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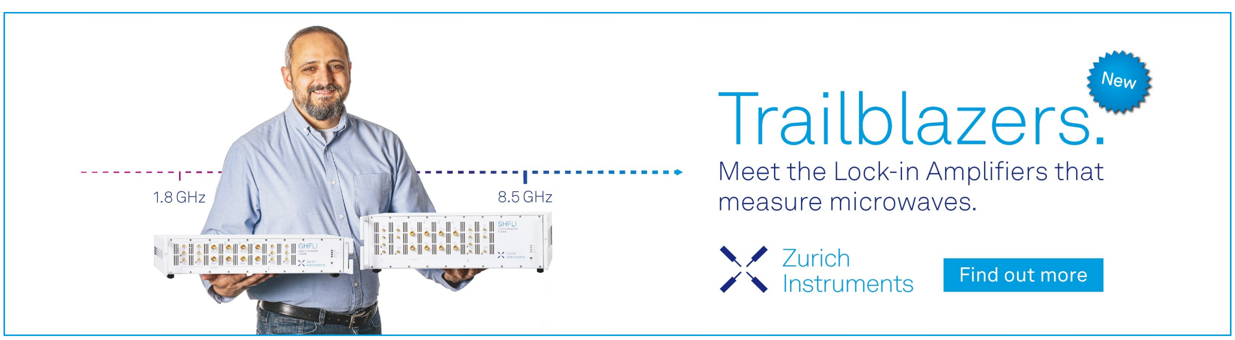

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Some Properties of Subclass of P-Valent Function With New Generalized Operator

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Abstract. We have introduced and investigated the subclass $\mathrm{T}[\mu,\tau;p]$ for p-valent functions defined by the new linear operator $\Upsilon_{\xi,\eta}^{m,s}$ in this paper. The main objective is to investigate many characteristics, such as coefficient estimates, theorems of distortion, closure theorems, neighborhoods and starlikenessradii, convexity and close-to-convexity of class $\mathrm{T}[\mu,\tau;p]$ functions.

Key words. P-valent functions, Distortion theorems, Integral operator, Starlike functions, Convex functions, Close-to-convex functions, Hadamard product.

INTRODUCTION AND DEFINITIONS

Let $\mathcal{L}=\{w\in\mathbb{C}:|w|<1\}$ be an open unit disc in $\mathbb{C}$. Let $H(\mathcal{L})$ be the analytic functions class in $\mathcal{L}$ and let $\mathcal{L}[a,\varepsilon]$ be the subclass of $H(\mathcal{L})$of the form $g(w) = a + a_\iota w^\iota + a_{\iota+1}w^{\iota+1} + \cdots$,

where $a\in\mathbb{C}$ and $\iota\in\mathbb{N}=\{1,2,\ldots\}$ with$H_0\equiv H[0,1]$ and $H\equiv H[1,1]$. Let $g(w)$ be an analytic function an open unit disc. If the equation $v=g(w)$ has never more than p-solutions in $\mathcal{L}=\{w\in\mathbb{C}:|w|<1\}$, then $g(w)$ is said to be p-valent in $\mathcal{L}$. The class of all analytic p-valent functions is denoted by $\mathcal{A}_p$, where g is expressed of the forms

$$g(w)=w^p+\sum_{\iota=p+\varepsilon}^{\infty}a_\iota w^\iota,\quad (p,\iota\in\mathbb{N}=\{1,2,3,\ldots\},w\in\mathcal{L}). \tag{1}$$

The Hadamard product for two functions in $\mathcal{A}_p$, such that

$$k(w)=w^p+\sum_{\iota=p+\varepsilon}^{\infty}c_\iota w^\iota,\quad (w\in\mathcal{L}) \tag{2}$$

is given by

$$g(w)*k(w)=w^p+\sum_{\iota=p+\varepsilon}^{\infty}a_\iota c_\iota w^\iota.\quad (w\in\mathcal{L}) \tag{3}$$

Definition 1. [1-5] For $g(w)\in\mathcal{A}_p$the generalized derivative operator $\mathrm{I}_{\xi,\eta}^{m,s}:\mathcal{A}_p\to\mathcal{A}_p$ is defined by

$$\mathrm{I}_{\xi,\eta}^{m,s}g(w)= w^p+\sum_{\iota=p+\varepsilon}^{\infty}\frac{\left(1+\xi(j-1)\right)^m}{\left(1+\eta(j-1)\right)^{m-1}}c(s,\iota)a_\iota w^\iota, \tag{4}$$

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where $s, m \in \mathbb{N}_0 = \{0, 1, \dots\}$, $\eta \ge \xi \ge 0$, and

$$c(s,\iota) = \binom{s+\iota-1}{s} = \frac{\Gamma(\iota+\varepsilon)}{\Gamma(\iota)} = \frac{(\iota+(\varepsilon-1))!}{\iota!\,(\varepsilon-1)!} = \frac{(s+1)_{\iota-1}}{(1)_{\iota-1}}.$$

It can easily be observed that

$$\mathrm{I}_{\xi,0}^{0,0}\mathcal{g}(w) = \mathrm{I}_{0,\eta}^{1,0}\mathcal{g}(w) = \mathcal{g}(w),$$

and

$$\mathrm{I}_{\xi,0}^{1,0}\mathcal{g}(w) = \mathrm{I}_{0,\eta}^{1,1}\mathcal{g}(w) = z\mathcal{g}'(w).$$

Also,

$$\mathrm{I}_{\xi,0}^{b-1,0}\mathcal{g}(w) = \mathrm{I}_{0,\eta}^{1,b-1}\mathcal{g}(w) \; where \; b = 1, 2, 3, \dots \; .$$

We can verify that

$$(1+s)\mathrm{I}_{\xi,\eta}^{m,s+1}\mathcal{g}(w) = z\left(\mathrm{I}_{\xi,\eta}^{m,s}\mathcal{g}(w)\right)' + s\left(\mathrm{I}_{\xi,\eta}^{m,s}\mathcal{g}(w)\right). \quad (5)$$

Definition 2. [6] For $\mathcal{g}(w) \in \mathcal{A}_p$ denoted by $D^{m+p-1}: \mathcal{A}_p \to \mathcal{A}_p$ the Ruscheweyh derivative of order $m+p-1$ is defined by

$$D^{m+p-1}\mathcal{g}(w) = \frac{w^p}{(1-w)^{\iota+p}} * \mathcal{g}(w) = \frac{w^p(w^{\iota-1}\mathcal{g}(w))^{m+p-1}}{(m+p-1)!} \quad (6)$$

if m is a grater of some integer than -p (see [6-10]).

Definition 3. For $\mathcal{g}(w) \in \mathcal{A}_p$the operator $\Upsilon_{\xi,\eta}^{m,s}: \mathcal{A}_p \to \mathcal{A}_p$ is defined by the Hadamard product of the generalized operator$\mathrm{I}_{\xi,\eta}^{m,s}$ and the Ruscheweyh derivative operatorD^{m+p-1}

$$\Upsilon_{\xi,\eta}^{m,s}\mathcal{g}(w) = \mathrm{I}_{\xi,\eta}^{m,s} * D^{m+p-1}\mathcal{g}(w)$$

$$\Upsilon_{\xi,\eta}^{m,s}\mathcal{g}(w) = w^p + \sum_{\iota=p+\varepsilon}^{\infty} \frac{\left(1+\xi(j-1)\right)^m}{\left(1+\eta(j-1)\right)^{m-1}} c(s,\iota)\mathrm{T}_\iota(\xi) a_\iota w^\iota, \quad (7)$$

where $s, m \in \mathbb{N}_0 = \{0, 1, \dots\}$, $\eta \ge \xi \ge 0$ and

$$\mathrm{T}_\iota(\xi) = \frac{\Gamma(\xi+\iota)}{\Gamma(\xi+p)(\iota-p)!}, \xi > -p.$$

Note that, the following are the unique operator$\Upsilon_{\xi,\eta}^{m,s}$ cases.

1. When $\mathrm{I}_{0,0}^{m,0} = 1$, include the Ruscheweyh derivative operator D^{m+p-1}[11].
2. When $D^{m+p-1} = 1$, include the Generalized Derivative operator $\mathrm{I}_{\xi,\eta}^{m,s}$ [1].
   - A. When $s = 0, \xi = 1, \eta = 0$, $\mathrm{I}_{\lambda,\eta}^{m,s}$ reduces to $\mathrm{I}_{1,0}^{m,0}$ which is introduced by Salagean Derivative operator[9].
   - B. When $s = 0, \eta = 0$, $\mathrm{I}_{\xi,\eta}^{m,s}$ reduces to$\mathrm{I}_{\xi,0}^{m,0}$ which is introduced by Generalized Salagean derivative operator introduced by Al-oboudi[12].
   - C. When $\eta = 0$, $\mathrm{I}_{\xi,\eta}^{m,s}$ reduces to $\mathrm{I}_{\xi,0}^{m,s}$ which is intr0duced by Generalized Al-Shaqsi and Darus Derivative operator[13].
   - D. When $\xi = 0, \eta = 0$, $\mathrm{I}_{\xi,\eta}^{m,s}$ reduces to $\mathrm{I}_{0,0}^{m,s}$ which is introduced by Srivastava Attiya Derivative operator [14].
   - E. When $m = 1 \; or \; m = 0, \xi = 0 \; or \; \eta = 0$, $\mathrm{I}_{\xi,\eta}^{m,s}$ reduces to $\mathrm{I}_{0,\eta}^{1,s} \equiv \mathrm{I}_{\lambda,0}^{0,s}$ which is introduced by Ruscheweyh Derivative operator[15].
   - F. When $m = 0 \; or \; m = 1$, $\mathrm{I}_{\xi,\eta}^{m,s}$ reduces to $\mathrm{I}_{\xi,\eta}^{0,s} \equiv \mathrm{I}_{\xi,\eta}^{1,s}$ which is introduced by Generalized Ruscheweyh Derivative operator[16].

Definition 4. Let the function $g(w)$ be of the form (1). Then $g(w)$ is said to be in the class $\mathrm{T}[\mu,\tau;p]$if it satisfies the following inequality:

$$\left|\frac{\frac{\mu w^2\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)''}{z\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)'}+1}{\frac{\mu w\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)'}{\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)}+(1-\mu)}-p\right|<\tau, \tag{8}$$

where $0\le\mu\le 1, 0<\tau\le 1, p\in\mathbb{N}=\{1,2,3,\dots\}, w\in\mathcal{L}$.

The normalized p-valent analytical functions of (for example) Srivastava and Patel[16], Sokol[17], Aouf[2],were extensively studied (see [7,8]).

First of all, we will deduce in this paper a necessary and adequate condition for a function $g(w)$ to be in class $\mathrm{T}[\mu,\tau;p]$.Then obtain for these functions the theorems of distortion and growth,closure theorems,neighborhood and radii of p-valent starlikeness,convexity and close-to-convexity of order $\sigma(0\le\sigma<1)$.

COEFFICIENT INEQUALITY

In this section, we provide an appropriate condition for a function g to be in class class$\mathrm{T}[\mu,\tau;p]$, which will function as one of the main findings of this paper to find other outcomes.

Theorem 1. A function $g(w)\in\mathcal{A}_p$ of the form (1) is in the class $\mathrm{T}[\mu,\tau;p]$ if it satisfies the following condition:

$$\sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\frac{\big(1+\xi(\iota-1)\big)^m}{\big(1+\eta(\iota-1)\big)^{m-1}}c(s,\iota)\mathrm{T}_\iota(\xi)|a_\iota|\le\tau\big(1+\mu(p-1)\big), \tag{9}$$

where

$$m\in\mathbb{N}_0=\{0,1,\dots\},\eta\ge\xi\ge 0, 0\le\mu\le 1, 0<\tau\le 1, p\in\mathbb{N}=\{1,2,3,\dots\}, w\in\mathcal{L}.$$

Proof. From Definition 1.4, we have

$$\left|\frac{\frac{\mu w^2\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)''}{z\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)'}+1}{\frac{\mu w\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)'}{\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)}+(1-\mu)}-p\right|<\tau$$

$$=\left|\frac{\mu w^2\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)''+w\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)'}{\mu w\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)'+(1-\mu)\left(\Upsilon_{\xi,\eta}^{m,s}g(w)\right)}-p\right|<\tau$$

$$=\left|\frac{\sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)(\iota-p)\frac{(1+\xi(\iota-1))^m}{(1+\eta(\iota-1))^{m-1}}c(s,\iota)\mathrm{T}_\iota(\xi)a_\iota w^\iota}{\big(1+\mu(p-1)\big)w^p+\sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)\frac{(1+\xi(\iota-1))^m}{(1+\eta(\iota-1))^{m-1}}c(s,\iota)\mathrm{T}_\iota(\xi)a_\iota w^\iota}\right|<\tau$$

$$\le\frac{\sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)(\iota-p)\frac{(1+\xi(\iota-1))^m}{(1+\eta(\iota-1))^{m-1}}c(s,\iota)\mathrm{T}_\iota(\xi)|a_\iota||w|^{\iota-p}}{\big(1+\mu(p-1)\big)+\sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)\frac{(1+\xi(\iota-1))^m}{(1+\eta(\iota-1))^{m-1}}c(s,\iota)\mathrm{T}_\iota(\xi)|a_\iota||w|^{\iota-p}}\le\tau. \tag{10}$$

We can see that by making w-1-through real values,

$$\sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\frac{\big(1+\xi(\iota-1)\big)^{m}}{\big(1+\eta(\iota-1)\big)^{m-1}}c(s,\iota)\mathrm{T}_{\iota}(\xi)|a_{\iota}|\le\tau\big(1+\mu(p-1)\big).$$

Thus, $g(w)\in\mathrm{T}[\mu,\tau;p]$.

The evidence is therefore complete.

Corollary 1. If a function $g(w)\in\mathcal{A}_p$ given by (1) is in the class $\mathrm{T}[\mu,\tau;p]$, then

$$|a_{\iota}|\le\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}{\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^{m}c(s,\iota)\mathrm{T}_{\iota}(\xi)}\ (\iota\ge p+\varepsilon,\varepsilon\in\mathbb{N}).\tag{11}$$

DISTORTION THEOREMS

Theorem 2. If $g(w)$ of the form (1) be in the class $\mathrm{T}[\mu,\tau;p]$, then for $|w|=r<1$, we have

$$r^{p}-\left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]r^{p+\varepsilon}\le|g(w)|,\tag{12}$$

and

$$|g(w)|\le r^{p}+\left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]r^{p+\varepsilon}.\tag{13}$$

The equalities in (12) and (13) are attained for the function $g(w)$ given by

$$g(w)=w^{p}+\left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]w^{p+\varepsilon}.\tag{14}$$

Proof. Via Theorem 2.1, we have

$$\sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\frac{\big(1+\lambda(\iota-1)\big)^{m}}{\big(1+\eta(\iota-1)\big)^{m-1}}c(s,\iota)\mathrm{T}_{\iota}(\xi)|a_{\iota}|\le\tau\big(1+\mu(p-1)\big).$$

Then, for $|w|=r<1$, we get

$$\begin{aligned}|g(w)|&\ge r^{p}-\sum_{\iota=p+\varepsilon}^{\infty}|a_{\iota}|\,r^{p+\varepsilon}\ge r^{p}-r^{p+\varepsilon}\sum_{\iota=p+\varepsilon}^{\infty}|a_{\iota}|\\&\ge r^{p}-\left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]r^{p+\varepsilon}.\end{aligned}$$

Also,

$$\begin{aligned}|g(w)|&\le r^{p}+\sum_{\iota=p+\varepsilon}^{\infty}|a_{\iota}|\,r^{p+\varepsilon}\le r^{p}+r^{p+\varepsilon}\sum_{\iota=p+\varepsilon}^{\infty}|a_{\iota}|\\&\le r^{p}+\left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]r^{p+\varepsilon}.\end{aligned}$$

The evidence is therefore complete.

Theorem 3. If $\mathcal{g}(w)$ of the form (1) be in the class $\mathrm{T}[\mu,\tau;p]$, then for $|w|=r<1$, we have

$$pr^{p-1}-\left[\frac{\tau(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]r^{p+\varepsilon-1}\le|\mathcal{g}'(w)|, \tag{15}$$

and

$$|\mathcal{g}'(w)|\le pr^{p-1}+\left[\frac{\tau(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]r^{p+\varepsilon-1}. \tag{16}$$

The equalities in (15) and (16) are attained for the function $\mathcal{g}(w)$ given by

$$\mathcal{g}'(w)=pw^{p-1}+\left[\frac{\tau(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]w^{p+\varepsilon-1}. \tag{17}$$

Proof. Since

$$|\mathcal{g}'(w)|\le p|w|^{p-1}-\sum_{\iota=p+\varepsilon}^{\infty}\iota\,|a_\iota||w|^{\iota-1}.$$

Form Theorem 2, we obtain

$$\sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\frac{\big(1+\xi(\iota-1)\big)^{m}}{\big(1+\eta(\iota-1)\big)^{m-1}}c(s,\iota)\mathrm{T}_\iota(\xi)|a_\iota|\le\tau\big(1+\mu(p-1)\big).$$

Then, for $|w|=r<1$, we obtain

$$|\mathcal{g}'(w)|\ge\ pr^{p-1}-r^{p+\varepsilon-1}\sum_{\iota=p+\varepsilon}^{\infty}\iota|a_\iota|$$
$$\ge pr^{p-1}-\left[\frac{\tau(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]r^{p+\varepsilon-1}.$$

We can get in a similar way,

$$|\mathcal{g}'(w)|\le p\,r^{p-1}+r^{p+\varepsilon-1}\sum_{\iota=p+\varepsilon}^{\infty}\iota|a_\iota|$$
$$\le pr^{p-1}+\left[\frac{\tau(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]r^{p+\varepsilon-1}.$$

So we have finished the proof of the theorem.

CLOSURE THEOREMS

Theorem 4. Let the functions $\mathcal{g}_e(e=1,2,\dots,l)$ defined by

$$\mathcal{g}_e(w)=w^p+\sum_{\iota=p+\varepsilon}^{\infty}|a_{\iota,e}|w^\iota,\quad\big(a_{\iota,e}\ge0\big) \tag{18}$$

be in the class $\mathrm{T}[\mu,\tau;p]$. Then the function $h(w)$ defined by

$$h(w) = \sum_{e=1}^{l} \beta_e \, g_e(w), \quad (\beta_e \geq 0) \tag{19}$$

is also in the class $\mathrm{T}[\mu, \tau; p]$, where

$$\sum_{e=1}^{l} \beta_e = 1.$$

Proof. We're able to compose

$$h(w) = \sum_{e=1}^{l} \beta_e \, w^p + \sum_{e=1}^{l} \sum_{\iota=p+\varepsilon}^{\infty} \beta_e |a_{\iota,e}| w^\iota$$
$$= w^p + \sum_{\iota=p+\varepsilon}^{\infty} \sum_{e=1}^{l} \beta_e |a_{\iota,e}| w^\iota.$$

In addition, since the functions $g_e (e = 1,2, \dots, l)$ are in the class $\mathrm{T}[\mu, \tau; p]$, then form Theorem 2.1, we have

$$\sum_{\iota=p+\varepsilon}^{\infty} \big(1 + \mu(\iota - 1)\big)(\iota - p - \tau) \frac{\big(1 + \xi(\iota - 1)\big)^m}{\big(1 + \eta(\iota - 1)\big)^{m-1}} c(s, \iota) \mathrm{T}_\iota(\xi) |a_\iota| \leq \tau\big(1 + \mu(p - 1)\big).$$

Thus, it is necessary to show that

$$\sum_{\iota=p+\varepsilon}^{\infty} \big(1 + \mu(\iota - 1)\big)(\iota - p - \tau) \frac{\big(1 + \xi(\iota - 1)\big)^m}{\big(1 + \eta(\iota - 1)\big)^{m-1}} c(s, \iota) \mathrm{T}_\iota(\xi) \left(\sum_{e=1}^{l} \beta_e |a_{\iota,e}|\right)$$
$$= \sum_{e=1}^{l} \beta_e \sum_{\iota=p+\varepsilon}^{\infty} \big(1 + \mu(\iota - 1)\big)(\iota - p - \tau) \frac{\big(1 + \xi(\iota - 1)\big)^m}{\big(1 + \eta(\iota - 1)\big)^{m-1}} c(s, \iota) \mathrm{T}_\iota(\xi) \, |a_{\iota,e}|$$
$$\leq \sum_{e=1}^{l} \beta_e \, \tau\big(1 + \mu(p - 1)\big) = \tau\big(1 + \mu(p - 1)\big).$$

The proof of the theorem is complete.

Corollary 2. Let the functions $g_e (e = 1,2)$ defined by (18) be in the class $\mathrm{T}[\mu, \tau; p]$. Then the function $h(w)$ defined by

$$h(w) = (1 - c) g_1(w) + c g_2(w) (0 \leq c \leq 1),$$

is also in the class $\mathrm{T}[\mu, \tau; p]$.

Proof. Form (18), we have

$$g_e(w) = w^p + \sum_{\iota=p+\varepsilon}^{\infty} |a_{\iota,e}| w^\iota, \quad (e = 1,2)$$

are in $\mathrm{T}[\mu, \tau; p]$. It is sufficient to demonstrate that h is a function defined by

$$h(w) = (1 - c) g_1(w) + c g_2(w) (0 \leq c \leq 1))$$

is in the class $\mathrm{T}[\mu, \tau; p]$, since

$$h(w) = w^p + \sum_{\iota=p+\varepsilon}^{\infty} \big[(1 - c)|a_{\iota,1}| + c|a_{\iota,2}|\big] w^\iota. \ (0 \leq c \leq 1) \tag{20}$$

In view of Theorem 2, we have:

$$\sum_{\iota=p+\varepsilon}^{\infty}\left(1+\mu(\iota-1)\right)(\iota-p-\tau)\frac{\left(1+\xi(\iota-1)\right)^{m}}{\left(1+\eta(\iota-1)\right)^{m-1}}c(s,\iota)\mathrm{T}_{\iota}(\xi)\left[(1-c)\left|a_{\iota,1}\right|+c\left|a_{\iota,2}\right|\right]$$

$$=(1-c)\sum_{\iota=p+\varepsilon}^{\infty}\left(1+\mu(\iota-1)\right)(\iota-p-\tau)\frac{\left(1+\xi(\iota-1)\right)^{m}}{\left(1+\eta(\iota-1)\right)^{m-1}}c(s,\iota)\mathrm{T}_{\iota}(\xi)\left|a_{\iota,1}\right|$$

$$+c\sum_{\iota=p+\varepsilon}^{\infty}\left(1+\mu(\iota-1)\right)(\iota-p-\tau)\frac{\left(1+\xi(\iota-1)\right)^{m}}{\left(1+\eta(\iota-1)\right)^{m-1}}c(s,\iota)\mathrm{T}_{\iota}(\xi)\left|a_{\iota,2}\right|$$

$$\leq(1-c)\tau\left(1+\mu(p-1)\right)+c\tau\left(1+\mu(p-1)\right)=\tau\left(1+\mu(p-1)\right),$$

which shows that $h(w)\in\mathrm{T}[\mu,\tau;p]$.

Theorem 5. Let

$$g_{p}(w)=w^{p}\ , \tag{21}$$

and

$$g_{\iota}(w)=w^{p}+\left[\frac{\tau\left(1+\mu(p-1)\right)\left(1+\eta(\iota-1)\right)^{m-1}}{\left(1+\mu(\iota-1)\right)(\iota-p-\tau)\left(1+\xi(\iota-1)\right)^{m}c(s,\iota)\mathrm{T}_{\iota}(\xi)}\right]w^{\iota}. \tag{22}$$

Then the function $g(w)$ of the form (1) is in the class$\mathrm{T}[\mu,\tau;p]$if and only if it is worthy of being articulated in the form:

$$g(w)=\beta_{p}g_{p}(w)+\sum_{\iota=p+\varepsilon}^{\infty}\beta_{\iota}\,g_{\iota}(w), \tag{23}$$

where $\beta_{p}\geq 0,\beta_{\iota}\geq 0,\iota\geq p+\varepsilon\ and\ \beta_{p}+\sum_{\iota=p+\varepsilon}^{\infty}\beta_{\iota}=1.$

Proof. Assume that the shape of $g(w)$ can be expressed in (23)

$$g(w)=\beta_{p}g_{p}(w)+\sum_{\iota=p+\varepsilon}^{\infty}\beta_{\iota}\,g_{\iota}(w)$$

$$=w^{p}+\sum_{\iota=p+\varepsilon}^{\infty}\frac{\tau\left(1+\mu(p-1)\right)\left(1+\eta(\iota-1)\right)^{m-1}}{\left(1+\mu(\iota-1)\right)(\iota-p-\tau)\left(1+\xi(\iota-1)\right)^{m}c(s,\iota)\mathrm{T}_{\iota}(\xi)}\beta_{\iota}w^{\iota}.$$

Therefore

$$\sum_{\iota=p+\varepsilon}^{\infty}\frac{\tau\left(1+\mu(p-1)\right)\left(1+\eta(\iota-1)\right)^{m-1}}{\left(1+\mu(\iota-1)\right)(\iota-p-\tau)\left(1+\xi(\iota-1)\right)^{m}c(s,\iota)\mathrm{T}_{\iota}(\xi)}$$

$$\times\frac{\left(1+\mu(\iota-1)\right)(\iota-p-\tau)\left(1+\xi(\iota-1)\right)^{m}c(s,\iota)\mathrm{T}_{\iota}(\xi)}{\tau\left(1+\mu(p-1)\right)\left(1+\eta(\iota-1)\right)^{m-1}}\beta_{\iota}$$

$$\sum_{\iota=p+\varepsilon}^{\infty}\beta_{\iota}=1-\beta_{p}\leq 1.$$

Hence $g(w)\in\mathrm{T}[\mu,\tau;p]$.

Conversely, assume that $g(w) \in \mathrm{T}[\mu,\tau;p]$.
Setting

$$\beta_\iota = \frac{\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^m c(s,\iota)\mathrm{T}_\iota(\xi)}{\tau\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}|a_\iota|, \quad (24)$$

since

$$\beta_p = 1 - \sum_{\iota=p+\varepsilon}^{\infty} \beta_\iota.$$

Therefore

$$g(w) = \beta_p g_p(w) + \sum_{\iota=p+\varepsilon}^{\infty} \beta_\iota\, g_\iota(w).$$

The evidence is therefore complete.

Corollary 3. The functions of the extreme points of class $\mathrm{T}[\mu,\tau;p]$ are

$$g_p(w) = w^p\ ,$$

and

$$g_\iota(w) = w^p + \left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}{\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^m c(s,\iota)\mathrm{T}_\iota(\xi)}\right] w^\iota.$$

INCLUSION AND NEIGHBORHOOD RESULTS

Following the work of [4,14,17,13]:

We identify the $(\varepsilon,\sigma)-$ neighborhood of a function $g(w) \in \mathcal{A}_p$ by

$$N_{\varepsilon,\sigma}(g) = \left\{\hbar \in \mathcal{A}_p : \hbar(w) = w^p + \sum_{\iota=p+\varepsilon}^{\infty} b_\iota w^p \ \ and \ \sum_{\iota=p+\varepsilon}^{\infty} \iota|a_\iota - b_\iota| \le \sigma\right\} \quad (25)$$

In particular, for $e(w) = w^p$

$$N_{\varepsilon,\sigma}(e) = \left\{\hbar \in \mathcal{A}_p : \hbar(w) = w^p + \sum_{\iota=p+\varepsilon}^{\infty} b_\iota w^p \ \ and \ \sum_{\iota=p+\varepsilon}^{\infty} \iota|b_\iota| \le \sigma\right\}. \quad (26)$$

In addition, form (1) a function is said to be in the class $\mathrm{T}^\gamma[\mu,\tau;p]$ if there is a $h(w) \in \mathrm{T}[\mu,\tau;p]$ function, such that

$$\left|\frac{g(w)}{h(w)} - 1\right| < p - \gamma, \qquad 0 \le \gamma < p.$$

Theorem 6. If

$$\frac{\big(1+\xi(\iota-1)\big)^m c(s,\iota)}{\big(1+\eta(\iota-1)\big)^{m-1}}\mathrm{T}_\iota(\xi) \ge \frac{\big(1+\xi(p+\varepsilon-1)\big)^m c(s,p+\varepsilon)}{\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}\mathrm{T}_{p+\varepsilon}(\xi), \quad (\ \iota \ge p+\varepsilon, \varepsilon \in \mathbb{N})$$

and

$$\sigma = \frac{\tau(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^m c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)},$$

then

$$\mathrm{T}[\mu,\tau;p] \subset N_{\varepsilon,\sigma}(e) .$$

Proof. Suppose $g(w) \in \mathrm{T}[\mu,\tau;p]$. Then we get into the use of Theorem 2.1 and the hypothesis

$$\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\frac{\big(1+\xi(p+\varepsilon-1)\big)^m}{\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)\sum_{\iota=p+\varepsilon}^{\infty}|a_\iota|$$

$$\leq \sum_{\iota=p+\varepsilon}^{\infty}\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\frac{\big(1+\xi(\iota-1)\big)^m}{\big(1+\eta(\iota-1)\big)^{m-1}}c(s,\iota)\mathrm{T}_\iota(\xi)|a_\iota| \leq \tau\big(1+\mu(p-1)\big),$$

hence

$$\sum_{\iota=p+\varepsilon}^{\infty}|a_\iota| \leq \frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^m c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)} . \tag{27}$$

Thus

$$\sum_{\iota=p+\varepsilon}^{\infty}\iota|a_\iota| \leq \frac{\tau(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^m c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)} = \sigma.$$

Hence $g(w) \in N_{\varepsilon,\sigma}(e)$.
The evidence is therefore complete.

Theorem 7. If $h(w) \in \mathrm{T}[\mu,\tau;p]$, then

$$N_{\varepsilon,\sigma}(h) \subset \mathrm{T}^{\gamma}[\mu,\tau;p],$$

where

$$\gamma = p - \frac{\sigma}{(p+\varepsilon)} \times \left[\frac{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^m c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^m c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi) - \big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}\right].$$

Proof. Assume that $g(w) \in N_{\varepsilon,\sigma}(h)$, then

$$\sum_{\iota=p+\varepsilon}^{\infty}\iota|a_\iota - b_\iota| \leq \sigma$$

Thus

$$\sum_{\iota=p+\varepsilon}^{\infty}|a_\iota - b_\iota| \leq \frac{\sigma}{p+\varepsilon}.$$

Since $h(w) \in \mathrm{T}[\mu,\tau;p]$, then form (27), we have

$$\sum_{\iota=p+\varepsilon}^{\infty}|b_\iota| \leq \frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^m c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}.$$

It is sufficient to demonstrate that

$$\left|\frac{g(w)}{h(w)} - 1\right| \leq \frac{\sum_{\iota=p+\varepsilon}^{\infty}|a_\iota - b_\iota|}{1-\sum_{\iota=p+\varepsilon}^{\infty}|b_\iota|} \leq \frac{\sigma}{(p+\varepsilon)\left[1 - \frac{\tau\left(1+\mu(p-1)\right)\left(1+\eta(p+\varepsilon-1)\right)^{m-1}}{\left(1+\mu(p+\varepsilon-1)\right)(\varepsilon-\tau)\left(1+\xi(p+\varepsilon-1)\right)^m c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}\right]}$$

$$\leq \frac{\sigma}{(p+\varepsilon)} \times$$

$$\left[\frac{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)-\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}\right]$$

$$=p-\gamma.$$

Therefore

$$\gamma=p-\frac{\sigma}{(p+\varepsilon)}\times$$
$$\left[\frac{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\lambda(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)}{\big(1+\mu(p+\varepsilon-1)\big)(\varepsilon-\tau)\big(1+\xi(p+\varepsilon-1)\big)^{m}c(s,p+\varepsilon)\mathrm{T}_{p+\varepsilon}(\xi)-\big(1+\mu(p-1)\big)\big(1+\eta(p+\varepsilon-1)\big)^{m-1}}\right].$$

This makes the proof of the theorem complete.

RADII OF STARLIKENESS, CONVEXITY AND CLOSE-TO-CONVEXITY

Theorem 8. Let the function $g(w)$, given by (1) , be in the class $\mathrm{T}[\mu,\tau;p]$. Then $g(w)$ is a p-valent starlike function of order $\sigma(0\leq\sigma<p)$for $|w|\leq r_1$, where

$$r_1=\inf_{\iota}\left[\frac{(p-\sigma)\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^{m}c(s,\iota)\mathrm{T}_{\iota}(\xi)}{\tau(j-\sigma)\big(1+\mu(p-1)\big)\big(1+\eta(j-1)\big)^{m-1}}\right]^{\frac{1}{\iota-p}}. \qquad (28)$$

The outcome is sharp for the function$g_j(w)$given by

$$g_{\iota}(w)=w^{p}+\left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}{\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^{m}c(s,\iota)\mathrm{T}_{\iota}(\xi)}\right]w^{\iota}.$$

Proof. Suffice it to demonstrate that

$$\left|\frac{wg'(w)}{g(w)}-p\right|\leq p-\sigma. \quad (|w|<r_1)$$

Since

$$\left|\frac{wg'(w)}{g(w)}-p\right|=\left|\frac{\sum_{\iota=p+\varepsilon}^{\infty}(\iota-p)a_{\iota}w^{\iota-p}}{1+\sum_{\iota=p+\varepsilon}^{\infty}a_{\iota}w^{\iota-p}}\right|\leq\frac{\sum_{\iota=p+\varepsilon}^{\infty}(\iota-p)|a_{\iota}||w|^{\iota-p}}{1-\sum_{\iota=p+\varepsilon}^{\infty}|a_{\iota}||w|^{\iota-p}}.$$

We need to demonstrate that, in order to prove the theorem,

$$\frac{\sum_{\iota=p+\varepsilon}^{\infty}(\iota-p)|a_{\iota}||w|^{\iota-p}}{1-\sum_{\iota=p+\varepsilon}^{\infty}|a_{\iota}||w|^{\iota-p}}\leq p-\sigma.$$

It is comparable to

$$\sum_{\iota=p+\varepsilon}^{\infty}(\iota-\sigma)|a_{\iota}||w|^{\iota-p}\leq p-\sigma,$$

via Theorem 2, we get

$$|w|\leq\left\{\frac{(p-\sigma)\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^{m}c(s,\iota)\mathrm{T}_{\iota}(\xi)}{\tau(\iota-\sigma)\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}\right\}^{\frac{1}{\iota-p}}.$$

The evidence is therefore complete.

Theorem 9. Let the function $g(w)$, given by (1) , be in the class $T[\mu,\tau;p]$. Then $g(w)$ is a p-valent convex function of order $\sigma(0 \le \sigma < p)$for $|w| \le r_2$, where

$$r_2 = \inf_{\iota} \left[\frac{p(p-\sigma)\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^m c(s,\iota)T_\iota(\xi)}{\tau(\iota-\sigma)(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}\right]^{\frac{1}{\iota-p}}. \quad (29)$$

The outcome is sharp for the function $g_j(w)$given by

$$g_\iota(w) = w^p + \left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}{\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^m c(s,\iota)T_\iota(\xi)}\right] w^\iota.$$

Proof. Suffice it to demonstrate that

$$\left|1+\frac{wg''(w)}{g'(w)}-p\right| \le p-\sigma. \quad (|w|<r_2)$$

Since

$$\left|1+\frac{wg''(w)}{g'(w)}-p\right| = \left|\frac{\sum_{\iota=p+\varepsilon}^{\infty}\iota(\iota-p)a_\iota w^{\iota-p}}{p+\sum_{\iota=p+\varepsilon}^{\infty}\iota a_\iota w^{\iota-p}}\right| \le \frac{\sum_{\iota=p+\varepsilon}^{\infty}\iota(\iota-p)|a_\iota||w|^{\iota-p}}{1-\sum_{\iota=p+\varepsilon}^{\infty}\iota|a_\iota||w|^{\iota-p}}.$$

We need to demonstrate that, in order to prove the theorem,

$$\frac{\sum_{\iota=p+\varepsilon}^{\infty}\iota(\iota-p)|a_\iota||w|^{\iota-p}}{1-\sum_{\iota=p+\varepsilon}^{\infty}\iota|a_\iota||w|^{\iota-p}} \le p-\sigma.$$

It is comparable to

$$\sum_{\iota=p+\varepsilon}^{\infty}\iota(\iota-\sigma)|a_\iota||w|^{\iota-p} \le p-\sigma,$$

using Theorem 2, we obtain

$$|w|^{\iota-p} \le \left\{\frac{p(p-\sigma)\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^m c(s,\iota)T_\iota(\xi)}{\tau(\iota-\sigma)(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}\right\},$$

or

$$|w| \le \left\{\frac{p(p-\sigma)\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^m c(s,\iota)T_\iota(\xi)}{\tau(\iota-\sigma)(p+\varepsilon)\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}\right\}^{\frac{1}{\iota-p}}.$$

The evidence is therefore complete.

Theorem 10. Let the function $g(w)$, given by (1) , be in the class $T[\mu,\tau;p]$. Then $g(w)$ is a p-valent close-to-convex function of order $\sigma(0 \le \sigma < p)$for $|w| \le r_3$, where

$$r_3 = \inf_{\iota} \left[\frac{\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^m c(s,\iota)T_\iota(\xi)}{\iota\tau\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}\right]^{\frac{1}{\iota-p}}. \quad (30)$$

The outcome is sharp for the function $g_j(w)$given by

$$g_\iota(w) = w^p + \left[\frac{\tau\big(1+\mu(p-1)\big)\big(1+\eta(\iota-1)\big)^{m-1}}{\big(1+\mu(\iota-1)\big)(\iota-p-\tau)\big(1+\xi(\iota-1)\big)^m c(s,\iota)T_\iota(\xi)}\right] w^\iota.$$

Proof. Suffice it to demonstrate that

$$\left|\frac{g'(w)}{w^{p-1}}-p\right| \le p-\sigma. \quad (|w|<r_3)$$

Then

$$\left|\frac{g'(w)}{w^{p-1}} - p\right| = \left|\sum_{\iota=p+\varepsilon}^{\infty} \iota a_\iota w^{\iota-p}\right| \leq \sum_{\iota=p+\varepsilon}^{\infty} \iota |a_\iota||w|^{\iota-p}.$$

Using Theorem 2, we obtain

$$|w|^{\iota-p} \leq \left\{\frac{(1+\mu(\iota-1))(\iota-p-\tau)(1+\xi(\iota-1))^m c(s,\iota)\mathrm{T}_\iota(\xi)}{\iota\tau(1+\mu(p-1))(1+\eta(\iota-1))^{m-1}}\right\},$$

or

$$|w| \leq \left\{\frac{(1+\mu(\iota-1))(\iota-p-\tau)(1+\xi(\iota-1))^m c(s,\iota)\mathrm{T}_\iota(\xi)}{\iota\tau(1+\mu(p-1))(1+\eta(\iota-1))^{m-1}}\right\}^{\frac{1}{\iota-p}}.$$

The evidence is therefore complete.

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