that any filed is an Euclidean ring and this leads to the following result:

**Theorem (3.2.5).**  Let K be a field. If $M$ is a K-divisible module, so $M$ is injective.

**Proof.**

The proof of this statement is easy, because; if $I$⊲K ∋ 0≠$ α$∈$I$, so every $β$∈K and hence: $β$=$(β α^{-1}$)$ α$∈$I$.

 Hence $I$=K. therefore only ideals of K are {0} = (0) and K= (1). Then K is P.I.D ($M $is injective because it is divisible).

**Theorem (3.2.6).**  Let R be an Euclidean ring. If

1. $M$ is a divisible R-module.
2. $I$ is a maximal ideal of R.

Then $M$ is $\frac{R}{I}$ –injective module.

**Proof.**

 Suppose that $I$⊲R ∋ $I$ is a maximal ideal of R. Suppose that $x+I $∈$ \frac{R}{I}$ and $x+I$ ≠0. To prove that ,∃ $y+I$ ∈$ \frac{R}{I}$ ∋ $(x+I)$($ y+I)$=$1+I$.

Let $J$= <$x$>$+I$ ∋ $J$ is an ideal. So $I⊆J⊆R$ and $I\ne J$ ($x\in I$).

We have $I$ is a maximal ideal. So $J$=R (Definition of maximal ideal). Then $1\in J$. Hence 1=$xy+z$ Ɐ y ∈R, $z\in I$. Therefor

$1+I$=($xy+z$)+$I$=$xy+I$=($x+I$)($y+I$).

Hence $\frac{R}{I}$ is a field. Then $\frac{R}{I}$ is an Euclidean ring, with $M$ is divisible module imply $M$ is injective.

**Example (3.2.7).**  Any divisible module $M$ over $\frac{z}{nz}$ is injective because $\frac{z}{nz}$ is a field.

 Recall that a domain R is a valuation ring ($v.r$) if R is not field and $a\in K$, $a^{-1}$∉R ∋ K is a field. Therefore we can define discrete valuation ring ($D.V.R)$ on K by the following:

**Definition (3.2.8).[33].**  We say the function $T$:K→Z is (D.V.R) if:

1. $T$ is onto.
2. $T$ ($ab$)=$ T(a)$+$ T(b)$.
3. $ T(a+b)$≥ min{$ T(a)$,$T(b)$.}.

**Remark (3.2.9). [2].**  An integral domain R which is (V.R) of (D.V) on K is a (D.V.R) and hence every (D.V.R) is an Euclidean domain.

**Proposition (3.2.10).**  Let $M$ be an R-module. If:

1. $M$ is divisible module;
2. R is (D.V.R) ∋ $T$:R\{0}→N is Euclidian.

Then $M$ is an injective module.

**Proof.**

Suppose that $a,b$ ∈ R\{0}. From condition (2) $T(a)$≤$T(b)$. If $T(a)$≥$T(b)$, so $T(\frac{a}{b}$)≥0 and $\frac{a}{b}$∈R. Take the following equation:

 a=$(\frac{a}{b}$)b+0 ∋ $T(a)$≥$T(b)$…………..(⁎)

 a= 0b+a ∋ $T(a)$<$T(b)$…………..(⁎⁎)

If (⁎) and (⁎⁎) satisfies Euclidean norm, so ∃ $h,r$ ∈R a=$hb+r$, then

$r=0$ or $T(r)$<$T(b)$. Hence R is an Euclidean domain. But from condition (1) $M$ is a divisible module. Thus $M$ is injective.

**Corollary (3.2.11). [ 9].** Let R be a ring. If:

1. R is UFD has a unique irreducible element.
2. $M$ is a divisible R-module.

Then $M$ is an injective module.

**§3: Some Rings Give injective module**

 In this section we study injective module over some other rings, such as Noetherian ring, local ring and Hereditary ring.

**Definition (3.3.1). [33].** For a module $M$ over a ring $R$ and for an ideal as a module of $R$; $M$ is called $I$-injective if $δ$: $I\rightarrow M$ is a homomorphism, then there exists an element $m\in M$ and $x\in I$ such that the image of $x$ is $mx$ (in other words $δ$ can be extended to homomorphism of $R$ into $M$).

**Remark (3.3.2).** Every module $M$ is injective if it is an $I$-injective such that$ I$ is a right ideal (in other words any $I$-injective module is injective).

 Note that if $I$ is a finitely generated, so it is clear that every f-injective module is injective over Noetherian ring $R$ ($M$ is f-injective module if it is $I$-injective). Also, any f-injective module $M$ over Noetherian ring $R$ is injective. By the next lemma, we can start with the main goal of this thesis, which is how to get the injective module.

**Lemma (3.3.3).** Let $R$ be a Noetherian ring. If $M$ is a f-injective, then it is injective module.

**Proof.**

 Suppose that $R$ is a Noetherian ring. So every ideal of $R$ is a f-generated. Hence every f-injective module is injective.

**Definition (3.3.4).[40].** Any $R$-module $M$ is called divisible if $M=Ma$ such that $0\ne a\in R$ and $M$ is $I$-divisible if $Ma=$ $ann\_{M}(r(a)); a\in R$. ($I$-divisible gives divisible over commutative ring).

**Theorem (3.3.5).** Let $R$ be a Noetherian ring. If $M$ is$ I$-divisible, so $M$ is an injective module.

**Proof.**

 From [30], Lemma 3.3.3, every $I$-divisible module is an $I$-injective. But every $I$-injective is f-injective. We have $R$ is a Noetherian ring. Hence $I⊆R$ is a finite generated ideal. Thus $M$ is an injective module.

**Corollary (3.3.6).** If $ann(P)=$ $M\_{p}$ Ɐ $p\in R$ and $ann(S∩T)=ann(S)+ann(T)$, then $M$ is injective module.

**Proof.**

Suppose that $δ:I\_{1}$→M ∋ $I\_{1}=k\_{1}R+k\_{2}R+…+k\_{n}R.$

 $I\_{2}=k\_{1}R+..…+k\_{n-1}R$ and $I\_{3}$=$k\_{n}R$. Take $δ\_{1}, δ\_{2}$ is a restriction of $δ $to $I\_{2}$ and $I\_{3}$ respectively. Hence

∃ $m\_{1}$, $m\_{2}$∋$δ\_{4}(y)$=$ m\_{1}y$ Ɐy∈$I\_{2 }$and $δ\_{2}(x)= m\_{2}x$ $Ɐx\in I\_{3 }$. Since $δ\_{1}, δ\_{2}$ coincide with $ δ$ on $I\_{2}∩I\_{3}$, so ($m\_{1}-m\_{2)}$($I\_{2}∩I\_{3})0$=$0 ∃$ $n\_{1}$∈$ann$($I\_{2 })$ and $n\_{2}\in ann$($I\_{3 })$ ∋ $m\_{1}-m\_{2}$=$n\_{1}-n\_{2}$. Assume that

$M$=$m\_{1}-n\_{1}(m\_{2}-n\_{2})$. Hence $My= δ(y)$ $Ɐy\in I\_{1 }$. Thus $M$ is $I$-injective. Therefore $M$ is injective module (Remark 3.3.2).

 We need to introduce more results about the relationship between injective module and this goal achieved by studying in the three rings in depth; Noetherian rings, D.V.R and local rings.

**Remark (3.3.7).** If $ I$ is a finitely generated ideal of $R,$ so it is normal $R$ is a Noetherian and vice versa and more, so when the ring is a P.I.D, we get same result.

**Proposition (3.3.8).** Let$ I$ be a finitely generated ideal of $R$. If $M$ is $I$-divisible, then $M$ is injective $\frac{R}{I}$ –module ($M$ is injective over the ring $\frac{R}{I}$).

**Proof.**

 The main goal of proof is how to prove that $\frac{R}{I}$ is a Noetherian ring. We have $I$ is a finitely generated ideal in $R$. So $R$ is a Noetherian ring. Suppose that $J$ is also an ideal in $\frac{R}{I}$ . We take the following set:

 $T=\{r\in R: r+I\in J\}.$

It is clear that $T$ is an ideal of $R$. Then $T$ is finitely generated and we say:

 $T$=($t\_{1}$,………$, t\_{m}$).

Assume that$ J$ can be write by the following

$$ J=(t\_{1}+I,………, t\_{m}+I).$$

If $t+I$ in $J$, so$ t\in T$ and hence t can be written as:

 $t=r\_{1}t\_{1}$+………$r\_{m}t\_{m }$; Ɐ $r\_{i}\in R$.

 Hence $t+I$ can be write it by $t+I=r\_{1}(t\_{1})$+………$r\_{m}(t\_{m }+I)$. So $\frac{R}{I}$ is a Noetherian ring. Thus $M$ is injective module (See Theorem 3.3.5).

Note that from (Hilbert Basis Theorem); if $R$ is a Noetherian ring (every ideal of $R$ is finitely generated), so also $R[X]$ is a Noetherian. Therefore we can introduce the following result:

**Corollary (3.3.9). [22].** Every $I$-divisible module over $R[X]$ is injective module.

**Definition (3.3.10). [7].** Any ring $R$ is called local if $R$ has a unique maximal ideal. In other words; every ideal is contained in some maximal ideal.

 Now from the definition of Noetherian ring and local ring, we can combine both concepts in order to obtain $M$ is injective module. See the following proposition:

**Proposition (3.3.11).** Let $M$ be an $R$-module. If.

1. $M$ is $I$-divisible.
2. $S$ is an irreducible element in $R$.
3. $R=k\_{1}S^{n}\ni k\_{1}$is a unite; $n\geq 0$.

Then $M$ is injective module.

**Proof.**

 First, the unique maximal ideal $M=\left(P\right)$ is principal. Suppose that $0\ne r$=$k\_{1}S^{n}$ and $M=\left(S\right)$. So every element in $R$-{$I$} is a unique and if $k\_{1}S^{n}$=$k\_{2}S^{m}$, then$ $ $k\_{1}S^{n-m}=k\_{2}$ is also a unit, Hence, therefore $n=m$ and $k\_{1}=k\_{2}$. Thus the uniqueness is true. Hence $R$ is local ring. Let $α\in R$ ∋ $α$ is an ideal. If $α\ne \left(1\right), $so $α$⊆$M \ni M$ is maximal ideal. Take n is a maximal element such that $n\in Z$ and $ α⊆M^{n}.$ We define the following set by:

 $ β=\{α\in R: S^{n}α\in α\}$.

 Note that $β$ is an ideal and $ α= β(S^{n}$). Now if $β=\left(1\right);∃ α\_{1}\in α \ni α\_{1}=S k\_{1}^{n}\ni $S is a unite otherwise $α⊆M^{n+1}$, so $S\in β.$ Hence any

ideal of $R$ is not equal zero and take the from $(S^{n}$), $n\geq 0$. Then all ideals of $R$ is finitely generated. Therefore $R$ is Noetherian ring. But from condition (1); $M$ is $I$-divisible. Thus $M$ is injective $R$-module.

**Definition (3.3.12).** **[22].** A ring $R$ is called D.V.R if:

$K: R-\{0\}\rightarrow N\ni K(r)=n$; $Ɐ r=k\_{1}S^{n}$:

1. $K(r)\geq 0, r\in R (K(0)=\infty ).$
2. $K(r) \geq 1 ⟺ K(r)=0, r\in M; K(r)=0 ⟺r $is a unit.
3. $K(rt)=K(r)+K(t), k,t \in R$.
4. $K(r+t)\geq m,n \{K(r),K(t)\}$.

Now from Definition 3.3.12; if $R$ is Noetherian and local ring, then $R$ is D.V.R. So:

**Corollary (3.3.13).** If $R$ is a D.V.R, then any $I$-divisible module $M$ over $R$ is injective.

**Proof.**

By Definition )3.3.12( and proposition )3.3.11(.

**Example (3.3.14).** If $R[X]$ is a ring and $U^{⸝}$is the all non unit, so any I-divisible module $M$ is injective, because if $I$ non unit ideal of $R$, then $R$ is a local ring and we know that $R[X]$ is a Noetherian if $R$ is a Noetherian ring.

**Example (3.3.15).** If $M$ is $I$-divisible over the ring $R$ and $I$ is a finitely prime generated ideal of $R$ , then $M$ is injective over $\frac{R}{I}$ where $R\_{I}$=$S^{-1}R$ and $S^{-1}R$ is all equivalence classes of fraction $\frac{α}{β}$; $α\in R, β\in S; S$ is the complement of $I$ and $R$ has no zero divisors.

**Example (3.3.16).** $\frac{Z}{(I)}$ is local ring such that it is a maximal finitely generated ideal ($\frac{Z}{(I)} $is the localized at the prime ideal (I) and $\frac{Z}{(I)}\in Z$ and $\frac{Z}{(I)}=\frac{α}{β}ϵQ: $I not divisible $β\}$.

**Example (3.3.17).** If $R=Z$ is a domain and $T=R-\{0\}.$ Then

$Q(R)=T^{-1}R$. So $Q(R)$ is a local ring and any ideal of $Q(R)$ is finitely generated, then $Q(R)$ is a Noetherian ring. Such that $T$= $\{n\in Z; n\ne 0\}$ is a multiplicative set ($T$ is called with $0\ne T$ and $1\in T$).

 Now we study injective module over hereditary (semi hereditary) rings.

**Definition (3.3.18).** **[8].** Any ring $R$ is called hereditary(semi-heredity) ring if every quotient module is injective (f-injective).

 Recall that any f-injective module is injective implies that $R$ is a Noetherian ring. Moreover; any f-injective over Noetherian ring $R$ is injective.

**Theorem (3.3.19).** Let $R$ be a semi-hereditary ring. Then every quotient module of injective module is injective.

**Proof.**

Suppose that $M\_{1}$ is f-injective and $g:M\_{1}\rightarrow M\_{2}$ be a homomorphism. Also, $π:I\rightarrow M\_{2}$ such that $I$ is a f-generated right ideal of $R$.

Such that i is inclusion mapping. Then $I $is finitely generated and projective. Hence $h: I\rightarrow M\_{1}$ ∋ $π$h=$g $. Note that $M\_{1}$ is f-injective. So ∃ $h^{⸝}$: $R$→$M\_{1}$ ∋ $h^{⸝}i$=$h$. Also $h^{⸝⸝}=πh^{⸝}$, so $h^{⸝⸝}i$=$πh^{⸝}i$=$ π$h=$ g$. Hence

f-injective. Then the quotation module of injective is also f-injective and hence is injective.

 Recall that a ring R is regular ring if for each $a\in R$ there exists $x\in R$ such that $a=axa$ [20].

**Lemma (3.3.20).** Every regular ring $R$ is a semi-hereditary self-injective.

**Proof.**

 We know that f-generated right ideal of $R$ is generated by an idempotent because $R$ is a regular ring. Hence $R$ is a semi-hereditary. Suppose that $δ:eR\rightarrow M\ni e$ an idempotent element. Suppose that $δ\left(e\right)=m$ . Then $δ(eey)$=$ δ(e)$. Also, $ey=mey, y\in R.$ Hence $M$ is f-injective. Thus $M$ is injective and $R$ is a self f-injective.

**Theorem (3.3.21).** Let $M$ an $R$-module. If $I$ is a maximal ideal of $R$ and the quotient ring $R\_{I}$ of $R$ is a field, then $M$ is an injective module.

**Proof.**

Let $r\in R$ and $B=\{b: br=0, b\in R$}. Since $R\_{I}$ is a field, then $B$ is not contained in any maximal ideal I containing r. Let $C=(r,B)$, so C is also not contained in any maximal ideal of $R$. We know that$ R=(r,B)$. Also since$ (r)B=0$, so $(r)$ is a direct summand of $R$. Now $(r)=(e) \ni e$ is idempotent of $R$. Also, $B=(1-e).$ If $D$=1-$e+r$. Then $R$ is a regular ring. By Lemma 3.3.3 $R$ is semi-hereditary self injective and hence $M$ is f-injective. Thus $M$ is an injective module.

**Remark. (3.3.22).** **[29].** Any ring $R$ is called $QF$(quasi-Frobenius) if every projective module is injective; or every injective module is discrete.

**Definition (3.3.23). [22].**  Any module $M$ is called free if it has a basis that is generating set consisting a linearly independent elements.

**Proposition (3.3.24).**  Let $R$ be a ring. If:

1. $R$ is a local ring.
2. $R$ is a $QF$ ring.
3. $M$ is a flat module.

 Then $M$ is injective.

**Proof.**

 First, we need to prove that $M$ is a free module. Assume that $M$ is not free. Suppose that $M$ is a flat module. Let $M$ be a minimum number of all elements such that m generating $M$ and t is the minimum of $0\ne b\_{i}\in R$ and the sum of $b\_{i}u\_{i}=(\sum\_{}^{}b\_{i}u\_{i}=0).$ Not that $u\_{i}=u\_{1},…….,u\_{n}$ is a minimum base of $M$. There is $S^{+}$ and we assume$\sum\_{}^{}b\_{i}u\_{i}=0$ and

 $M=(u\_{1},…….,u\_{s}$, $u\_{s+1},…….,u\_{n}$).

 So $u\_{i}=$ $\sum\_{}^{}r\_{i }\acute{u}\_{j}$and $\sum\_{}^{}r\_{i}b\_{i}=0$; $r\_{i}$∈R, $\acute{u}\_{j}\in M$.

 If $\acute{u}\_{j}=\sum\_{}^{}\acute{r}\_{jk}u\_{k} ,$ $\acute{r}\_{jk}\in R, $so $u\_{i}=$ $∑\sum\_{}^{}r\_{ij }r\_{jk }u\_{k}.$ Since $(u\_{1},…….,u\_{n})$ is a minimal, then $\sum\_{}^{} r\_{sj }\acute{r}\_{j}$ is a unite of $R$. If $S=1$, then $ a\_{1}=0 C! .$

For $S>1$ then $b\_{s}=\sum\_{}^{}h\_{i}b\_{i },$ $h\_{i}\in R$ as $\sum\_{}^{}r\_{i}b\_{i}=0$. For $1\leq i\leq S-1$,

so $\sum\_{}^{}b\acute{u}\_{i}=$0 such that $\acute{u}\_{i}=u\_{1}+h\_{i}u\_{s}$. Hence

 $M=$( $\acute{u}\_{1}$,…,$ \acute{u}\_{s-1}, u\_{s},…u\_{n}$). C!

 So $M$ is a free module. Therefore $M$ is a projective. But $R$ is local and $QF$ ring. Then every projective module is injective.

**Proposition (3.3.25).** Let $M$ be any module over a semi prime ring $R$. If $T(R)$ is a lattice and [b) has a complement in $T$($R$), $b\in R$, then $M$ is injective module.

**Proof.**

 Suppose that $R$ is a semi prime ring and [$b$) has a complement in $T$($R$). So by [$19],$ Lemma 4], (Lemma 1 in $[19])$, $\sqrt{(b)}$ =$\sqrt{(e)}=(e)$. Now in $\sqrt{(b)}$ and $e$=$e^{n}$ in $(b).$ Hence $(b)=$($e$). So $R$ is a regular ring. Thus $R$ is a semi hereditary ring and hence $M$ is injective (see Theorem 3.3.21).

**Example (3.3.26).** The integer number $Z$ is a semi prime ring so it is a regular ring ($Z$ is semi prime ring). Hence any module over$ Z$ is injective.

In the next result, we study the relationship between Noetherian ring and semi prime ring.

**Corollary (3.3.27).**  Any semi prime ring $R$ is Noetherian and local ring such that $T$($R$) is a Boolean algebra and then every $R$-module $M$ over $R$ is injective.

**Proof.**

 Because $R$ is a semi prime ring and $T(R)$ is a Boolean algebra. From proposition 3.3.8, $R$ is a regular. If P is a prime ideal in $R$, then

 $R-P$ is a saturated set. $T$($R$) is a Boolean algebra then by (Lemma 4 in $[20]);$ $R-P=[e$)∋$e^{2}=e$. So $P=(1-e)$. But prime ideal gives principal. Then $R$ is Noetherian and local. So every module $M$ over $R$ is injective.

**Definition (3.3.28). [14].** Let $M$ be a module over integral domain R and $m\in M$ is called torsion element if $,∃ 0\ne r\in R$ such that $rm=0$ the set of all torsion elements in $M$ denoted by

T($M)=\{m\in M; rm=0 for som r\in R\}⊆M$. if T($M$)=$M$ then $M$ is called torsion module and if T($M$)=0 then $M$ is called torsion free module.

**Definition (3.3.29). [4]**.Any ring R is called a multiplication ring if all ideals are multiplication. Such that an ideal A is multiplication if every ideal B⊆A, ∃ an ideal C s.t B=AC.

**Proposition (3.3.30).**  Let R be a semi prime ring, if b is $0\ne b$ is a non unit in R and any completely irreducible saturated set is completely prime, then R is a Noetherian and a regular ring.

 Recall that from [19]; any set $φ\ne F⊆R$ is called saturated in R $( T(R) )$ if$ Ɐ a,b \in R, ab\in F⟺a,b\in F$. Also an element F in $T(R)$ is completely irreducible if: F=$⋂F\_{i}\ni F\_{i}\in T\left(R\right).$ So F=$F\_{i}.$ And F is called prime if $F\_{1}⋂F\_{2}⊆F⟹F\_{1}⊆F$ or $F\_{2}⊆F.$ Therefor $F\in T(R)$ is completely prime if $⋂F\_{i}⊆F(\{F\_{1}\}⊆T(R) ⟹F\_{i}⊆F$. Now we return to prove proposition 3.3.30

**Proof.**

Take R is a semi prime ring and $0=b\in R$ is a non unit and every completely irreducible saturated set is completely prime. To prove $T(R$)is dual semi-complemented. Suppose that $F\in T(R)\ni F\ne [1)$. So ∃ b$\in F\ni b$ is a non unit. Then $b\_{k}=0, 0\ne k\in R.$ It is clear that $F v [k)=[0)$ and $[k)\ne 0$ (semi-prime property). Hence $T(R)$ is dual semi-complemented. Now from the condition (every completely irreducible saturated set is completely prime, Theorem 3 and Lemma 5 in[20]; we obtain that $T(R)$ is a Boolean algebra. Also from R is a semi prime and Theorem 2 in [20]; we get R is a Noetherian and regular rings.

**Corollary (3.3.31).**  If any ring R satisfies all conditions of proposition 3.3.30, then every R-module $M$ is injective.

**Proof.**

See proposition )3.3.30(and corollary )3.3.27(.

**Proposition (3.3.32).**  Let R be a ring. If:

1. R is multiplication ring or hereditary ring;
2. R is P.I.D;
3. $M$ is P-injective module.

 Then $M$ is an injective module.

**Proof.**

Assume that $M=T(M)$ and P-injective module over multiplication ring or hereditary ring. Suppose that H≤M. We have $T(M)=M$, so from [27]; any submodule of torsion module is torsion (H is torsion submodule). Also, we have $M$ is P-injective, then H is P-injective submodule in $M$. Now H is torsion and P-injective submodule over multiplication or hereditary ring. Hence H is Q-injective ($M$ is injective). But R is P.I.D. Thus $M$ is injective module.

**Corollary (3.3.33).** If $(T(M)=0)$ and P-injective, then any $M$ over P.I.D is injective.

Now we introduce another ring which has a good relation to injective module namely P.P-ring. We say that R is a P.P-ring if is P.I. of R is projective module. Also, we know that R is a (QF) ring if every projective module is injective.

**Lemma (3.3.34). [33].**  Let R be a ring. If

1. R is a P.P. ring;
2. $M$ is a free module;
3. N≤$M$ is acyclic.

Then $M$ is a projective module.

**Theorem )3.3.35(.**  Let R be a ring. If:

1. R is a P.P-ring;
2. R is a QF;
3. N submodule of free module $M$.

Then $M$ is an injective module.

**Proof.**

From conditions (1) and (3) $M$ is a projective module. But from condition (2) R is a QF. So $M$ is an injective module.

From[32]; the following statements are equivalent for any R-module $M$:

1. $M$ is a divisible module.
2. $M$ is an I-divisible module.
3. $M$ is an f-divisible module.

Therefore; the following result is true:

**Corollary )3.3.36(.**  Let R be a ring. If:

1. R is a P.P-ring;
2. R is a P.I.D;
3. $M$ is I-divisible module. Then $M$ is an injective module.

**Proof.**

Clear. From[33]; we have R is a P.P. ring and $M$ is I-divisible module. So $M$ is f-divisible. But every f-divisible module is divisible. We have R is P.I.D. Then $M $is an injective module.

Conclusions

Conclusions

In this thesis we worked on how to obtain the injective module through some modules as well as some characteristics such as Including divisible module, Noetherian and regular module, Quasi-injective module, uniform module, non-singular module, semi simple module and semi hollow module. There are some of the main proofs and results that we have proven in chapter two:

1. Let R be a P.I.D if $M$ is an injective R-module and $N\leq M,$ then $\frac{M}{N}$ is injective.
2. Let $M\_{1}$ be a divisible module. If $M\_{2}$ is a projective module, then the homomorphism from $M\_{2}$into $M\_{1}$is also a divisible module.
3. Let $M $be a regular R-module. If $M$ is semi hollow and Rad($M$) is a Noetherian module, so $M$ is injective.

Also we got the injective module through some domains, including Dedekind domain and UFD, as well as some rings, including Euclidean ring, semi-hereditary ring and semi prime ring. There are some of the main proofs and results that we have proven in chapter three:

1. Let $R$ be a $Dedekind domain$ and it is a UFD. If every element of $M$ is divisible, so $M$ is an injective module.
2. Let $R$ be an integral domain and has no invers. If $R$ is a $Ν$oetherian local $ring with$ P.M.I, then $R$ is a $Dedekind domain$.
3. Every divisible over Euclidean domain is injective module.
4. Let $R$ be a semi-hereditary ring. Then every quotient module of injective module is injective.

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**REFERENCES**

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الملخص

الهدف الرئيسي من هذا البحث هو دراسة العلاقة بين المقاس الغامر وبعض المفاهيم ذات العلاقة مثل المقاس القابل للقسمة والمقاسات الأرتيرنية والنيوثرية والمنظمة, قدمنا خواص مهمة حول المقاس الغامر وكذلك اعتمدنا على التعاريف الاساسية في الفصل الاول والتي استخدمت في النتائج. لدينا كل مقاس غامر هو قابل للقسمة ولكن العكس يحتاج الى شرط الحلقة تكون ساحة مثاليات رئيسية (P.I.D). النتيجة المهمة الاخرى هي اذا كان المقاس $M$ شبه بسيط على حلقة شبه بسيطة فأن $M$ تكون مقاس غامر. كذلك حصلنا على اذا كان المقاس يحمل صفتين المنتظمة والدوار فيكون غامر. وايضا اذا كان $M $ منتظم وN مقاس جزئي منه ومنته التولد فيكون ال $M$ مقاس غامر.

الجزء المهم الاخر هو دراسة العلاقة بين المقاس الغامر وحلقة ديدكند حيث كل مقاس قابل للقسمة على حلقة (D.V.R) يكون غامر. وكذلك كل مقاس غامر كاذب مع R ديدكيند يحقق ال M مقاس غامر.

الجزء الاخير من الدراسة تمثل بكيفية الحصول على المقاس الغامر اذا كانت الساحات تحمل الصفات التالية:

ساحة اقليدية – ساحة نيوثرية.

ومن هذا الجزء درسنا المقاس I-divisible وعلاقته مع المقاس الغامر.

 **جمهورية العراق**

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رسالة مقدمة

إلى مجلس كلية التربية للعلوم الصرفة - جامعة الانبار

وهي جزء من متطلبات نيل شهادة الماجستير في الرياضيات

من قبل

 **فوزي نوري حماد**

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 **بإشراف**

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