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Study of certain subclasses of analytic functions involving convolution operator

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Abstract: The purpose of the present paper is to introduce new operator SR^n using the Salagean operator S^n and Ruscheweyh operator R^n for analytic functions. We study the differential subordinations in the general case and investigate differential subordination properties regarding the operator SR^n. Moreover, we determine dominants and best dominants of differential subordinations integral operators are also considered.

Key words: Differential subordination, Analytic function, Univalent function, Convex function, Dominant, Best dominant, Ruscheweyh operator, Salagean operator, Convolution product.

INTRODUCTION

Let $E = \{z \in \mathbb{C} : |z| < 1\}$ be an open unit disc in $\mathbb{C}$ (complex plane) and by $G(E)$ the space of analytic functions in E and let $G[b, m]$ be the subclass of $G(E)$ of the form

$$g(z) = b + b_m z^m + b_{m+1} z^{m+1} + \cdots , \tag{1}$$

where $b \in \mathbb{C}$ *and* $m \in \mathbb{N}$ *with* $G_0 \equiv G[0, 1]$
and $G \equiv G[1, 1]$and let

$$\mathcal{A}_m = \{g \in G(E), g(z) = z + b_{m+1}z^{m+1} + \cdots, z \in E\}.$$

S denote of the all functions is univalent in E.
Let

$$S^* = \left\{g \text{ is univalent}, \Re e \frac{zg'(z)}{g(z)} > 0, (z \in E)\right\}, \tag{2}$$

denote the class of starlike functions in E and

$$K = \left\{g \text{ is univalent}, \Re e \left(\frac{zg''(z)}{g'(z)} + 1\right) > 0, (z \in E)\right\}, \tag{3}$$

denote the class of convex functions in E. Let g *and* F *be members of* $G(E)$. The function g is said to be subordinate to F, if there is a Schwarz function d analytic in E, with

$$\mathrm{d}(0) = 0 \text{ and } |\mathrm{d}(z)| < 1, (z \in E) \text{ such that}$$

$g(z) = F(d(z))$.We denote this subordination by. We denote this subordination by

$$g(z) \prec F(z) \text{ or } g \prec F.$$

If F is univalent in E, then we get the following [3-4]

$$g(z) \prec F(z) \Leftrightarrow g(0) = F(0) \text{ and } g(E) \subset F(E).$$

Let Θ *and* Λ be sets in $\mathbb{C}$, let $\phi : \mathbb{C}^3 \times E \to \mathbb{C}$ and h be univalent in E. If q is analytic in E with $q(0) = a$ with generalizations of implication

$$\{\phi(q(z), zq'(z), z^2q''(z); z)\} \subset \Theta \Longrightarrow q(E) \subset \Lambda, \tag{4}$$

with satisfies the second-order differential subordination

$$\phi(q(z), zq'(z), z^2q''(z); z) \prec f(z), \tag{5}$$

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then q is said to be solution of subordination. The univalent function w is said to be dominant of the solution of subordination, or dominant, if $q \prec w$ for q satisfying (5). A dominant $\widetilde{w}$ that satisfying $\widetilde{w} \prec w$ for dominants w of (5) is called a best dominant of (5).

Definition 1.1.([5]) The W is denoted by the set of w, such that $W \, and \, w$ are analytic with injective on $\bar{E}/T(w)$, for

$$T(w) = \left\{x \epsilon \, \partial E \, ; \, \lim_{z \to x} w(z) = \infty\right\}, \tag{6}$$

such that $w'(x) \neq 0$ with $x \epsilon \, \partial E/t(w)$**.** The $W(c)$ is denoted by the subclass of W which $w(0) = c$.

Definition 1.2.([2]) If the function $g(z)$ belonging to the class $\mathcal{A}$, given by

$$g(\mathrm{z}) = \mathrm{z} + \sum_{h=m+1}^{\infty} b_h z^h \,, \quad (z \in E) \tag{7}$$

for $\in \mathcal{A}$, $0 \leq \lambda_1 \leq \lambda_2, z \in E, n \in \mathbb{N} \cup \{0\}$ and m positive natural number, the operator $S^n_{\lambda_1,\lambda_2}: \mathcal{A}_m \to \mathcal{A}_m$ is defined by

$$S^n_{\lambda_1,\lambda_2} g(\mathrm{z}) = z + \sum_{h=m+1}^{\infty} \left(\frac{1 + (\lambda_1 + \lambda_2)(j-1)}{1 + \lambda_2(j-1)}\right)^n b_h z^h. \tag{8}$$

From (8) we have

$$z\left(S^n_{\lambda_1,\lambda_2} g(\mathrm{z})\right)' = (\lambda_1 + \lambda_2) S^{n+1}_{\lambda_1,\lambda_2} g(\mathrm{z}) - \left((\lambda_1 + \lambda_2) - 1\right) S^n_{\lambda_1,\lambda_2} g(\mathrm{z}). \tag{9}$$

Remark 1.3. Note that is the special case of $S^n_{\lambda_1,\lambda_2}$. for $\lambda_1 = 1, \lambda_2 = 0$, $S^n_{\lambda_1,\lambda_2}$ reduces to S^n which is introduced by Salagean [8].

$$S^n g(\mathrm{z}) = z + \sum_{h=m+1}^{\infty} \left(1 + (j-1)\right)^n b_h z^h. \tag{10}$$

For $g \in \mathcal{A}_m, m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}$, and

$$S^0 g(z) = g(z)$$
$$S^1 g(z) = zg'(z)$$
$$\dots$$
$$S^{n+1} g(z) = z\left(S^n g(z)\right)'. \; (z \in E)$$

Definition 1.4 ([7]) Let $g \in \mathcal{A}_m, m \in \mathbb{N}$,
$n \in \mathbb{N} \cup \{0\}$ and the operator R^n is defined by $R^n: \mathcal{A}_m \to \mathcal{A}_m$,

$$R^0 g(z) = g(z)$$
$$R^1 g(z) = zg'(z)$$
$$\dots$$
$$(n+1) R^{n+1} g(z) = z\left(R^n g(z)\right)' + nR^n g(z). \, (z \in E) \tag{11}$$

Remark 1.5 If $g \in \mathcal{A}_m$, $g(z) = z + \sum_{h=m+1}^{\infty} b_h z^h$, *then*

$$R^n g(z) = z + \sum_{h=m+1}^{\infty} C^n_{n+h-1} b_h z^h. \quad (z \in E)$$

Lemma 1.6 ([6]) If f is a convex function in E with
$k(z) = f(z) + m\beta z f'(z)$, $(z \in E)$
where $\beta > 0 \; and \; m \in \mathbb{N}$. If

$$t(z) = f(0) + t_m z^m + t_{m+1} z^{m+1} + \cdots, (z \in E)$$

is analytic in E and

$$t(z) + \beta z t'(z) \prec k(z),$$

then

$$t(z) \prec f(z),$$

such that result is sharp.

DIFFERENTIAL SUBORDINATION RESULTS

Definition 2.1 ([1]) For $g \in \mathcal{A}$, $0 \leq \lambda_1 \leq \lambda_2$, $z \in E$ and let $n \in \mathbb{N} \cup \{0\}$. Denote by $SR^n_{\lambda_1,\lambda_2}$

the operator given by the Hadamard product (the convolution product) of the Salagean operator$S^n_{\lambda_1,\lambda_2}$ and the Ruscheweyh operator R^n,

$$SR^n_{\lambda_1,\lambda_2}: \mathcal{A}_m \to \mathcal{A}_m,$$
$$SR^n_{\lambda_1,\lambda_2} g(z) = (S^n_{\lambda_1,\lambda_2} * R^n) g(z).$$

Remark 2.2 If $g \in \mathcal{A}_m, g(z) = z + \sum_{h=m+1}^{\infty} b_h z^h$, *then*

$$SR^n_{\lambda_1,\lambda_2} g(z) = z + \sum_{h=m+1}^{\infty} C^n_{n+h-1} \left(\frac{1 + (\lambda_1 + \lambda_2)(j-1)}{1 + \lambda_2 (j-1)} \right)^n b_h z^h. \tag{12}$$

For $\lambda_1 = 1, \lambda_2 = 0$, $SR^n_{\lambda_1,\lambda_2}$ we have *the Hadamard product* SR^n [1] of the Salagean operatorS^n and the Ruscheweyh operator R^n.

Theorem 2.3. Let f be a convex function with
$f(0) = 1$. Let $k(z) = f(z) + zf'(z)$, for
$g \in \mathcal{A}_m, z \in E, n \in \mathbb{N} \cup \{0\}$, and the differen-tial subordination

$$\frac{1}{z}\ SR^{n+1}_{\lambda_1,\lambda_2} g(z) + \frac{((\lambda_1 + \lambda_2) - 1)}{(\lambda_1 + \lambda_2)} z \left(SR^n_{\lambda_1,\lambda_2} g(z) \right)'' \prec k(z), \quad (z \in E) \tag{13}$$

Then

$$\left(SR^n_{\lambda_1,\lambda_2} g(z) \right)' \prec f(z), (z \in E) \tag{14}$$

and this result is sharp.

Proof. *Let*

$$t(z) = \left(SR^n_{\lambda_1,\lambda_2} g(z) \right)' = 1 + \sum_{h=m+1}^{\infty} C^n_{n+h-1} \left(\frac{1 + (\lambda_1 + \lambda_2)(j-1)}{1 + \lambda_2 (j-1)} \right)^{n+1} b_h z^{h-1},$$

with $t(0) = 1$, *and*

$$g(z) = z + \sum_{h=m+1}^{\infty} b_h z^h, \quad (z \in E)$$

we have

$$t(z) + zt'(z) = \frac{1}{z}\ SR^{n+1}_{\lambda_1,\lambda_2} g(z) + \frac{((\lambda_1 + \lambda_2) - 1)}{(\lambda_1 + \lambda_2)} z \left(SR^n_{\lambda_1,\lambda_2} g(z) \right)''.$$

We get $t(z) + zt'(z) \prec k(z) = f(z) + zf'(z)$, for $z \in E$. By using Lemma 1.6, we have

$$t(z) \prec f(z),$$

in other words

$$\left(SR^n_{\lambda_1,\lambda_2} g(z) \right)' \prec f(z),$$

and this result is sharp.

Corollary 2.4. . Let f be a convex function with $f(0) = 1$. Let $k(z) = f(z) + zf'(z)$, for
$n \in \mathbb{N} \cup \{0\}, z \in E, g \in \mathcal{A}_m$. If $\lambda_1 = \lambda_2 = 1$
and the differential subordination

$$\frac{1}{z} SR^{n+1} g(z) + \frac{1}{2} z (SR^n g(z))'' \prec k(z), \tag{15}$$

then

$$(SR^n g(z))' \prec f(z). \quad (z \in E)$$

Theorem 2.5. Let f be a convex function with
$f(0) = 1$ and let k be the function

$$k(z) = f(z) + zf'(z).\ If\ n \in \mathbb{N} \cup \{0\}\ and\ z \in E,$$

$g \in \mathcal{A}_n$ confirm the differential subordination

$$\left(SR^n_{\lambda_1,\lambda_2} g(z)\right)' \prec k(z), \quad (z \in E) \tag{16}$$

then

$$\frac{SR^n_{\lambda_1,\lambda_2} g(z)}{z} \prec f(z), \quad (z \in E) \tag{17}$$

and the function $f(z)$ is the best dominant.

Proof. For $g \in \mathcal{A}_n$, $g(z) = z + \sum_{h=m+1}^{\infty} b_h z^h$, we get

$$SR^n_{\lambda_1,\lambda_2} g(z) = z + \sum_{h=m+1}^{\infty} C^n_{n+h-1}\left(\frac{1+(\lambda_1+\lambda_2)(j-1)}{1+\lambda_2(j-1)}\right)^n b_h z^h.$$

Let

$$t(z) = \frac{SR^n_{\lambda_1,\lambda_2} g(z)}{z}$$

$$= \frac{z + \sum_{h=m+1}^{\infty} C^n_{n+h-1}\left(\frac{1+(\lambda_1+\lambda_2)(j-1)}{1+\lambda_2(j-1)}\right)^n b_h z^h}{z}$$

$$= 1 + \sum_{h=m+1}^{\infty} C^n_{n+h-1}\left(\frac{1+(\lambda_1+\lambda_2)(j-1)}{1+\lambda_2(j-1)}\right) b_h^2 z^{h-1},$$

we obtain

$$t(z) + zt'(z) = \left(SR^n_{\lambda_1,\lambda_2} g(z)\right)'. \quad (z \in E)$$

Then

$$\left(SR^n_{\lambda_1,\lambda_2} g(z)\right)' \prec k(z), \qquad (z \in E)$$

become

$$t(z) + zt'(z) \prec k(z) = f(z) + zf'(z). (z \in E)$$

By Lemma 1.6, we get

$$t(z) = \frac{SR^n_{\lambda_1,\lambda_2} g(z)}{z} \prec f(z). \ (z \in E)$$

Since $t(z) = f(z)$, we deduce $f(z)$ is best dominant.

Example 2.6. Let $f(z) = 1 + \frac{z}{3}$, $k(z) = f(z) + zf'(z) = \frac{3+2z}{3}$. If $t(z) + zt(z)$ is analytic in E and satisfies

$$t(z) + zt'(z) \prec k(z) = 1 + z$$

then

$$t(z) \prec f(z) = \frac{3+2z}{3},$$

and the function $f(z)$ is the best dominant.

Theorem 2.7. Let f be a convex function,
$f(0) = 1$, *let* $k(z) = f(z) + zf'(z)$,
$z \in E$. *If* $n \in \mathbb{N} \cup \{0\}$ *and* $z \in E, g \in \mathcal{A}_n$
confirm the differential subordination

$$\left(\frac{zSR^{n+1}_{\lambda_1,\lambda_2} g(z)}{SR^n_{\lambda_1,\lambda_2} g(z)}\right)' \prec k(z), \quad (z \in E) \tag{18}$$

then

$$\frac{SR_{\lambda_1,\lambda_2}^{n+1}g(z)}{SR_{\lambda_1,\lambda_2}^{n}g(z)} \prec f(z), \quad (z \in E) \tag{19}$$

Proof. Let $g(z) = z + \sum_{h=m+1}^{\infty} b_h z^h$ *and* $g \in \mathcal{A}_n$, we obtain

$$SR_{\lambda_1,\lambda_2}^{n}g(z) = z + \sum_{h=m+1}^{\infty} C_{n+h-1}^{n}\left(\frac{1+(\lambda_1+\lambda_2)(j-1)}{1+\lambda_2(j-1)}\right)^n b_h z^h.$$

Take

$$t(z) = \frac{SR_{\lambda_1,\lambda_2}^{n+1}g(z)}{SR_{\lambda_1,\lambda_2}^{n}g(z)} = \frac{z + \sum_{h=m+1}^{\infty} C_{n+h-1}^{n+1}\left(\frac{1+(\lambda_1+\lambda_2)(j-1)}{1+\lambda_2(j-1)}\right)^{n+1} b_h z^h}{z + \sum_{h=m+1}^{\infty} C_{n+h-1}^{n}\left(\frac{1+(\lambda_1+\lambda_2)(j-1)}{1+\lambda_2(j-1)}\right)^{n} b_h z^h}$$

$$= \frac{1 + \sum_{h=m+1}^{\infty} C_{n+h-1}^{n+1}\left(\frac{1+(\lambda_1+\lambda_2)(j-1)}{1+\lambda_2(j-1)}\right)^{n+1} b_h z^{h-1}}{1 + \sum_{h=m+1}^{\infty} C_{n+h-1}^{n}\left(\frac{1+(\lambda_1+\lambda_2)(j-1)}{1+\lambda_2(j-1)}\right)^{n} b_h z^{h-1}}$$

and we get

$$t'(z) = \frac{\left(SR_{\lambda_1,\lambda_2}^{n+1}g(z)\right)'}{SR_{\lambda_1,\lambda_2}^{n}g(z)} - t(z)\,.\frac{\left(SR_{\lambda_1,\lambda_2}^{n}g(z)\right)'}{SR_{\lambda_1,\lambda_2}^{n}g(z)}.$$

Then

$$t(z) + zt'(z) = \left(\frac{zSR_{\lambda_1,\lambda_2}^{n+1}g(z)}{SR_{\lambda_1,\lambda_2}^{n}g(z)}\right)'.$$

Relationship (18) becomes

$$t(z) + zt'(z) \prec k(z) = f(z) + zf'(z)$$

and by using Lemma 1.6,we have $t(z) \prec f(z)$, in other words

$$\frac{SR^{n+1}g(z)}{SR^{n}g(z)} \prec f(z). \qquad (z \in E)$$

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