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**Ministry of Higher Education**

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**University of Anbar**

**College of Education for Pure Sciences**

**Department of Mathematics**

**Study of Some Properties on Analytic Function Theory**

A Thesis Submitted to the Council of the College of Education for Pure Sciences,

University of Anbar in Partial Fulfillment of the Requirements for the Degree of Master in Mathematics

**By**

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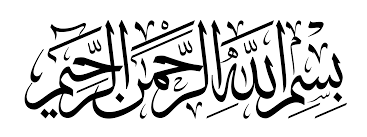
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**Dedication**

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*Nihad 2021*

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**List of Publications**

1. Coefficient Bounds of m-Fold Symmetric Bi-Univalent Functions for Certain Subclasses**, International Journal of Nonlinear Analysis and Applications, (Scopus),12 (2021): 71-82.**
2. Third Order Differential Subordination for Analytic Functions Involving Convolution Operator,[Baghdad Science Journal](https://bsj.uobaghdad.edu.iq/index.php/BSJ/index)**, (2021), (Scopus & Clarivate),** **(Accepted for publication).**
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4. Quasi Subordination of Bi-Univalent Functions Involving Convolution Operator**, AIP Conference Proceedings (2021), (Scopus),** **(Accepted for publication).**
5. Coefficient Bounds for Certain Subclasses for Meromorphic Functions Involving Quasi Subordination**, AIP Conference Proceedings (2021), (Scopus),** **(Accepted for publication).**
6. Study of Some Applications on a class of Analytic Functions Involving Convolution operator, **IEEE, Conference Proceedings (2021), (Scopus),** **(Accepted for publication).**

|  |  |
| --- | --- |
| **Symbol**  **List of Symbols** | **Description** |
|  | Complex plane. |
|  | \{0}. |
|  | Set of all real numbers. |
|  | The Integer numbers. |
|  | Set of natural numbers. |
|  | . |
|  | Open unit disk . |
|  | Punctured pen unit disk . |
|  | Boundary of unit disk }. |
|  | . |
|  | Class of normalized analytic univalent functions in the open unit disk. |
|  | The class of all univalent functions in |
|  | Class of meromorphic univalent functions in . |
|  | Class of bi-univalent functions in |
| **Symbol** | **Meaning** |
|  | Real partof a complex number. |
|  | Class of normalized starlike functions in . |
|  | Class of normalized starlike functions f order . |
|  | Class of normalized convex functions in . |
|  | Class of normalized convex functions of order . |
|  | Class of close-to-convex univalent functions in . |
|  | Hadamard product (or convolution) of functions and . |
|  | The class of analytic functions in open unit disk U of the form: |
|  | is subordinate to |
|  | Majorization |
|  | Quasi subordination |
|  | Class of admissible function |
|  | Class of admissible function |
|  |  |

**Abstract**

Abstract

The purpose of this thesis is to study some of the properties on analytic function theory. Some results third-order differential subordination and superordination results for analytic functions in the open unit disk were obtained. Also, it aims to make use of convolution operator in the class and shed light on some geometric properties, such as coefficient inequality, distortion, growth theorems closure theorems , integral operators, radii of close to convexity, convexity, and starlikeness for functions in class . Here, two new subclasses of a class m-fold symmetric bi-univalent functions in have been studied. Coefficient bounds for the Taylor-Maclaurin coefficients are obtained. Further , functional problems for functions in and are solved. The study also sheds light on the investigation of two new subclasses and of meromorphic functions which are defined by terms of a quasi-subordination are introduced. The coefficient estimates inclusive the classical functional for functions belonging to this class are then derived. Investigating a majorazation problem for the classes meromorphic functions associated with these classes are also pointed out. The discussion was shown for quasi subordination of bi-univalent functions involving convolution operators.

Finally, we introduced and investigated two new subclasses of class bi univalent functions in, which includes the convolution between Hurwitz-Lerch Zeta and the generalized derived operator and satisfies quasi subordination conditions. The coefficient estimates and were determined in these subclasses. Also, certain new subclass of univalent functions with sigmoid function is introduced and applications with problems in are obtained.

Introduction

Introduction

The theory of geometric function is an old mathematics branch, and specifically, many mathematicians and researchers are interested in complex analysis because of its geometrical dimensions and numerous study opportunities. A study of the theory of analytic univalent and multivalent functions is an old branch of mathematics, particularly complex analysis attracting a large number of researchers owing to sheer beauty of its geometrical aspects and a lot of avenues for research work. One of the most important branches of complex analysis of single variables is the study of univalent functions, several variables too. The classical study of this subject has been engaging the attention of researchers at least until as early 1907. This field captioned as geometric function theory is found to be a hybrid or an interplay of geometry and analysis. An analytic univalent and multivalent functions are a holomorphic or meromorphic function. The theory of univalent and multivalent functions is one of the important areas of study in this field that links geometry and analysis.

Indicate by the class of normalized functions satisfying the condition and given by next Taylor expansion:

which are analytic in the open unit disk

Geometrically, the normalization amounts to only a interpretation of the image domain and corresponds to rotation and stretching or shrinking of the image domain. The basics of this theory were put in the 19th century, one with paper of Koebe [57] in 1907, Geomwell' s Proof of the area theorem in 1914. Bieberbach [29] studied the second coefficients of a function in 1916. He demonstrated , with equality if and only if is rotation of the koebe function and he also stated is generally valid . This conjecture was settled by Louis de Branges in (1984). During the period (1916- 1984) many researchers were engaged in settling this conjecture. This conjecture posed a challenge to many mathematicians and inspired several scholars to create a variety of new approaches in complex analysis. In 1923 Lowner [ 65] proved the Bieberbach conjectured for , many investigations have of n. Eventually, in (1985) Branges [27]was established this conjecture. Also, when talking about geometric function theory, we must talk about the study of operators that play a major role in mathematics in generally and especially in geometric function theory. Dziok-Srivastava and Libera (1969) introduced an integral operator and examined specific properties of starlike functions under this operator Salegean (1983) studied the class of analytic functions defined by differential , fractional differential and linear operators these are important in geometric function theory. Subordination between two analytic functions initiated by Littlewood[62 ,63 ] and Lindelöf [ 61], where Rogosinski [ 87 ,88 ] introduced the term and established the basic results involving subordination. Ma and Minda [ 66 ] showed that many of these properties can be obtained by unified method research scholars and mathematicians, internationally recognized, like Ruscheweyh, Srivastava, Miller, Mocanu, Duren, Silverman, Owa, Jahangiriet al., have paved the way for new directions in the field of complex analysis, especially geometric function theory. In fact univalent function theory has played a central role in the development of complex function theory. This thesis is divided into four chapters as follows;

In Chapter onelist of all relevant definitions of analytic, univalent, meromorphic, bi-univalent functions, some examples we need during has been given.

Chapter two consists of two sections, section one deals with third-order differential subordination results for univalent functions associated with convolution operators. Here, we obtain new results for third-order differential subordination and superordination in the open unit disk. Section two introduces a subclass of analytic functions defined by the convolution operator. We obtain some geometric characterizations like coefficient estimates, distortion and growth theorems, closure theorems and integral operators, radii of close to convexity, convexity, and starlikeness for functions in this class.

Chapter three falls into two sections, section one is concerned with coefficient bounds of m-fold symmetric bi-univalent functions for certain subclasses. Here, we introduced two new subclasses and of a class m-fold symmetric bi-univalent functions in open unit disk. Coefficient bounds for the Taylor-Maclaurin coefficients are obtain . Also, we solve functional problems for functions in and . Section two deals with the coefficient estimates for certain subclasses for meromorphic functions involving quasi subordination . Here, we introduce two new subclasses of meromorphic functions which are defined in terms of a quasi-subordination. The coefficient estimates inclusive the classical functional for functions belonging to this class are then derived. Investigate a majorazation problem for the classes meromorphic functions associated with these classes is also pointed out.

Chapter four includes two sections. The first section discusses the coefficients assessment for certain subclasses of bi-univalent functions related to quasi-subordination. Here, we submit and investigate specific new subclasses of the function class Σ of bi-univalent functions defined in the open unit disk, which is connected with the quasi subordination. We're finding estimates on the Taylor - Maclaurin coefficiency and for functions in these subclasses. Already pointed out are some documented and new implications of those findings. Section two deals with we introduced the applications of quasi subordination associated with generalized Sakaguchi type functions. Here, we introduced and investigated a new class of analytic functions which is defined by terms of a quasi-subordination. The coefficient estimates inclusive the classical inequality of functions belonging to this class are then derived. Also several certain special improver results for the associated classes involving subordination are presented .

**Chapter One**

**Basic Definitions and Fundamental Results**

**Chapter One**

**Basic Definitions and Fundamental Results**

**Introduction**

The essential subject of this chapter is the fundamentals of geometric function theory, which is obtained by combining geometric and analytical methods. This chapter includes two sections with some examples. In the first section we have mentioned all the required definitions, some examples, of analytic functions facts about geometric function theory, it is obtained from mixing geometric and analysis, univalent, meromorphic and bi-univalent functions and also subordination, superordination, quasi-subordination which are needed in subsequent chapters for research, with some examples can be found in standard textbooks see Goodman [37], Duren [29], and Miller and Mocanu [47]. In section two, basic lemmas and theorems have been mentioned; they are essential for the Proofs of our main results.

* 1. **Basic Definitions**

**Definition 1.1.1 [29]:** Suppose that indicates the open unit disk in complex plane . A function of the complex variable is analytic at a point if its derivative exists not only at but each point z in some neighborhoods of . It is analytic in domain if it is analytic at every point in .

**Definition 1.1.2 [93]:** Let be the class of analytic functions in open unit disk U. For and define the subclass by

with 1=[1,1] and .

**Definition 1.1.3 [29]:**An analytic function in a domain is said to be univalent if it does not take the same value twice, that is for all pairs of distinct points and in . In other words is one to one mapping of onto another domain *.* The theory of univalent functions is so deep, so, we need certain simplifying assumption. The most obvious one in our study is to replace the arbitrary domain by one that is convenient, and is the open unit disk The class of all univalent functions is denoted by

**Definition 1.1.4 [29]:** A function is said to be locally univalent at a point if it is univalent in some neighborhood of . For analytic functions , the condition is equivalent to local univalence at . A function univalent in a domain is locally univalent at each of the points in ; but the converse is not true in general.

**Example 1.1.5 [47]:**Consider the function in the domain Since for , it follows that is locally univalent in But so this function is not univalent in the whole domain However, is univalent on (where, denote the real part of )

**Definition 1.1.6 [69]:**Let denote the class of all analytic functions in U to satisfy the condition and write in the form

which are univalent in open unit disk

**Definition 1.1.7 [29]:** A function is said to be normalized if satisfies the conditions

**Definition 1.1.8 [69]:** Let denotes the class of functions of the following form

which are meromorpich univalent in the puncture unit disk

.

**Definition 1.1.9 [29]:** A function is said to be conformal at a point if it preserves the angle between oriented curves passing through in magnitude as well as in sense. Geometrically, images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at both in magnitude and direction. A function is said to be conformal in the domain if it is conformal at each point of the domain. Any analytic univalent function is a conformal mapping because of its angle – preserving property.

**Definition 1.1.10 [29]:** A bilinear transformation or , is a rational function of the form

,

where are fixed and

**Example 1.1.11 [29]:**The leading example of a function of class is the Koebe function defined by:

which is an extremal function for many subclasses of the class of univalent functions. It maps the disk , one to one and conformally onto the entire complex plane minus the part of the negative real axis. This is best seen by writing

We now prove that the image of the unit disk under is a slit domain, which is a domain consisting of the whole complex plane except for a slit cut out of it, for this purpose for all. Consider the following sequence of functions to evaluate .

, , .

Noting that maps the unit disk conformally onto the right half-plane ; see Fig.(1.1.1)



Figure 1.1.1. Image of the unit disk under the Koebe function.

**Definition 1.1.12 [29]:** A set in the complex plane is called starlike with respect to interior point of if the line segment joining any other interior point of to lies in the interior of . In a more picturesque language, the requirement is that every point of is visible from, i.e.

Furthermore, a function which maps the open unit disk onto a starlike domain is called a starlike function, the set of all starlike functions is denoted by which is analytically expressed as

It is well known that if any analytic function satisfies above equation and , then is univalent and starlike in .

**Example** **1.1.13 [29]:** The function defined on the unit disk is a starlike function. In Figure (1.1.2) we can see the image of this function, which is a starlike domain.

****

Figure 1.1.2

**Definition 1.1.14 [37]:** A set in the complex plane is called convex set if it is starlike with respect to each of its points, that is, the line segment joining any two interior points of lies in the interior of , i.e.

Let and let be the function in the open unit disk Then maps onto a convex domain, if and only if

Such function is said to be convex in The set of all convex functions is denoted by . Thus .

**Example 1.1.15 [37]:** The function is a convex function. Below is given the image of this function,



Figure 1.1.3

**Definition 1.1.16 [72]:** The class of starlike and convex functions of order respectively which are defined by

In particular

It is clear that from the above function is convex if and only if is starlike.

**Definition 1.1.17 [29]:** A function is said to be close to convex if there is a convex function such that

An equivalent formulation would involve existence of a starlike function such that

we denote by the class of all close-to-convex functions in

**Definition 1.1.18 [29]:** A function is said to be close-to-convex of order if there is a convex function such that

An equivalent formulation would involve existence of a starlike function such that

we denote by the class of all close-to-convex functions of order . For we have the class of all close-to-convex function in . We note that . Every convex function is obviously close-to-convex. More generally, every starlike function is close-to-convex. Indeed, each has the form for some , and

Then from the function above, we conclude that

and this means that, every close-to-convex function is univalent.

**Definition 1.1.19 [47]:** Let we denote by , and the subclasses of that are meromorphic univalent, meromorphically convex functionsorder of and meromorphically starlike functions of order,respectively. Analytically,

afunction if and only if

similarly, a function if and only if

**Definition 1.1.20 [29]:** Radius of starlikeness of a function is the largest for which it is starlike in

**Definition 1.1.21 [29]:** Radius of convexity of a function is the largest for which it is convex in

**Definition 1.1.22 [29]:** If the functions belonging to the class , given by

and

then the Hadamard product (or the convolution) of and denoted by is defined by

**Definition 1.1.23 [69]:** Let be a topological vector space over the field and let be a subset of . A point is called an extreme point of if it has no representation of the form as proper convex combination of two distinct points and in

**Definition 1.1.24 [29]:** A function is called Schwarz function, if for each , then , where capital O is defined as follows:

Let and be any two sequence and , for each . If there is a constant number , such that , then we write

**Definition 1.1.25 [69]:** For two analytic functions and in U. A function is subordinate to (or superordinate to , written as :

if there exists a Schwarz function which is analytic in U satisfies the following conditions :

such that = ( ( ).

Indeed it is known that,

In a special case, if is a univalent function in open unit disk then the reverse implication also holds true

.

**Definition 1.1.26 [14]:** Let 4 and be univalent in . If be an analytic function in open unit disk that satisfies the next third order differential subordination:

From(1.16), is namely a solution of the differential subordination. Moreover, q(z) is a univalent function which is namely a dominant of the solution of the differential subordination (1.16), or more simply, a dominant if q(z) for all satisfying (1.16). A dominant satisfies q(z) for all dominates q(z) of (1.16) is called the best dominant. Note that the best dominant is unique up to a rotation of U.

**Definition 1.1.27 [14]:** Let 4 and be analytic in U. If the function (z) is analytic in that satisfies the next third order differential subordination :

Then , is namely a solution of the differential superordnation given by (1.17). Moreover, an analytic function q(z) is namely a subordinate of the solutions of the differential superordination given by (1.17), or more simply a subordinate if q(z) subordination for all it should be satisfy (1.17). A univalent subordinate that satisfies for subordination q(z) of (1.17) is called the best subordinate of the differential superordination given by (1.17). Note that the best subordinate is unique up to a rotation of .

**Definition 1.1.28 [10]:** For , is defined by

where and

.

It would be easily seen that and

For , it is easy from (1.18), that

(1.19)

and

(1.20)

**Definition 1.1.29 [95]:** For the Srivastava – Attiya operator is defined by

, (1.21)

where .

From (1.21) it is easy that

(1.22)

**Definition 1.1.30 [84]:** Is symbolized as **Q,** a collection of all functions q that is univalent and analytic on closed unit disk except and denote where is closed unit disk

and

E(q) **=** (1.23)

Such that for **.** **Q(**b) denote the subclass of **Q** for which with **Q**(0) **= Q0** and **Q**(1)  = **Q**1.

**Definition 1.1.31 [7]:** Let be a set in complex plane Also let and , be the set of positive integers . The class of admissible function consists of those functions that satisfy the next admissibility conditions :

,

whenever, v = q() , u =  ,  and ,

where and .

**Definition 1.1.32 [26]:** Let  be a set in complex plane , let be in the subclass and . The class of admissible function consists of function that satisfies the next admissibility conditions:

,

whenever, ,   and ,

where and

**Definition 1.1.33 [86]:** If and be two analytic functions we can say that is quasi-subordination to in and can written as the form

if there exist and analytic functions with , such that

Note that , when , then , so that

in .

Furthermore, if , then and in this case is majorized by ,written

in .

In this case,

Therefore, quasi-subordination is a generalization of subordination and also of majorization .

**Definition 1.1.34 [94]:** An analytic function is said to be bi-univalent in a domain , if and are both univalent in . The class of all bi-univalent analytic functions in is denoted by .

**1.2 Some Basic Results**

The following lemmas and theorems will be used to prove our results in the next chapters .

**Lemma 1.2.1[7]:** Let with and satisfies the next condition:  and  where and . If is a set in complex plane , and

, (1.24)

then .

**Lemma 1.2.2[7]** : Let be in this subclass with . If , be univalent function in open unit disk U and satisfying the next condition

 and  ,

where and ,

then, (1.25)

it means that, , (.

**Lemma1.2.3[56]** Let be the Schwarz function given by

then

(1.27)

where .

**Lemma 1.2.4[33]** Symbolized as to a sigmoid function and

then is a modified sigmoid function .

**Lemma 1.2.5** **[33]** Let

and .Then .

**Lemma 1.2.6** **[66]** If a function and be a complex number with positive real part, then

.

**Lemma 1.2.7** **[82]** If ,then | for all *i* , where *P* is the family of all function analytic in *U* , for which *,* where (*z* ∈*U*) *.*

**Lemma 1.2.8[29]: (Schwarz Lemma)**

Let be analytic in the unit disk with, and in .Then

in Strict inequality holds in both estimates unless is a rotation of the disk

**Theorem 1.2.9 [29]: (Alexander's Theorem)**

Let be an aalytc function in with . Then if and only if.

**Theorem1.2.10 [29] :( Distortion Theorem)**

For each

For each equality occurs if and only if is a suitable rotation of the Koebe function. We call the upper and lower bounds for as distortion bounds.

**Theorem 1.2.11 [29] :( Growth Theorem)**

For each

For each equality occurs if and only if is a suitable rotation of the Koebe function.

**Theorem 1.2.12 [29] :( Bieberbach Conjecture)**

The coefficients of each satisfy for *n=2,3*,….. The strict inequality holds for all n unless is the Koebe function or one of its rotation.

The next theorem is about coefficient of functions in and due to Littlewood [62-63].

**Theorem 1.2.13[29] :( Littlewood's Theorem)**

For the constant , the coefficient of each function satisfy for

**Theorem 1.2.14 [29]**: Assume that is analytic and is not constant in a domain of the complex -plan. For any point for which ,this mapping is conformal, that is, it preserves the angle between two differentiable arcs.

**Theorem 1.2.15 [29]: (Maximum Modulus Theorem)**

Suppose that a function is continuous on a boundary of ( any disk or region). Then the maximum value of which always reached, occurs somewhere on the boundary of and never in the interior.

**Theorem 1.2.16 [62]** **:** If and are analytic in with , then for and

**Chapter Two**

**Some Results on Differential Subordination of Analytic Functions Involving Convolution Operator**

**Chapter Two**

**Some Results on Differential Subordination of Analytic Functions Involving Convolution Operator**

**Introduction**

In the functions of a real variable, the concept of differential subordination is extremely significant. This concept, therefore, allows one to investigate the variety of initial functions. There are some differential applications of complex-valued function theory in which the characterization of a function is defined by a differential condition. Miller and Mocanu [69] have conducted a number of papers on differential subordination. The study of differential subordination stems out from textbooks by Duren [29] Goodman [37] and Pommerenke [82]. This chapter consists of two sections. Section one is interested in the effects of third-order differential subordination and superordination for analytic functions involving convolution operator, like . Let . If the function and satisfying the following condition :

 and 

and ,

then .

Section two is dedicated to the study of some applications on a class of analytic functions involving convolution operator also we give some geometric characterizations like coefficient estimates, distortion and growth theorems, closure theorems and integral operators, radii of close to convexity, convexity and starlikeness for functions in the class .

**2.1** **Third Order Differential Subordination for Analytic Functions Involving Convolution Operator**

Let be the open unit disk

,

and let be the class of analytic functions in U.

For and define the subclass by

with 1=[1,1] and .

Let denote the subclass of analytic functions in U to satisfy the condition and write in the form

**Definition 2.1.1 [10].** For

(2.2)

where and

.

It would be easily seen that and

For , it is easy from (2.2), that

(2.3)

and

(2.4)

**Definition 2.1.2 [95].** For the Srivastava – Attiya operator is defined by

, (2.5)

where .

From (2.5) it is easy that

. (2.6)

**Definition 2.1.3.**For , the operator is defined by convolution of the Srivastava – Attiya operator and the generalized

and

(2.7)

From (2.7), the following identity relations can be obtained

, (2.8)

also

(2.9)

and

. (2.10)

Note that, the following are special cases of operator

1. When includes the Srivastava-Attiya operator (see [95]).
2. When includes the (see [25]).
3. When which is introduced by Ruscheweyh derivative operator (see [8*9*]).
4. When , reduces to which is introduced by Salagean derivative operator(see [92]).
5. When s = 0 , reduces to which is introduced by generalized Salagean derivative operator (or Al-Oboudi derivative ) (see [90]).
6. When m = 0 , reduces to which is introduced by generalized Ruscheweyh derivative operator (or Al-Shaqsi – Darus derivative operator ) (see [91]).
7. When , reduces to which is introduced by Srivastava- Attiya derivative operator(see [95]).
8. When k=1 , c = 0 , reduces to which is introduced by Alexander integral operator (see [3] ).
9. When k = 1 , , reduces to which is introduced by Bernardi integral operator (see [18]).
10. When , reduces to which is introduced by Jung-Kim-Srivastava integral operator (see [35]).

Ponnusamy is the first authors to investigated third order, published in 1992.The process of admissible functions (also known as the differential subordinations method) was first introduced by Miller and Mocanu in 1978 (see [71]), and the theory started to improve in 1981 (see [69]). for details see( [6-8], [14],[36], [48-49],[51], [77], [84], [95]).

In this section, some results of differential subordination are obtained. Also studding the class of admissible functions involving the and Srivastava – Attiya operator defined by (2.8)

**Definition** **2.1.4**. Let be a set in complex plane and . The class of admissible function consists of those functions 4  that satisfy next admissibility conditions:

,

whenever

, ,  and

,

where and .

**Theorem 2.1.1**. Let be in this class . If the function and **Q0** satisfying the following condition :

 and  (2.11)

and , . (2.12)

then . (2.13)

Proof: From relation between (2.7) and (2.8), we have

. (2.14)

Let g(z) be analytic in open unit disk defined by

g(z) = . (2.15)

Then

. (2.16)

Based on that

(2.17)

and

= . (2.18)

Now, the transformation from 4 to defined by

a(v, u, t, r) = v , , (2.19)

and

. (2.20)

Let

. (2.21)

The Proof shall make use of Lemma 1.2.1.By using (2.13) to (2.18), and from equation (2.21), we get :

=

Therefore, (2.12) gives

.

Such that 

and .

Now, since the admissibility condition for given in Definition 1.1.31 with n = 2.

Thus, using (2.11) and Lemma 1.2.1, implies that .

The next consequence is expansion of Theorem 2.1.1 for the case where the attitude of q(z) on is unknown.

**Corollary 2.1.1** . Let be an univalent function in open unit disk U with and let . Let for some , where . If the function belongs to and satisfies :

 and , (2.23)

where and and

, (2.24)

then .

Proof:

Since is an univalent function in U therefore, and .

The class is an admissible class and from Theorem 2.1.1 yields

The consequences certained by Corollary 2.1.1 is just a summary from the next subordination quality

.

Since , get .

The Proof of Corollary 2.1.1 is now complete.

In case this means it is a simply connected domain, then for some conformal mapping in open unit disk U onto . In this case, the class , it can be formed as . As a result, Theorem 2.1.1 has the following immediate effect.

**Theorem 2.1.2 .** Let . If the function and satisfies the following condition :

 ,  , (2.25)

and , (2.26)

then

**Corollary 2.1.2 .**  Let be an univalent function in open unit disk U with . Also let be a subset of the complex plane and for some , where . If the function and satisfies :

 and ,

where and and

,

then . .

Proof :

Case 1. To prove this corollary by using Theorem 2.1.1, have

and since Implies that .

Case 2. Let . Then

By applying Theorem 2.1.1, deduce

where is any mapping from U into itself, for . By , get .

Hence .

**Theorem 2.1.3 .** Let μ(z) be a univalent function in open unit disk. Also let 4 and be given by (2.21). Suppose that

(2.27)

differential equation has a solution which satisfies (2.11). If belongs to satisfies condition (2.26) and if

is analytic in open unit disk U, then

, q(z) is therefore, the best dominant.

Proof: From Theorem 2.1.1, it views q as a dominant of (2.26) because q satisfies (2.27), and also a solution of (2.26). Thus, q shall be dominated by all dominants. Hence q is the best dominate. This completes the Proof Theorem 2.1.3.

From Definition 2.1.4, see that the special case when q(z) = Mz (M > 0), the class , is stated as follows.

**Definition 2.1.5 .** Let be aset in complex plane and M > 0 . The class of admissible function consists of the function 4 such that

, (2.28)

where

**Corollary 2.1.3 .** Let be in this class. If the function , then it satisfies the following conditions:

and

,

then

The special case of the above Corollary 2.1.3 when ,the class is simply symbolized by its . Corollary 2.1.4 can be rewritten as follows.

**Corollary 2.1.4 .** Let be in this class. If is a function it would satisfy following conditions:

and

,

then **.**

**Corollary 2.1.5 .** A non-zeros belongs to the complex plane , let and M > 0. If it would satisfy the conditions :

and ,

then .

Proof: Let be equal and

where , ( M > 0) .

Use Corollary 2.1.3, that can be shown as . This means that, the admissibility condition (2.28), is satisfied. This follows easily, because

= ,

where . The required results now follows from Corollary 2.1.3.

**Definition** **2.1.6** . Let and be a set in complex plane . The class of admissible functions consists of those functions 4 that satisfies the next admissibility conditions:

,

whenever

, ,  and

,

where and

**Theorem 2.1.4** . Let If a function belongs to and q belongs to **Q1** satisfying the following condition :

 and  (2.29)

and

(2.30)

then .

Proof: From the relation between (2.7) and (2.8), we have

. (2.31)

Let g(z) be analytic in open unit disk defined by

g(z) = . (2.32)

Then

. (2.33)

Based on that

(2.34)

.  > (2.35)

Now, the transformation from4 to defined by

a( v, u, t, r) = v , , (2.36)

and (2.37)

Let . (2.38)

The Proof will make use of Lemma 1.2.1 .Using (2.32) to (2.35) , from (2.38), we get

=

Therefore,(2.30) gets

(2.40)

such that



and .

And since the admissibility condition for , given in Definition 2.1.6 with n = 2 .

Thus, using (2.29) and Lemma 1.2.1, it implies that

.

In case is a simply connected domain, then for some conformal mapping in open unit disk onto . In this case, the class is rewritten as . We can show below the result of Theorem 2.1.4.

**Theorem 2.1.5.** Let be in this class. If the function and q belongs to **Q1** satisfying the following condition :

 and  (2.41)

and

, (2.42)

then .

In a particular case where q(z) = Mz , where ( M > 0 ), and in saw of Definition 2.1.6, the class of admissibility functions and symbol by is :

**Definition 2.1.7.** Let be a set in complex plane and M > 0. The class of admissible function consists of the function 4  such that

(2.43)

where

**Corollary 2.1.6.** If and let satisfies the following conditions:

and

, then

The particular case of the above Corollary 2.1.6 when , the class is simply denoted by . Corollary 2.1.6 can be rewritten as shape.

**Corollary 2.1.7:** Let be in this class . If the function belongs to satisfying the following conditions:

and

then

In this part, note that, making use of the recurrence relation (2.9), certain differential subordination consequences associated with the operator are obtained. Because the Proofs of the consequences contained in this part are similar to those from the prior part, they will be omitted.

**Definition** **2.1.8**. Let and  be a set in complex plane . The class of admissible function consists the function 4 that satisfies the next conditions:

,

whenever , ,



and



where and .

**Theorem 2.1.6.** If the function . Let and q that belongs to **Q0** satisfies the next condition :

 and  (2.44)

and

, . (2.45)

then .

In a particular case . This means it is a simply connected domain, then for some conformal mapping in open unit disk U onto . In this status, the class is written as . The next consequence is immediate of Theorem2.1.6.

**Theorem 2.1.7 .** Let . If is a function and satisfy next condition :

 and  (2.46)

and

(2.47)

then .

In a particular case when q(z) = Mz (M > 0), the class , is stated as follows.

**Definition 2.1.9 .** Let M > 0 and be a set in the complex plan . The class of admissible function consists of those function 4 such that

(2.48)

where

**Corollary 2.1.8.** Let be in this class. If the function belongs to that satisfies the following conditions:

and

, then

The particular case of the above Corollary 2.1.8 when

,

the class simply symbolizes it. Corollary 2.1.9 can be expressed in the following way.

**Corollary 2.1.9.** Let . If , satisfies the following conditions

and

then

**.**

**Corollary 2.1.10.**  A non-zeros that belong to the complex plane , let and M > 0. If satisfies conditions :

and

,

then .

**Definition 2.1.10.** Let and  be a set in complex plane . The class of admissible function consists of those functions 4 that satisfies the next conditions:

,

whenever

, , 

and



where and .

**Theorem 2.1.8**. Let be in this class . If and q belongs to **Q1** satisfying the following condition :

 and  (2.49)

and

,. (2.50)

then .

In this part, the third order differential superordination for the operator defined in (2.7) is obtained and proving many theorems, for the purpose , consider the next class of admissible functions.

**Definition 2.1.11**. Let with not equal zero and  be a set in complex plane . The class  of admissible function consists of those function 4 that satisfies the next admissibility conditions:

,

whenever

, , 

and



where and .

**Theorem 2.1.10**. Let . If belongs to ,with **Q0 ,** and satisfying the following condition :

and  (2.51)

and is an univalent in open unit disk U, then

. (2.52)

It means that .

Proof. Let g(z) be a function defined by (2.15), and defined by (2.21). Since , from (2.22) and (2.52) yield,

.

From (2.19) and (2.20) deduce that the admissibility condition for given in Definition 1.1.32 with n = 2. Subsequently  and, by using (2.52) and Lemma 1.1.2, get

this equivalently,

In case this means it is a simply connected domain, then for some conformal mapping of open unit disk U on to . In this case, the class is formed as . The next is instant consequence of Theorem 2.1.10.

**Theorem 2.1.11.** Let and be analytic in unit disk U. If the function belongs to and , and if

satisfying the condition(2.51) and the function

is univalent in unit disk U ,then

. (2.53)

It means that .

The following theorem establishes the existence of the strongest subordination of (2.53) for sufficient. .

**Theorem 2.1.12** . Let and be univalent function in unit disk U, let be given by (2.23). Suppose that :

(2.54)

differential equation has a solution q(w) belongs to **Q0** and if with not equal zero satisfying the condition (2.51) and

is univalent in unit disk U, then

.

It means that

and q(z) is the best dominant.

Proof: To Proof this theorem using Theorem 2.1.10 and Theorem 2.1.11, draw q is a subordination of (2.53). Since that q satisfies (2.54), it is also a solution of (2.53) and, subsequently, q will be subordinate by all subordinates. Hence q is the best subordination.

**Definition** **2.1.12.** Let be a set in complex plane and with . The class . of admissible function consists of those functions 4 that satisfies the next admissibility conditions:

,

whenever

, ,  and



where and .

**Theorem 2.1.13**:Let . If with **Q1** and q belongs to with not equal to zero satisfying the condition:

 and  (2.55)

and the function is univalent in open unit disk U, then

(2.56)

It means that .

Proof: Let g(z) be a function defined by (2.32) and defined by (2.38). Since , from (2.39) and (2.56) yield,

.

From (2.36) and (2.37), deduce that the admissibility condition for belongs to given in Definition 1.1.32 with . Subsequently and, by using (2.55) and Lemma 1.1.2, it implies that

.

This is the complete Proof of Theorem2.1.13

In case this means that it is a simply connected domain, then for some conformal mapping μ(z) of open unit disk U onto . In this case, the class  is formed as  . The next is an instant consequence of Theorem 2.1.13.

**Theorem 2.1.14**. Let , and let be analytic function in unit disk U. If belong to and q belong to with  not equal to zero satisfying the condition:

 and  (2.57)

and the function is univalent in unit disk U, then

. (2.58)

It means that .

This completes the Proof of this theorem.

Collect Theorem 2.1.2 and Theorem 2.1.11. The next sandwich-type theorem is obtained.

**Theorem 2.1.15:** Let and be analytic functions in U, is an univalent function in open unit disk U, with and . If the function with and the function

is univalent in open unit disk , and if the conditions (2.14) and (2.51) are satisfied

, it means that

. (2.59)

**2.2 Study of Some Applications on a class of Analytic Functions Involving Convolution operator**

Indicate by the class of normalized functions satisfying the condition and given by next Taylor expansion:

which are analytic in the open unit disk where is complex number .

The general Hurwitz defined by

(2.61)

where .

The Hurwitz-Lerch Zeta function of Srivastava and Choi [23] has found some very interesting features. In [93], Srivastava and Attiya introduce the following operator:

where

where for convenience

(2.64)

By using (2.60), (2.63) and (2.64), we have

.

In [58] the authors introduced the Komatu integral operator

where .

The convolution operator between Srivastava-Attiya operator and Komatu integral operator

where

**Definition 2.2.1 .** Let  be the class function as the form

and satisfying the following

where ).

Further for real parameter we defined the subclasses of the class

and

Several authors worked above will class in different ways (see[52],[74],[92]).

**Theorem 2.2.1.** A function defined by (2.70) is in the class if and only if

(2.74)

where .

Proof.

Let . Then we obtain

when by Schwarz analytic function and for every , then

Thus,

Putting

we get

Let . Then the necessary result .

On the other hand, merely demonstrating that

.

Selecting

we get

then, .

The Proof of this theorem is complete .

**Corollary 2.2.1.** A function given by (2.70) is in the class if and only if

In class , we derived  distortion and growth theorems. .

**Theorem 2.2.2.** Let be a function given by (2.70), in the class . Then for

we have

and

Proof .

Since , from Theorem 2.2.1, we can write

From theorem 2.2.1, is non-descreasing positive sequence

By (2.70), getting

and

Now, we obtain

and

Take the function

The Proof of this theorem is complete .

**Theorem2.2.3.** Let be a function given by (2.70) is in class .Then

.

provided the sequence

is non-decreasing and positive and

provided the sequence

is non-decreasing and positive. The result is sharp.

**Proof.**

Let . Then by assumption, we have

and

we get

.

Also,

then, if , while if we have

is negative and decreasing, therefore we obtain

we get the same result by using the same methodology.

Take and

The Proof of this theorem is complete .

The closure theorems for the classis given by the following

Take defined by

, , . (2.78)

**Theorem 2.2.4.** Let be a function defined by (2.78) in the class . Then the function defined by

is a member of the class, where

**Proof.**

Sinceits follows from Theorem 2.2.1, that

for all .

Thus

This implies that

The Proof of this theorem is complete .

**Theorem 2.2.5.** The class is closed under convex combination.

**Proof.**

Assume that the function given by (2.70)in class , its suffices to prove that the function

is also in the class.

Since, for

we observe that

.

Hence .

The Proof of this theorem is complete .

In this prat, we provided the integral operator property and we find the extreme points for the class.

**Theorem 2.2.6.** The function defined by

where is real number such that , belong to the class , also, the function

**Proof.**

From (2.82) it follows that

where

Therefore ,

By assumption .Then by Theorem 2.2.1, we get . The Proof of this theorem is complete .

**Theorem 2.2.7.** Let and

Then if it is can be expressed in the form

where and .

**Proof.**

Let where and . Then

Therefore, by (2.70) we get

Using Theorem 2.2.1, we have

Conversely, let . Then

putting

and , we obtain .

The Proof of this theorem is complete .

**Definition 2.2.2.** A function is called closed-to-convex of order is satisfies

for some and for all . Also, a function is called starlike of order if it satisfies

for some and for all . Further , a function is called convex of order ,if and only if is starlike of order , that is if

for all , and for all

**Theorem 2.2.8.** The function is close-to-convex of order in , where

**Proof.**

It is necessary to prove that

and if

Thus by Theorem2.2.1, (2.91) is true if

or if

The Proof of this theorem is complete .

**Theorem 2.2.9.** Let be a function defined in (2.70) in class . Then is starlike of order , in , where

**Proof.**

If is starlike is sufficient to show that , for

.

Since

Since and if

Therefore, by Theorem2.1, (6.7) if this is the case

or if

The theorem is readily deduced from the previous equation.

**Theorem 2.2.10.** Let be a function defined in (2.70) in class . Then is convex of order , in , where

**Proof.**

If is convex is sufficient to show that , for .

Since

Since and if

Therefore, by Theorem 2.2.1, (2.94) if this is the case

or if

**Chapter Three**

**Coefficient Bounds for Certain Subclasses of m-Fold Bi-Univalent and Meromorphic Functions involving Quasi Subordination**

**Chapter Three**

**Coefficient Bounds for Certain Subclasses of m-Fold Bi-Univalent and Meromorphic Functions involving Quasi Subordination**

**Introduction**

Many authors provided articles in an attempt to further investigation of the idea of quasi-subordination, Darus and Mohd [73] presented the limits of functional of some well-known subclasses of analytical functions identified by the notion also, Gurusamy et.al.[43] presented an approximation of root transformation for certain analytical and univalent involving quasi subordinations in [40], while inequalities were obtained by El-shwah and anas in[31] with such quasi-subordinated subclasses of complex order analytical univalent functions. In 2016, the class of bi-univalent defined by quasi subordination was introduced by Magesh et.al.[68] and the coefficient bounds were obtained . In [78],a new class of analytical functions involving quasi-subordination was introduced in spatially modified sigmoid functions .

This chapter consists of two sections, section one is concerned with the coefficient bounds of m-fold symmetric bi-univalent functions for certain subclasses. Here, we introduced two new subclasses and of a class m-fold symmetric bi-univalent functions in open unit disk. Coefficient bounds for the Taylor-Maclaurin coefficients are obtain . Also, we solve functional problems for functions in and . Section two deals with the coefficient bounds for certain subclasses for meromorphic functions involving quasi subordination . Here, we introduce two new subclasses of meromorphic functions which are defined in terms of a quasi-subordination. The coefficient estimates inclusive, the classical functional for functions belonging to this class are then derived. Investigating a majorazation problem for the classes meromorphic functions associated with these classes is also pointed out.

**3.1 Coefficient Bounds of m-Fold Symmetric Bi-Univalent Functions for Certain Subclasses**

Indicated by the class of normalized functions satisfying the condition and given by next Taylor expansion :

which are analytic in the open unit disk

Further, let indicate the class of all functions in which are univalent in open unit disk .

The Koebe one – Quarter Theorem [29] ensures that the image of under every univalent function contains a disk of radius . therefore , every univalent functions has an inverse define , (and

where

If both functions and are univalent in is known to be bi-univalent functions. Indicate for the class of bi-univalent functions in ,which are normalized by (3.1) .

Lewin [59] obtained a coefficient bound given by for each and investigated the class of bi-univalent functions .Thereafter, stimulated by Lewin works [59] , Clunie and Brannan [19] guessed that for each .

Actually , in recent years Srivastava et al.[94] have actually enliven the study of bi-univalent and analytic functions, by Bulut [21] it was followed by such work, Adegani

and et al.[1], Guney et al. [42], and other (see, for example [12,14,53]). We notice that the is note empty . For example , the functions

are members of . However, the Koebe functions is a member of . Until now , the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients , ( ) ,for functions is as yet an open problem (see ,for specifics,[16] .

For all , function define by ) is maps and univalent in into region with m-fold symmetry. A function is called m-fold symmetric (see [81]) if the condition of normalized is hold and written as the form :

Class m-fold symmetric univalent functions indicate by which are normalized by above series expansion (3.4) . In particular if , the function in class are one-fold symmetric. Similar to the notion of m-fold symmetric, one can think of the notion of m-fold symmetric bi-univalent functions in a normal way. For all positive integer , each function in class creates an m-fold symmetric bi-univalent function. The normalized form of is define as in (1.4) and is defined as follows:

where .

Class m-fold symmetric bi-univalent functions denoted by . For , the formula (3.5) synchronized with the formula (3.3) of the class . Indicated by of the class function of the form :

such that

In view of the work of Pommerenke [81] , the m-fold symmetric function h in the class of the form

. (3.6)

Throughout our present investigation , it is assumed that analytic function with positive real part in such that and and is symmetric with regard to the real part . Such a function has an expanded series of the form:

Let two analytic functions and in with

Assume that :

Observe that

. (3.10)

By simple computations , we have :

(3.11)

and

(3.12)

**Definition 3.1.1:** Let be a function, given in (3.4) ,be in the class , if satisfied the following conditions

where and ,

and

where be a function defined by (3.5) .

Note that the particular cases of the above class

1. when reduces to the classes
2. when and reduces to the classes introduces to the class starlike function [44].
3. when and reduces to the classes introduces to the class convex function [44].

**Theorem 3.1.1:** Let be a function, defined by (3.4), in class Then

and

(3.14)

where

and

Proof :

Let . Then there are two analytic functions and with , satisfying the next conditions:

and

We get :

and

From (3.11),(3.12),(3,16) and (3.17), we find that :

, (3.18)

(3.20)

and:

From (3.18) and (3.20), we get :

(3.22)

and

By adding (3.19) and (3.21) and , up on some calculations using (3.18) and (3.22), we obtain :

Moreover, the equations (3.22) , (3.23), jointly with (3.10), yield :

Now, from (3.18) and (3.24), we get:

Where:

as certain in (3.13).

Subtracting (3.21) from (3.19), and using (3.22) and (3.18), we get:

Therefore, by using equation (3.10) in (3.25) , we obtain:

Since

where

Up on substituting (3.27) with (2.26), we are led easily to the assertion (3.14) of Theorem 3.1.1.

In case of one-fold symmetric functions of Theorem 3.1.1 we get the next results .

**Corollary 3.1.1:** Let be a function, defined by (3.4), in the class . Then

and

where

and

**Theorem 3.1.2.** Let be a function, defined by (3.4), in class . Then

where

and

.

Proof. From (3.23) we get

where, .

Subtract (3.21) from (3.19), we get

From (3.29) and (3.30), it follows that

(3.31)

where

and

.

Because each n (real ) and 1 , this implies that get (3.28) .

In cases of one-fold functions symmetric, Theorem 3.1.2 reduces to the next

**Corollary 3.1.2.** Let be a function, define by (3.4), in the class . Then

In Theorem 3.1.2 in case , we get the following corollary

**Corollary3.1.3** Let be a function, define by (3.4), in a class .Then

**Remark 3.1.1:** In case of one-fold symmetric, then Corollary 3.1.3 reduces to the next corollary.

**Corollary 3.1.4.** Let be a function, define by (1.4), be in the class . Then

**Definition 3.1.2.** Let be a function, define by (3.4) , in a class if satisfied next conditions

,

where

and

,

where be a function define by (3.5) .

Note in the above definition in case the class reduce by .

**Theorem 3.1.3 .** Let be a function , defined by (3.4), in the class . Then

and

where

and

Proof :

Let . Then there are two analytic functions and with , such that satisfying the next conditions:

, (3.34)

and

(3.35)

Since

and

Now, from (3.11) ,(3.12),(3.34) and (3.35) we obtain

and

From (3.36) and (3.38), we get

, (3.40)

adding (3.37) and (3.39) and up on some calculations use (3.36) and (3.40),we obtain

Also, from (3.9) and (3.41), jointly with (3.10), implies that

Now, from (3.36) and (3.42), conclude that

where

as asserted in (3.32).

Next, subtracting (3.39) from (3.37), we find :

By using (3.10) and (3.36) in (3.43), if follows that :

(3.44)

Which ,implies that in view of (3.36).

By applying (3.32) in (3.34),we get (3.33).

Theorem 3.1.3 is complete .

**Remark 3.1.2**: in case one-fold symmetric functions, Theorem 3.1.3 which we recall as next Corollary.

**Corollary 3.1.5 .** Let be a function , defined by (3.4), be in the class .Then

and

where

and

**Theorem 3.1.4.** Let be a function, define by (3.4), in a class . Then

where

and

Proof .

Adding (3.37) and (3.39), we get

Subtracting (3.39) from (3.37), we get :

From (3.46) and (3.47), it follows that:

(3.48)

where

.

Since all i  are real and , which implies the assertion (3.45) .

In case of the one- fold symmetric function , Theorem 3.1.4 reduce to the next .

**Corollary 3.1.6.** Let be the function , define by (3.4), in a class . Then

where

and

In Theorem 3.1.4 in case , as a result, we get the following corollary:

**Corollary 3.1.7.** Let be a function, define by (3.4), in a class . Then

where

In case of one-fold symmetric, then Corollary 3.1.7 reduces to the next Corollary

**Corollary 3.1.8.** Let be a function, define by (3.4), in a class . Then

.

**3.2 Coefficient Bounds for Certain Subclasses for Meromorphic Functions Involving Quasi Subordination**

Indicated by the class of normalized functions satisfying the condition and given by next Taylor expansion :

which are analytic and univalent in the open unit disk where is complex plane .

Furthermore, let indicate the subset of in that is analytic and univalent function. The coefficient for a function in class is well known to be bound by . Information on a geometric properties of the functions is given by the bounds of these coefficients . For example, the limit for the mean that two coefficient easily yields the distortion and growth limits for the class . in the investigation of univalent analytical functions , the functional coefficient also usually exists. In mathematics, the is an inequality for the coefficients of univalent analytic functions found by in year (1933) [34], for , then the functional for normalized univalent function given by (3.49).

In 1970 Robertson [86] the concept of quasi-subordination introduced . Moreover, if and be two analytic function , a functions is quasi-subordination to a function in and can written as the follows

if there is and an analytic functions with , such that

Note that , in case , implies that , so that

in .

Furthermore, when substitute if , then and in this case is said that majorized by ,written

in .

In this case,

As a result, it is clear that quasi subordination is a generalization of both subordination and majorization. (see [28,53] ).

Indicate by the class of meromorphic univalent functions and written as the design

which are univalent and analytic in the open punctured unit disk

If for each , then, a function is meromorphic starlike

There are a large number of studies on meromorphic univalent functions. Estimates of the coefficient of meromoriphic univalent functions remain one of the key subjects of interesting. To mention a few of them, Aouf et al.[32] investigate inequality for certain class of meromorpheic function and estimates of certain functional coefficients for meromerphic star like functions are derived from Ravichandran and Ali [5]. Suppose in this work that is analytical in with 

**Definition 3.2.1.** Let , ( ) be the class consist of functions , satisfying quasi subordination

For , there is and be two analytic functions, with and by the concept of quasi-subordination, so that

.

Let and so

.

**Definition 3.2.2.** Let ,( ) be the class consist of functions , satisfying quasi-subordination

, .

For , there is and be two analytic functions, with and by the concept of quasi-subordination, so that

Let and such that

The purpose of this section was to obtain the boundary coefficients for the functional coefficient of Fekete-Szego involving quasi-subordination for the meromorphic function subclass.

Let

and .

To prove our results we need the next Lemma.

**Theorem 3.2.1.** Let be a function, belong to the class hen for any complex number ,

.

.

Proof.

If , then there is and be two analytic functions, with , and by the concept of quasi-subordination, such that

(3.51)

the function define by

or this equivalently,

.

Subsequently,

and so

(3.52)

Since

its follows form (3.51) and (3.52) we obtain

and

hence

where

Then

Since is bounded and analytic in , then , (refer to[76],page170). Therefore, be a function with positive real part in given . (refer to[29].p.41). Thus

An application of Lemma 1.2.6 yield

Observe that

Hence

.

**Remark 3.2.1**. In case (3.56) reduce to

**Theorem 3.2.2.** Let be a function, belongs to the class .hen for any complex number ,

Proof.

If , then there is and be two analytic functions, with and (by definition of quasi-subordination), such that

Define the function by

or this equivalently ,

.

Subsequently,

and so

(3.58)

Since

, (3.59)

its follows form (3.57) and (3.58) we obtain

and

,

hence

where

Then

Next the same argument as in Theorem 3.2.1, where and by using Lemma 1.2.6 to the inequality, the required result is obtained .

**Remark 3.2.2 .** In case (3.56) reduce to

**Theorem 3.2.3.** Let belongs to the class satisfies

For any complex number , then

Proof. If satisfies

then by the definition of majorization

Since

from (3.63) and (3.53) it follows that

hence

And then the consequence follows.

**Theorem 3.2.4.** Let belongs to the class satisfies

For any complex number,then

Proof.

The Proof for this theorem is comparable to that given in Theorem 3.2.3.

**Chapter Four**

**Quasi Subordinatio of Bi-Univalent Functions and Sakaguchi Type Functions**

**Chapter Four**

**Quasi Subordination of Bi-Univalent Functions and Sakaguchi Type Functions**

**Introduction**

In 1967 Lewin [59] obtained a coefficient bound given by ≤ 1*.*51 for each and investigated class of bi-univalent functions.Thereafter, stimulated by the work of Lewin[59] , Clunie and Brannan [19] guessed that for each . Actully , in recent years Srivastava et al.[94] have actually enlivened the study of bi-univalent and analytic functions, by Bulut [21]it was followed by such work, Adegani and et al.[29], Guney et al. [42],(see, for example [4] ,[12] ,[16], [19] ,[42] ,[54] ,[82],[94]).

This chapter includes two sections. The first section discusses the coefficients assessment for certain subclasses of bi-univalent functions related to quasi-subordination. Here, we submit and investigate specific new subclasses of the function class Σ of bi-univalent functions defined in the open unit disk, which is connected with the quasi subordination. We're finding estimates on the Taylor - Maclaurin coefficiency and for functions in these subclasses. Already pointed out are some documented and new implications of those findings. Section two deals with we introduced the application of quasi subordination associated with generalized Sakaguchi type functions. Here, we introduced and investigated a new class of analytic functions identified in terms of a quasi-subordination. of analytic functions which is defined by terms of a quasi-subordination. The coefficient estimates inclusive the classical inequality of functions belonging to this class are then derived. Also several certain special improver results for the associated classes involving subordination are presented .

**4.1 Quasi Subordination of Bi-Univalent Functions Involving Convolution Operator**

Indicated by the class of normalized functions satisfying the condition and given by next Taylor expansion :

which are analytic in the open unit disk where is complex plane .

Further, let denote all functions class in which are univalent in open unit disk . For more details on univalent functions .

The Koebe one – Quarter Theorem [29] ensures that the image of under every univalent function contains a disk of radius . therefore , every univalent functions has an inverse define by , (and

,

where

. (4.2)

If both functions and are univalent in U, is known to be bi univalent functions. Indicate for the class of bi-univalent functions in U by Σ ,which are normalized by (4.1).

The general Hurwitz defined by

(4.3)

where

The Hurwitz-Lerch Zeta function of Srivastava and Choi [23] has found some interesting features. In [93], the following operator is introduced by Srivastava and Attiya :

(4.4)

where

where for convenience

(4.6)

By using (4.1),(4.5) and (4.6), we have

.

In [2] the authors introduce the generalized derived operator

where

.

The convolution operator between Srivastava-Attiya operator and generalized derived operator

(4.9)

where

Note that

1. When includes the Srivastava-Attiya operator Ξs,k [95].
2. When includes the generalized derivative operator [25]
3. When reduces towhich is introduced by Salagean derivative operator [92].
4. When reduces to which is introduced by generalized Salagean derivative operator (or Al-Oboudi derivative ) [9].
5. When , reduces to which is introduced by generalized Ruscheweyh derivative operator (or AlShaqsi – Darus derivative operator ) [26].
6. When λ reduces by which introduce by Sirvastava-Attiya operator [93]
7. When , reduces to which is introduced Ruscheweyh derivative operator[11].
8. When reduces to which is introduced by Alexander integral operator [3].
9. When reduces to which is introduced by Bernardi integral operator [18].
10. When reduces to which is introduced by Jung-Kim-Srivastava integral operator [55].

Robertson in 1970 [86] the concept of quasi-subordination introduced . Moreover, if and be two analytic function , a function is quasi-subordination to a function in and can written as the form

if there is and an analytic functions with , such that

Note that , when , then , so that

in .

Furthermore, when substitute if , then and in this case is said that majorized by ,written

in .

In this case,

Notice, however, that quasi subordination is both a widespread subordination and majorization . (see [86] ).

In 2016, the class of bi-univalent defined by quasi subordination was introduced by Magesh et al.[68] and the coefficient bounds were obtained. In [66] , Ma and Minda introduce and studied the unified classes :

where is analytic and univalent function with positive real part in the unit disk U, satisfying and is a starlike region with respect to 1, and symmetric with respect to real axis . and Φ(*U*) is a starlike region with respect to 1, and symmetric with respect to real axis . In classes and the function is a starlike area in relation to 1, and symmetrical in relation to the real axis. The function in the classes *T*(Φ) and *S*(Φ) are namely starlike of Ma-Minda type or convex of Ma-Minda type respectively (see [32]). In this investigation we suppose that

and

**Definition 4.1.1:** Let *f* ∈ Σ be a function defined in (4.1) in the class if it satisfies the following quasi subordination conditions

and

(4.14)

where .

For particular values to parameter .We will be receiving new and well-known classes.

**Remark 4.1.1:** For and a function *f* ∈ Σ, defined in (4.1) is said to be in the class

and

where with is inverse function of .

**Remark 4.1.2.**: For , a function , defined in (4.1) is said to be in the class then next condition are satisfied

and

where .

**Remark 4.1.3** : For , a function , defined in (4.1) is said to be in the class then the following condition are satisfied

and

where .

**Remark 4.1.4**: For a function *f* ∈ Σ, defined in(4.1) is namely if it satisfies the following quasi subordination conditions

and

**Theorem 4.1.1**: Letgiven in (4.1) be in class . Then

where

and

Proof.

Since then there exist analytic function such that , and , satisfying

and

Define the functions andby

or, equivalently

Then and analytic in *U*, with τ(z) = = 1. Since and have positive real part in unit disk and

In view (4.21),(4.22) in (4.17) and (4.18) clearly, we have

and

Using (4.21) and (4.22) together with (4.11) and (4.12), it is evident that

(4.25)

From (4.23), (4.24) , (4.25) and (4.26), we get it as follows

and

From (4.27) and (4.29), we obtain

it follows that

and

Adding (4.28) and (4.30), we get

Applying Lemma 1.2.7 , for the coefficients its follows from (4.32) and (4.33),getting

and

which yield coefficient in (4.15).

Subtracting (4.30) from (4.28) , we obtain

Upon substituting from (4.33) and (4.34) putting (4.37) and applying Lemma 1.2.7 , we find

(4.38) and (4.39) yields the estimate in (4.16). This completes the Proof.

**Corollary 4.1.1**: Let be in class.Then

**Corollary 4.1.2**: Let be in classThen

**Corollary 4.1.3**: Let be in classThen

**Corollary 4.1.4**: Let *f* be in class.Then

.

**Definition 4.1.2:** Let *f* ∈ Σ be a function defined in (4.1) in the class, if it satisfies the following quasi subordination conditions

and

where .

For particular values to parameter .We will be receiving new and well-known classes.

**Remark 4.1.5:** For and a function , defined in (4.1) is said to be in the class if it satisfies the following quasi subordination conditions

and

where with is inverse function of .

**Remark 4.1.6**: For and a function , defined in (4.1) is called be in the class ,then the following condition are satisfied

and

where .

**Remark 4.1.7**: For a function ,, defined in (4.1) is namely if it satisfies the following quasi subordination conditions

and

where .

**Theorem 4.1.2**: Let given in (4.1) be in class . Then

where

and

**Proof.**

Steps of the Proof of the theorem are the same in the Theorem 4.1.1 , we can get the relations as follows

and

From (4.44) and (4.46), we obtain that

it follows that

and

Adding (4.28) and (4.30), we get|

Applying Lemma 1.2.7 , for the coefficients its follows from (4.49) and (4.50),getting

and

which yield coefficient in (4.42).

Subtracting (4.47) from (4.45) , we obtain

Upon substituting (4.50) and (4.51) putting (4.54) and applying Lemma 1.2.7 , we find that

and

(4.53) and (4.54) yields the estimate in (4.43). This completes the Proof

**Corollary 4.1.5:** Let *f* be in class .Then

and

.

**Corollary 4.1.6 :** Let be in class. Then

**Corollary 4.1.7**: Let be in class.Then

**4.2 Application of Quasi Subordination Associated With Generalized Sakaguchi Type Functions**

Let symbol the collection of normalized functions satisfying the condition and given by Taylor expansion :

which are analytic in the unit disk

where is a complex plane .

Furthermore , let symbol the class of all functions in which are univalent in unit disk . Let w(z) be analytic function in unit disk with all coefficients are real and , such that

Also, let univalent and analytic function with positive real part in unit disk with and, which maps the unit disk onto a zone starlike with respect to 1 and symmetric with respect real axis . The Taylorʼs expansion with all coefficients are real and can written as the form

such that C1 > 0 .

Let the class of functions and written in the following form

Robertson in 1970 introduced [86] the concept of quasi-subordination . Moreover, if and be two analytic functions we say that is quasi-subordination to in and can be written as the form

if there exist and analytic functions with ,

such that

Note that , when , then (see [10-11]), so that

in .

Furthermore, if , then and in this case is majorized by ,written

in .

In this case,

Therefore, quasi-subordination is a generalization of subordination and also of majorization . For example (see [50],[53],[67],[85] and [86]).

Sakaguchi in [90], introduced of class starlike functions with respect to symmetric points in unit disk , for satisfy

.In the same way, in [39] wang et al [80] introduced of the class convex functions with respect to symmetric points in unit disk , for satisfy (see[78]).

In mathematics, the Fekete-Szego is an inequality for the coefficients of univalent analytic functions found by Fekete-Szego in year (1933) [34], for ,then the Fekete-Szego functional for normalized univalent function given by (4.57), (see[20,22,38,91]).

The aim of this section is to introduc a new class of univalent functions applying the generalized Salagean operator (see[9,39]) .

Define the following differential operator as below

If is given by (4.57) , then from (4.61) ,we see that

where and .

Using a special function is sigmoid function and which is differentiable and bounded and real function that is defined for all real input value and has non-negative derivative at each point, we can written sigmoid function as form

 (4.63)

The sigmoid function is salutary since it has very properties(see [46] ), as them, a sigmoid function is a monotonic, and has a first derivative which is bell shaped, it outputs real numbers between zero and one and since sigmoid function is one-one then it never loses information .

**Lemma 4.2.1.**[56] . Let be the Schwarz function given by

then

where .

**Definition 4.2.1.** A function given by (4.57) is said to be in class if the next quasi-subordination holds :

where

From above definition we note that if and only if there exist be analytic function with , such that

If, in condition (4.66), in case , then the class is symbol it satisfying the condition



Note that,

… (4.68)

**Theorem 4.2.1.** Let of the form (4.57) , be a function in the class . Then

and for some ,

(4.70)

Proof :

Let function in class , we get

Let . From Definition 4.1.1 we can write

a modified sigmoid function H(z) is given as below

Combining (4.58), (4.64) and (4.72) we obtain

Now , using the series expansion from (4.57), and the expansion given by (4.68),we get

> ()

From the expansion (4.71) and (4.74) , on equating the coefficients of z and z2 in (4.74), we will get

, ()

Now, from (), we get

From (),it follows that

.

Therefore,

For some , from (30) and (31) we obtain

We have given by (4.58) is bounded and analytic in unit disk , thus , up on using [75](page 172) , for some

. ()

Putting the value of from (32) into (33) , we get

(4.81)

Now , if in () , we get

Otherwise , if in (), define a function

. (4.83)

The (4.83) is polynomial in and hence analytic in . The maximum occurs at , . Thus

Therefore , by using Lemma 4.2.1 , we get

.

In case , we has the following

**Corollary 4.2.1.** Let of the form (4.57) , be a function in the class . Then

and for any complex number ,

.

The consequence is sharp .

In Corollary 4.2.1 in case , we obtain the next corollary.

**Corollary 4.2.2.** Let of the form (4.57), be a function in the class . Then

and for any complex number ,

.

The consequence is sharp .

In case in Corollary 4.2.2, we get the following

**Corollary4.2.3**. Let of the form (4.57) , be a function in the class . Then

and for any complex number ,

The consequence is sharp .

In case in Corollary 4.2.2, we deduce the following

**Corollary 4.2.4:** Let of the form (4.57) , be a function in the class . Then

**.**

The consequence is sharp**.**

**Conclusions**

**Conclusions**

We conclude from this study, many results. In the case of applying the new operator to the geometric functions, it was concluded that they remain to preserve their geometric features. We can get new results inside the unit disk by putting some constraints on the functions that belong to these subclasses characteristics. Also, we derived some geometric properties like coefficient boundary, coefficient inequality, distortion theorem, closure theorem, extreme points, radii of starlikeness, convexity, close to convex, and values of integration. Also, in the case of applying the two new classes m-fold symmetric bi univalent to the geometric functions, we have derived the estimates of the Taylor-Maclaurin coefficients and and functional problems for functions belonging to these new subclasses. The results presented in this study have been shown to considerably improve the earlier results. Also, in the case of applying the two new subclasses of meromorphic functions to the geometric functions, we derived the functional problems for functions and Investigate the problems of majorazation for the classes' meromorphic functions as well, it is useful in complex analysis.

Finally, in the case of applying the convolution operator between the Srivastava-Attia operator and the generalized derivative operator, and we applied the two new classes bi univalent to the geometric functions on these operators, it was determined and | for all classes bi-univalent. Also, we conclude by using Generalized Sakaguchi Type Functions some results function.

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**المستخلص**

**الغرض من هذه الرسالة هو دراسة بعض خصائص نظرية الدالة التحليلية. تم الحصول على بعض نتائج التبعية التفاضلية من الدرجة الثالثة ونتائج التبعية الفائقة للدوال التحليلية في قرص الوحدة المفتوح. يهدف أيضًا ، من خلال استخدام مؤثر الالتفاف** في الصنف , تم ألقاء الضوء على بعض الخواص الهندسية مثل مبرهنات التشويه ومتراجحة المعاملات بالمعامل ونظريات النمو ونظريات الإغلاق والعوامل المتكاملة ،انصاف الاقطار المحدبة ,المحدبة ,النجمية للدوال في الصنف . هنا ، تمت دراسة صنفان جزئيان جديدان من الدوال ثنائية التكافؤ لعدد من الصفوف المتماثلة m في U, تخمينات المعامل لمعاملات تايلور-ماكلورين تم الحصول عليهم. علاوة على ذلك ، تم حل المشكلات لدوال للدوال في و. تلقي الدراسة أيضًا الضوء على التحقيق في صنفان جزئيان جديدان من الدوال الميرومورفيه التي تم تحديدها من خلال شروط شبه التبعية. تشمل تخمينات المعامل الدوال الكلاسيكية للدوال التي تنتمي إلى هذه الفئة يتم اشتقاقها بعد ذلك. التحقيق في مشكلة التخصص للدوال الميرومورفية للفئات المرتبطة بهذه الفئات تمت الإشارة إليها أيضًا. تم عرض المناقشة لشبه التبعية للدوال الثنائية التكافؤ التي تتضمن مؤثر الالتفاف.

أخيرًا ، قدمنا ودرسنا صنفان جزئيان جديدان من الدوال ثنائية التكافؤ في U ، والتي تتضمن الالتفاف بين Hurwitz-Lerch Zeta والمؤثر المشتق المعمم والتي تحقق شروط شبه التبعية. تخمينات المعامل و تم تحديدهما في هذه الاصناف الجزئية. أيضًا ، يتم تقديم صنف جزئي جديد معين من الدوال أحادية التكافؤ المرتبطة بالدالة السينية ويتم الحصول على تطبيقات مع مسألة في

 **جمهورية العراق**

**وزارة التعليم العالي والبحث العلمي**

**جامعة الانبار**

**كلية التربية للعلوم الصرفة**

**قسم الرياضيات**

**دراسة بعض خصائص نظرية الدالة التحليلية**

رسالة مقدمة

إلى مجلس كلية التربية للعلوم الصرفة - جامعة الانبار

وهي جزء من متطلبات نيل شهادة ماجستير في الرياضيات

من قبل

**نهاد حميد شهاب**

بكالوريوس رياضيات- كلية التربية للعلوم الصرفة - جامعة الأنبار- 2013

**بإشراف**

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