Wiley
Journal of Function Spaces
Volume 2024, Article ID 5532624, 9 pages
https://doi.org/10.1155/2024/5532624

Research Article

New Results of Fixed-Point Theorems and Their Applications in Complete Complex $\mathscr{D}_c^*$-Metric Spaces

Nassar Aiman Majid,[1,2] **Alaa. M. F. AL. Jumaili**,[3] **Zhen Chuan Ng**,[1] **and See Keong Lee**[1]

[1]*School of Mathematical Sciences, Universiti Sains Malaysia, George Town, 11800 Penang, Malaysia*
[2]*Department of Ecology Sciences, College of Applied Sciences-Hit, University of Anbar, Ramadi, Iraq*
[3]*Department of Mathematics, College of Education for Pure Sciences, University of Anbar, Ramadi, Iraq*

Correspondence should be addressed to Nassar Aiman Majid; aimanalhity@uoanbar.edu.iq

Received 26 March 2024; Revised 16 August 2024; Accepted 27 August 2024

Academic Editor: Bilal Bilalov

The first objective of the present manuscript is to introduce the notion of complex-valued D_c^*-metric spaces as generalizing and improving the idea of $\mathscr{D}^*$-metric spaces by using the context of complex-valued metric space and $\mathscr{D}^*$-metric spaces. The principle of contraction mappings has recently been established. Furthermore, the second objective is devoted to establishing various new fixed-point theorems in complete complex-valued $\mathscr{D}_c^*$-metric spaces that are improved and generalized by Azam et al. Banach's contraction principle and various distinguished outcomes are also presented in the literatures. Additionally, some implementations of common fixed-point theory on complete complex-valued $\mathscr{D}_c^*$-metric spaces have been offered by applying open and closed balls. Additionally, we provide some illustrative examples as an implementation for our main results.

Keywords: complex D^*-metric spaces; D^*-metric spaces; contraction mappings; coupled fixed-point

1. Introduction and Preliminaries

The metric spaces provide a more general setting for researchers in various fields such as pure and applied mathematical analysis, mathematical modelling, optimization, and economic theories. As motivated by the impact of metric spaces to mathematical analysis, various researchers extended the concept of metric spaces to various other abstract spaces such as G-metric, G_b-metric, and cone metric $\mathscr{D}^*$-metric and complex $\mathscr{D}_c^*$-metric spaces. On the other hand, the complex-valued metric spaces are advantageous in several branches of pure and applied mathematics, including algebraic geometry and number theory. Banach's contraction principle provides a technique for solving a variety of pure and applied problems in mathematical sciences, physics, and engineering and has been expanded and enhanced in various directions, and it is commonly cited as the origin of metric fixed-point theory. Thus, the Banach contraction principle has several extensions. In 2004, Ran and Reurings [1] considered an important extension of the Banach contraction principle, and Nieto and Rodriguez-Lopez in [2] described and highlighted the presence and uniqueness of fixed-points for this contractive condition for the equivalent elements of X. It also presented the Banach contraction principle in a metric space endowed with a partial order. Following that, other writers took into account various reports of integral contractions and came up with fixed-point conclusions on these contractions in various metric spaces [3, 4]. The concept of *complex metric space* began via Azam, Fisher, and Khan [5] to take advantage of the notion of complex-valued Hilbert and complex-valued normed spaces to prove the existence of fixed-points under certain contractive condition. Sitthikul and Saejung [6] verified various fixed-point outcomes via extending the contractive conditions in setting of complex spaces. Several authors contributed diverse ideas in complex metric spaces, and we refer readers to see [7–12]. After that, Öztürk and Kaplan [13] proved the existence and uniqueness of common-coupled fixed-points for maps in the setting of complex-valued G_b-metric. As well, in the same year, Dubey, Shukla, and Dubey [14] obtained some fixed-point results for a map satisfying rational expression in complex-valued b-metric space. Subsequently, Singh et al. [15] established several common fixed-points of pair of maps satisfying more general contractive conditions portrayed via rational expression in complex metric spaces. In 2018, several ideas, such as coincidence points and compatible and occasionally weakly compatible maps, were presented via Ege and Karaca [16] in the setting of complex-valued G_b-metric space. Recently, Rashwan, Hammad, and Mahmoud [17] established several common fixed-point theorems utilizing four weakly compatible maps in complex-valued G-metric space. The purpose of the present manuscript is to present and study the complex-valued $\mathscr{D}_c^*$-metric spaces and establish a contraction principle. In addition, we verify various novel fixed-point results in complete complex $\mathscr{D}_c^*$-metric that are extended and generalized to Azam et al.'s, Banach's contraction principle, and various distinguished outcomes in the literatures. Moreover, in view of closed balls, various implementations of common fixed-point theory on complete complex $\mathscr{D}_c^*$-metric spaces have been introduced. In our research, first, we introduce different types of definitions and essential conclusions by using the term of $\mathscr{D}^*$-metric spaces, since we put in our consideration that such type of examining will provide the readers a chance to be fully understand in the following subsequent sections.

In 2007, Shaban, Nabi, and Haiyun [18] presented a new version of generalized metric spaces, namely, $\mathscr{D}^*$-metric spaces, as follows.

Definition 1. Let $\mathscr{X}$ be a nonempty set. The $\mathscr{D}^*$-metric is a function $\mathscr{D}^* : \mathscr{X}^3 \longrightarrow [0,\infty)$,that verification of the next statements:

$(\mathscr{D}_1^*)$ $\mathscr{D}^*(\mathrm{x}, y, z) \geq 0$,for each $\mathrm{x}, y, z \in \mathscr{X}$;

$(\mathscr{D}_2^*)$ $\mathscr{D}^*(\mathrm{x}, y, z) = 0 \Longleftrightarrow z = y = x$;

$(\mathscr{D}_3^*)$ $\mathscr{D}^*(x, y, z) = \mathscr{D}^*(\mathscr{P}\{x, y, z\})$, where $\mathscr{P}$ a permutation function,

$(\mathscr{D}_4^*)$ $\mathscr{D}^*(x, y, z) \leq \mathscr{D}^*(x, y, \mathrm{b}) + \mathscr{D}^*(\mathrm{b}, z, z)$, for each $x, y, z, \mathrm{b} \in \mathscr{X}$.

In that case, the function $\mathscr{D}^*$ is called an $\mathscr{D}^*$-metric, and the pair $(\mathscr{X}, \mathscr{D}^*)$ is called an $\mathscr{D}^*$-metric space.

Remark 1. In a $\mathscr{D}^*$-metric space, $\forall x, y \in \mathscr{X}, \mathscr{D}^*(x, x, y) = \mathscr{D}^*(x, y, y)$.

Example 1. Direct examples of such a function are the following:

1. $\mathscr{D}^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
2. $\mathscr{D}^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on $\mathscr{X}$.

Definition 2. Suppose that $z_1 \& z_2 \in \mathbb{C}$. A partial order $\precsim$ on $\mathbb{C}$ is defined as $z_1 \precsim z_2$ iff $\mathscr{R}e(z_1) \leq \mathscr{R}e(z_2) \& Im(z_1) \leq \mathrm{I}m(z_2)$.

It follows that $z_1 \precsim z_2$ if the following conditions are satisfied:

$(\mathscr{C}_1)$ $\mathscr{R}e(z_1) = \mathscr{R}e(z_2)$ and $\mathrm{I}m(z_1) = \mathrm{I}m(z_2)$;

$(\mathscr{C}_2)$ Re $(z_1) <$ Re (z_2) and $\mathrm{I}m(z_1) = \mathrm{I}m(z_2)$;

$(\mathscr{C}_3)$ Re $(z_1) = \mathscr{R}e(z_2)$ and $\mathrm{I}m(z_1) < \mathrm{I}m(z_2)$;

$(\mathscr{C}_4)$ Re $(z_1) <$ Re (z_2) and $\mathrm{I}m(z_1) < \mathrm{I}m(z_2)$.

In private, write $z_1 \precnsim z_2$ if $z_1 \neq z_2$, one of $(\mathscr{C}_2)$, $(\mathscr{C}_3)$, & $(\mathscr{C}_4)$ is satisfied, and $z_1 \prec z_2$ if only $(\mathscr{C}_4)$ is satisfied.

Remark 2. We acquired that the following statements hold:

(i) If $a, \mathrm{b} \in \mathbb{R}$ and $a \leq \mathrm{b} \Longrightarrow |az \prec \mathrm{b}z| \forall z \in \mathbb{C}$.

(ii) If $0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$.

(iii) If $z_1 \precsim z_2 \& z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$.

Azam, Fisher, and Khan [5] extended the notion of abstract metric spaces by proposing the complex-valued metric spaces and provide several fixed-point results for maps satisfying a rational inequality to exploit the notion of complex-Hilbert and complex-normed spaces as follows.

Definition 3. Let $\mathscr{X}$ be a nonempty set. Suppose that the function $\mathscr{D}^* : \mathscr{X}^2 \longrightarrow \mathbb{C}$, satisfies the following conditions:

$(\mathscr{C}_1)$ $0 \preccurlyeq \mathrm{d}(x, y)$, for each $x, y \in \mathscr{X}$ and $\mathrm{d}(x, y) = 0$ iff $x = y$,

$(\mathscr{C}_2)$ $\mathrm{d}(x, y) = \mathrm{d}(y, x)$, for each $x, y \in \mathscr{X}$,

$(\mathscr{C}_3)$ $\mathrm{d}(x, y) \preccurlyeq \mathrm{d}(x, z) + \mathrm{d}(z, y)$, for each $x, y, z \in \mathscr{X}$.

In that case, d is regarded to be a complex-valued metric on $\mathscr{X}$, while $(\mathscr{X}, \mathrm{d})$ is referred to as a complex-valued metric space.

Next, we present the idea of complex-valued $\mathscr{D}^*$-metric space similar to the concept of complex metric space [5] to extend the idea of complex metric and $\mathscr{D}^*$-metric spaces as follows.

Definition 4. Let $\mathscr{X}$ be a nonempty set. A complex $\mathscr{D}_c^*$-metric on $\mathscr{X}$ is a function $\mathscr{D}_c^* : \mathscr{X}^3 \longrightarrow \mathbb{C}$, that satisfies the following conditions:

$(\mathscr{C}\mathscr{D}_1^*)$ $0 \precsim \mathscr{D}_c^*(x, y, z)$, for each $x, y, z \in \mathscr{X}$;

$(\mathscr{C}\mathscr{D}_2^*)$ $\mathscr{D}_c^*(\mathrm{x}, y, z) = 0$, if and only if $x = y = z$;

$(\mathscr{C}\mathscr{D}_3^*)$ $\mathscr{D}_c^*(x, y, z) = \mathscr{D}_c^*(\mathscr{P}\{x, y, z\})$, where $\mathscr{P}$ a permutation function,

$(\mathscr{C}\mathscr{D}_4^*)$ $\mathscr{D}_c^*(x, y, z) \precsim \mathscr{D}_c^*(x, y, \mathrm{b}) + \mathscr{D}_c^*(\mathrm{b}, z, z)$, for each $x, y, z, \mathrm{b} \in \mathscr{X}$.

In that case, $\mathscr{D}_c^*$ is said to be a complex-valued $\mathscr{D}^*$-metric, and the pair $(\mathscr{X}, \mathscr{D}_c^*)$ is said to be a complex-valued $\mathscr{D}^*$-metric space.

Remark 3. The following proposition follows readily by utilizing Definition (2), part $(\mathscr{C}\mathscr{D}_4^*)$.

Lemma 1. *Assume that $(\mathscr{X}, \mathscr{D}_c^*)$ is complex $\mathscr{D}^*$-metric space. Then, the next statements hold for every $x, y, z \in \mathscr{X}$:*

1. $\mathscr{D}_c^*(x, y, z) \precsim \mathscr{D}_c^*(x, y, y) + \mathscr{D}_c^*(x, z, z)$,
2. $\mathscr{D}_c^*(x, x, y) \precsim 2\mathscr{D}_c^*(x, y, x)$.

3. $\mathscr{D}_c^*(x, x, y) = \mathscr{D}_c^*(x, y, y)$.

Definition 5. Presume that $(\mathscr{X}, \mathscr{D}_c^*)$ is a complex-valued $\mathscr{D}^*$-metric space; then:

1. A sequence $\{x_s\}$ is complex $\mathscr{D}^*$-convergent to point $x \in \mathscr{X}$ if for each $c \in \mathbb{C}$&$0 \prec c$, $\exists$ a positive integer s_0 such that $\mathscr{D}_c^*(x, x_s, x_r) \prec c$ for each $s, r \geq s_0$; we refer to x as the limit point of $\{x_s\}$ and write $x_s \longrightarrow x$.
2. A sequence $\{x_s\}$ is a complex $\mathscr{D}^*$-Cauchy sequence, if $\forall c \in \mathbb{C}$, and $0 \prec c$, $\exists$ is a positive integer s_0, where $\forall s, r \geq s_0$, $\mathscr{D}^*(x_s, x_s, x_r) \prec c$.
3. A complex $\mathscr{D}^*$-metric $(\mathscr{X}, \mathscr{D}_c^*)$ is called complex-valued $\mathscr{D}^*$-complete if each complex-valued $\mathscr{D}^*$-Cauchy sequence is complex $\mathscr{D}^*$-convergent of $(\mathscr{X}, \mathscr{D}_c^*)$.

Definition 6. Suppose $(\mathscr{X}, \mathscr{D}_c^*)$ is complex $\mathscr{D}_c^*$-metric. For $0 \prec m$ and $x \in \mathscr{X}$, the open ball $\mathscr{B}_{D^*}^c(x, m)$ and closed ball $\mathscr{B}_{D^*}^c[x, m]$ with center x and radius m are described as follows:

$$\mathscr{B}_{\mathscr{D}^*}^c(x, m) = \{y \in \mathscr{X} : \mathscr{D}_c^*(y, y, x) \prec m\},$$
$$\mathscr{B}_{D^*}^c[x, m] = \{y \in \mathscr{X} : \mathscr{D}_c^*(y, y, x) \precsim m\}.$$

1. $x \in \mathscr{X}$ is said to be interior point of $\mathscr{A} \subseteq \mathscr{X}$, if $\exists 0 \prec m \in \mathbb{C}$, where

$$\mathscr{B}_{\mathscr{D}^*}^c(x, m) \subseteq \mathscr{A}.$$

2. A point $x \in \mathscr{X}$ is said to be limit point of $\mathscr{A}$ whenever $\mathscr{B}_{\mathscr{D}^*}^c(x, m) \bigcap (\mathscr{A} - \{x\}) \neq \varnothing$, for each $0 \prec m \in \mathbb{C}$.

Definition 7. A subset $\mathscr{A}$ of complex-valued $\mathscr{D}^*$-metric space $(\mathscr{X}, \mathscr{D}_c^*)$ is called:

1. Open when every element of $\mathscr{A}$ is interior point of $\mathscr{A}$.
2. Close whenever every limit point of $\mathscr{A}$ belongs to $\mathscr{A}$.

Definition 8. Suppose that $(\mathscr{X}, \mathscr{D}_c^*)$&$(\mathscr{X}^*, \mathscr{D}_c^{**})$ are complex $\mathscr{D}^*$-metric spaces. In that case, $F : (\mathscr{X}, \mathscr{D}_c^*) \longrightarrow (\mathscr{X}^*, \mathscr{D}_c^{**})$ is complex-valued $\mathscr{D}^*$-continuous at $a_o \in \mathscr{X}$ if $F^{-1}(\mathscr{B}_{\mathscr{D}_c^{**}}(Fa_o, m))$ is open in $(\mathscr{D}_c^*) \forall m > 0$. We say F is complex-valued $\mathscr{D}^*$-continuous if F is $\mathscr{D}^*$-continuous at each point of $\mathscr{X}$.

From the fact that complex $\mathscr{D}^*$-metric topologies are metric topologies, the next lemma follows.

Lemma 2. *Suppose that $(\mathscr{X}, \mathscr{D}_c^*)$&$(\mathscr{X}^*, \mathscr{D}_c^{**})$ are two complex $\mathscr{D}^*$-metric spaces. In that case, $F : (\mathscr{X}, \mathscr{D}_c^*) \longrightarrow (\mathscr{X}^*, \mathscr{D}_c^{**})$ is complex $\mathscr{D}^*$-continuous at $a_o \in \mathscr{X}$ iff it is complex $\mathscr{D}^*$-sequentially continuous at a_o; that is, when a sequence $\{x_s\}$ is complex $\mathscr{D}^*$-convergent to point a_o, obtained Fx_s is complex $\mathscr{D}^*$-convergent to point Fa_o.*

2. First Results in Complex-Valued $\mathscr{D}_c^*$-Metric Spaces

This section is devoted to establishing the principle of contraction mappings in complex $\mathscr{D}_c^*$-metric space.

Proposition 1. *If $(\mathscr{X}, \mathscr{D}_c^*)$ is a complex-valued $\mathscr{D}_c^*$-metric space and $\{x_s\}$ is a sequence in $\mathscr{X}$, so $\{x_s\}$ is complex-valued $\mathscr{D}_c^*$-convergent to x iff $|\mathscr{D}^*(x, x_s, x_r)| \longrightarrow 0$ as $s, r \longrightarrow \infty$.*

Proof 1. Assume that a sequence $\{x_s\}$ is complex $\mathscr{D}_c^*$-convergent to $x \in \mathscr{X}$. Let

$$o = \frac{\mu}{\sqrt{2}} + i\frac{\mu}{\sqrt{2}}, \text{for } \mu > 0,$$

Therefore, $0 \prec o \in \mathbb{C}$, and $\hbar \in \mathbb{N}$ such that $\mathscr{D}^*(x, x_s, x_r) \prec o$ for all $s, r \geq \hbar$. Consequently, $|\mathscr{D}^*(x, x_s, x_r)| < |o| = \mu$ for all $s, r \geq \hbar$. It follows that $|\mathscr{D}^*(x, x_s, x_r)| \longrightarrow 0$ as $s, r \longrightarrow \infty$.

Conversely, assume that $|\mathscr{D}^*(x, x_s, x_r)| \longrightarrow 0$ as $s, r \longrightarrow \infty$. Then, given that $o \in \mathbb{C}$ with $0 \prec o, \exists \delta > 0$ is a real number (s.t.) for $z \in \mathbb{C}$, $|z| < \delta \Longrightarrow z \prec o$.

For δ, there is $\hbar \in \mathbb{N}$ such that $|\mathscr{D}^*(x, x_s, x_r)| < \delta$ for all $s, r \geq \hbar$. This means that $\mathscr{D}^*(x, x_s, x_r) \prec o$ for all $s, r \geq \hbar$. Therefore, $\{x_s\}$ is complex-valued $\mathscr{D}_c^*$-convergent to x. □

Proposition 2. *Let $(\mathscr{X}, \mathscr{D}^*)$ be a complex-valued $\mathscr{D}_c^*$-metric space; then, for a sequence $\{x_s\}$ in $\mathscr{X}$ and point $x \in \mathscr{X}$. The following are equivalent:*

1. *$\{x_s\}$ is complex-valued $\mathscr{D}^*$-convergent to x.*
2. *$|\mathscr{D}^*(x_s, x, x)| \longrightarrow 0$ as $s \longrightarrow \infty$.*
3. *$|\mathscr{D}^*(x_s, x_s, x)| \longrightarrow 0$ as $s \longrightarrow \infty$.*
4. *$|\mathscr{D}^*(x, x_s, x_r)| \longrightarrow 0$ as $r, s \longrightarrow \infty$.*

Proof 2. The proof is a consequence of Lemma 1 and Propositions 1. □

Proposition 3. *If $(\mathscr{X}, \mathscr{D}^*)$ is complex-valued $\mathscr{D}_c^*$-metric space and $\{x_s\}$ is a sequence in $\mathscr{X}$, so $\{x_s\}$ is complex $\mathscr{D}_c^*$-Cauchy iff $|\mathscr{D}^*(x_s, x_s, x_r)| \longrightarrow 0$ as $s, r \longrightarrow \infty$.*

Proof 3. Assume $\{x_s\}$ is a complex $\mathscr{D}_c^*$-Cauchy sequence. Let

$o = (\mu/\sqrt{2}) + i(\mu/\sqrt{2})$, for a real number $\mu > 0$,

Then, $0 \prec o \in \mathbb{C}$, and there is $\hbar \in \mathbb{N}$ such that $\mathscr{D}^*(x_s, x_s, x_r) \prec o \forall s, r \geq \hbar$. Consequently, $|\mathscr{D}^*(x_s, x_s, x_r)| < |o| = \mu \forall s, r \geq \hbar$. It follows $|\mathscr{D}^*(x_s, x_s, x_r)| \longrightarrow 0$ as $s, r \longrightarrow \infty$.

Conversely, assume that $|\mathscr{D}^*(x_s, x_s, x_r)| \longrightarrow 0$ as s, $r \longrightarrow \infty$. So, given $o \in \mathbb{C}$ with $0 \prec o$, $\exists \delta > 0$ (s.t.) for $o \in \mathbb{C}$, $|z| < \delta \Longrightarrow z \prec o$.

For $\delta, \exists$, a natural number $\hbar$ (s.t.) $|\mathscr{D}^*(x_s, x_s, x_r)| < \delta$ for all s, $r \geq \hbar$. This means that $\mathscr{D}^*(x_s, x_s, x_r) \prec o$ for all s, $r \geq \hbar$. Thus, $\{x_s\}$ is complex $\mathscr{D}^*$-Cauchy. □

Proposition 4. *If* $(\mathscr{X}, \mathscr{D}^*)$ *is complex-valued* $\mathscr{D}_c^*$*-metric space. So,* $\mathscr{D}_c^*(x, y, z)$ *is jointly continuous mapping for each three of its variables.*

Proof 4. Assume a sequence $\{x_{\hbar}\}$, $\{y_r\}$, and $\{z_s\}$ are complex-valued $\mathscr{D}_c^*$-convergent to $x, y, \& z$, respectively, in that case, utilizing $(\mathscr{C}\mathscr{D}_4^*)$ obtains: $\mathscr{D}_c^*(x, y, z) \precsim \mathscr{D}_c^*(y, y, y_r) + \mathscr{D}_c^*(y_r, x, z)$,

$\mathscr{D}_c^*(z, x, y_r) \precsim \mathscr{D}_c^*(x, x, x_{\hbar}) + \mathscr{D}_c^*(x_{\hbar}, y_r, z)$, and

$\mathscr{D}_c^*(z, x_{\hbar}, y_r) \precsim \mathscr{D}_c^*(z, z, z_s) + \mathscr{D}_c^*(z_s, y_r, x_{\hbar})$. Consequently,

$$\mathscr{D}_c^*(x, y, z) - \mathscr{D}_c^*(x_{\hbar}, y_r, z_s) \precsim \mathscr{D}_c^*(y, y, y_r) + \mathscr{D}_c^*(x, x, x_{\hbar}) + \mathscr{D}_c^*(z, z, z_s).$$

In the same way, we obtain:

$$\mathscr{D}_c^*(x_{\hbar}, y_r, z_s) - \mathscr{D}_c^*(x, y, z) \precsim \mathscr{D}_c^*(y_r, y_r, y) + \mathscr{D}_c^*(x_{\hbar}, x_{\hbar}, x) + \mathscr{D}_c^*(z_s, z_s, z).$$

Utilizing Lemma 1, we obtain

$$|\mathscr{D}_c^*(x_{\hbar}, y_r, z_s) - \mathscr{D}_c^*(x, y, z)| \precsim 2|\mathscr{D}_c^*(x, x, x_{\hbar}) + \mathscr{D}_c^*(y, y, y_r) + \mathscr{D}_c^*(z, z, z_s)|.$$

Therefore, $|\mathscr{D}_c^*(x_{\hbar}, y_r, z_s) - \mathscr{D}_c^*(x, y, z)| \longrightarrow 0$ as $\hbar$, r, s $\longrightarrow \infty$.when utilizing Lemma 2, the result holds. □

Now, our main result in this section is to prove a contraction principle in complex-valued $\mathscr{D}^*$-metric space, as follows.

Example 2. Suppose that $\mathscr{X} = \mathbb{C}$ and $\mathscr{D}_c^* : \mathscr{X}^3 \longrightarrow \mathbb{C}$ are defined by

$$\mathscr{D}_c^*(z_1, z_2, z_3) = (|a_1 - a_2| + |a_2 - a_3| + |a_3 - a_1|) + i(|b_1 - b_2| + |b_2 - b| + |b_3 - b_1|),$$

where $z_i = a_i \text{-} i b_i \in \mathbb{C}$ for every $i \in \{1, 2, 3\}$. Thereafter, $(\mathscr{X}, \mathscr{D}_c^*)$ is a complex-valued $\mathscr{D}^*$-metric space. Define the mapping $\mathscr{T} : \mathscr{X} \longrightarrow \mathscr{X}$ as $\mathscr{T}z = 1/4z$. Then,

$\mathscr{T}$ satisfy $\mathscr{D}_c^*(\mathscr{T}z_1, \mathscr{T}z_2, \mathscr{T}z_3) = \mathscr{D}_c^*(1/4z_1, 1/4z_2, 1/4z_3) = 1/4(|a_1 - a_2| + |a_2 - a_3| + |a_3 - a_1|) + 1/4i(|b_1 - b_2| + |b_2 - b_3| + |b_3 - b_1|) \precsim \hbar\mathscr{D}_c^*(z_1, z_2, z_3)$. Hence,

$\mathscr{D}_c^*(\mathscr{T} z_1, \mathscr{T}z_2, \mathscr{T}z_3) \precsim \hbar\mathscr{D}_c^*(z_1, z_2, z_3) \forall z_1, z_2, z_3 \in \mathscr{X}$, such that $1/4 \leq \hbar < 1$, thus $z = 0$ is a unique fixed-point.

3. Main Results of Fixed-Point Theorems in Complete Complex-Valued $\mathscr{D}_c^*$-Metric Spaces

This section is devoted to establishing various new fixed-point outcomes in complete complex-valued $\mathscr{D}_c^*$-metric.

First, the following outcomes will be needed.

Definition 9 (see [19–22]). A mapping $\Phi : [0,\infty) \longrightarrow [0, 1)$ is called an MT-map (or $\mathfrak{R}$ – map) if

$$\lim_{w \longrightarrow t^+} \Phi(w) < 1 \forall t \in [0,\infty). \tag{1}$$

Remark 4. (see [17]). Clearly, if $\Phi : [0,\infty) \longrightarrow [0, 1)$ is a nondecreasing map or a nonincreasing map, then Φ is an $\mathfrak{R}$-map. Therefore, the set of $\mathfrak{R}$-maps is a wealthy collection.

Theorem 1. *The following statements are equivalent for mapping* $\Phi : [0,\infty) \longrightarrow [0, 1)$:

1. Φ *is an* $\mathfrak{R}$*-mapping.*
2. $\forall t \in [0,\infty), \exists m_t^{(1)} \in [0, 1)$ *and* $\varepsilon_t^{(1)} > 0$ *such that* $\Phi(w) \leq m_t^{(1)} \forall w \in (t, t + \varepsilon_t^{(1)})$.
3. $\forall t \in [0,\infty), \exists m_t^{(2)} \in [0, 1)$ *and* $\varepsilon_t^{(2)} > 0$ *such that* $\Phi(w) \leq m_t^{(2)} \forall w \in [t, t + \varepsilon_t^{(2)}]$.
4. $\forall t \in [0,\infty), \exists m_t^{(3)} \in [0, 1)$ *and* $\varepsilon_t^{(3)} > 0$ *such that* $\Phi(w) \leq m_t^{(3)} \forall w \in (t, t + \varepsilon_t^{(3)}]$.
5. $\forall t \in [0,\infty), \exists m_t^{(4)} \in [0, 1)$ *and* $\varepsilon_t^{(4)} > 0$ *such that* $\Phi(w) \leq m_t^{(4)} \forall w \in (t, t + \varepsilon_t^{(4)}]$.
6. *For any nonincreasing* $\{x_s\}_{s \in \mathbb{N}}$ *in* $[0, \infty)$, *we have* $0 \leq \sup_{s \in \mathbb{N}} \Phi(x_s) < 1$.
7. Φ *is a mapping of the contractive factor; for any strictly decreasing* $\{x_s\}_{s \in \mathbb{N}}$ *in* $[0, \infty)$, *we have* $0 \leq \sup_{s \in \mathbb{N}} \Phi(x_s) < 1$.

Now, we present one new fixed-point theorem, which is one of the important main results of our work.

Theorem 2. *Let* $\mathscr{T} : \mathscr{X} \longrightarrow \mathscr{X}$ *be a mapping in a complete complex-valued* $\mathscr{D}^*$*-metric space* $(\mathscr{X}, \mathscr{D}_c^*)$. *Assume that there exists a* $\mathfrak{R}$*-mapping* $\Phi : [0,\infty) \longrightarrow [0, 1)$ *such that*

$$\mathscr{D}_c^*(\mathscr{T}_x, \mathscr{T}_y, \mathscr{T}_z) \precsim \Phi(|\mathscr{D}_c^*(x, y, z)|)\mathscr{D}_c^*(x, y, z), \forall x, y, z \in \mathscr{X}. \tag{2}$$

Then, $\mathscr{T}$ *has a unique fixed-point on* $\mathscr{X}$.

Proof 5. Assume that $x_0 \in \mathscr{X}$ is given. The sequence $\{x_s\}$ is defined as

$$x_s = \mathscr{T}^s x_0 = \mathscr{T} x_{s-1} \forall s \in \mathbb{N}. \tag{3}$$

By Remark 4, $\forall s \in \mathbb{N}$, we have

$$\mathscr{D}_c^*(x_s x_s, x_{s+1}) \precsim \Phi(|\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)|)\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s). \tag{4}$$

This implies that

$$|\mathscr{D}_c^*(x_s, x_s, x_{s+1})| \le \Phi(|\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)|)|\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)|. \tag{5}$$

Suppose that $\lambda_s = |\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)|$ for each $s \in \mathbb{N}$. Then, by (5), we obtain

$$\lambda_{s+1} \le \Phi(\lambda_s)\,\lambda_s < \lambda_s \forall s \in \mathbb{N}. \tag{6}$$

Consequently, we realize that $\{\lambda_s\}$ is strictly decreasing in $[0, \infty)$. Stratifying (7) of Theorem (1), we obtain

$$0 \le \sup_{s\in\mathbb{N}} \Phi(\lambda_s) < 1. \tag{7}$$

This mean that

$$0 \le \sup_{s\in\mathbb{N}} \Phi(|\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)|) < 1. \tag{8}$$

Assume that

$$\delta = \sup_{s\in\mathbb{N}} \Phi(|\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)|). \tag{9}$$

Then, $\delta \in [0, 1)$. Via (5) once more, $\forall s \in \mathbb{N}$, we obtain

$$\begin{aligned}|\mathscr{D}_c^*(x_s, x_s, x_{s+1})| &\le \Phi(|\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)|)|\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)|\\ &\le \delta|\mathscr{D}_c^*(x_{s-1}, x_{s-1}, x_s)| \le \delta^2|\mathscr{D}_c^*(x_{s-2}, x_{s-2}, x_{s-1})|\\ &\le \cdots \le \delta^n|\mathscr{D}_c^*(x_0, x_0, x_1)|.\end{aligned} \tag{10}$$

For every $s, r \in \mathbb{N}$ with $r > s$, by the previous inequality and employing $(\mathscr{C}\mathscr{D}_4^*)$, we obtain

$$\begin{aligned}|\mathscr{D}_c^*(x_s, x_s, x_r)| &\le |\mathscr{D}_c^*(x_s, x_s, x_{s+1})| + |\mathscr{D}_c^*(x_{s+1}, x_{s+1}, x_{s+2})|\\ &\quad + \cdots + |\mathscr{D}_c^*(x_{r-1}, x_{r-1}, x_r)|\\ &\le \left(\delta^s + \delta^{s+1} + \cdots + \delta^{r-1}\right)|\mathscr{D}_c^*(x_0, x_0, x_1)|\\ &< \frac{\delta^s}{1-\delta}\,|\mathscr{D}_c^*(x_0, x_0, x_1)|.\end{aligned}$$

Since $\delta \in [0, 1)$, $\lim_{s\longrightarrow\infty} (\delta^s/(1-\delta))|\mathscr{D}_c^*(x_0, x_0, x_1)| = 0$. Thus, via the previous inequality, we obtain

$$|\mathscr{D}_c^*(x_s, x_s, x_r)| \longrightarrow 0 \; as\, r, s \longrightarrow \infty. \tag{11}$$

For every $s, r \in \mathbb{N}$, via Lemma 1, we obtain

$$\mathscr{D}_c^*(x_s, x_s, x_r) \precsim \mathscr{D}_c^*(x_s, x_r, x_r) + \mathscr{D}_c^*(x_r, x_s, x_s), \tag{12}$$

which implies

$$|\mathscr{D}_c^*(x_s, x_s, x_r)| \le |\mathscr{D}_c^*(x_s, x_r, x_r)| + |\mathscr{D}_c^*(x_r, x_s, x_s)| \tag{13}$$

Via inequalities (4) and (6), we obtain $|\mathscr{D}_c^*(x_s, x_s, x_r)| \longrightarrow 0$ as $r, s \longrightarrow \infty$. By applying proposition Theorem 1, $\{x_s\}$ is complex $\mathscr{D}^*$-Cauchy sequence. Via the completeness of $(\mathscr{X}, \mathscr{D}_c^*)$, $\exists q \in \mathscr{X}$, (s.t.) $\{x_s\}$ is complex-valued $\mathscr{D}^*$-convergent to q.

Next, establishing that $\mathscr{T} q = q$, assuming that $\mathscr{T} q \ne q$, for all $s \in \mathbb{N}$, *by* 4, via (2), we have

$$\mathscr{D}_c^*(x_{s-1}, x_{s-1}, \mathscr{T} q) \precsim \Phi(|\mathscr{D}_c^*(x_s, x_s, q)|)\mathscr{D}_c^*(x_s, x_s, q) \tag{14}$$

This concludes

$$\begin{aligned}|\mathscr{D}_c^*(x_{s+1}, x_{s+1}, \mathscr{T} q)| &\le \Phi(|\mathscr{D}_c^*(x_s, x_s, q)|)|\mathscr{D}_c^*(x_s, x_s, q)|\\ &< |\mathscr{D}_c^*(x_s, x_s, q)|\end{aligned} \tag{15}$$

Letting $x_s \longrightarrow q$ as $n \longrightarrow \infty$ and $\mathscr{D}^*$ be continuous in each three of its mutables, so by Proposition 4 and utilizing the limit from both sides of inequality (15), we obtain

$$|\mathscr{D}_c^*(q, q, \mathscr{T} q)| \le |\mathscr{D}_c^*(q, q, q)| = 0. \tag{16}$$

Since $0 \precsim \mathscr{D}_c^*(q, q, \mathscr{T} q)$, via Remark 1, we have

$$0 < |\mathscr{D}_c^*(q, q, \mathscr{T} q)|. \tag{17}$$

Therefore, considering (16) and (17), we have

$$0 < |\mathscr{D}_c^*(q, q, \mathscr{T} q)| \le 0. \tag{18}$$

This is a contradiction. Consequently, $\mathscr{T} q = q$ or $q \in F(\mathscr{T})$.

Lastly, illustrate that the fixed-point of $\mathscr{T}$ is unique, since $q \in F(\mathscr{T})$ has demonstrated that $F(\mathscr{T}) = \{q\}$.Let $p \in \mathscr{R}(\mathscr{T})$ can be explained. Assume $p \ne q$. Via (2), we get

$$\mathscr{D}_c^*(q, p, p) = \mathscr{D}_c^*(\mathscr{T} q, \mathscr{T} p, \mathscr{T} p) \precsim \Phi(|D_c^*(q, p, p)|)\mathscr{D}_c^*(q, p, p), \tag{19}$$

which implies

$$|\mathrm{D}_c^*(q, p, p)| \le \Phi(|\mathrm{D}_c^*(q, p, p)|)|\mathscr{D}_c^*(q, p, p)|. \tag{20}$$

Via (20), we get

$$(1 - \Phi(|D_c^*(q, p, p)|))|\mathscr{D}_c^*(q, p, p)| \le 0. \tag{21}$$

$$\text{Since } \Phi(|\mathscr{D}_c^*(q, p, p)|) \in [0, 1), \text{ we obtain } |\mathscr{D}_c^*(q, p, p)| \le 0. \tag{22}$$

This concludes

$$|\mathscr{D}_c^*(q, p, p)| = 0. \tag{23}$$

Thus, $D_c^*(q, p, p) = 0$. This is a contradiction.

Therefore, it should be $p = q$, and so $F(\mathcal{T}) = \{q\}$; the evidence is completed.

The following result is a special case of Theorem 2, and its proof is similar to that of Theorem 2. □

Corollary 1. *Let $(\mathcal{X}, \mathcal{D}_c^*)$ be a complete complex value $\mathcal{D}_c^*$-metric space and*

$\mathcal{T} : (\mathcal{X}, \mathcal{D}_c^*) \longrightarrow (\mathcal{X}, \mathcal{D}_c^*)$ is a contraction mapping on $\mathcal{X}$, such that $\mathcal{D}_c^*(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \precsim \hbar \mathcal{D}_c^*(x, y, z)$ for each x, $y, z \in \mathcal{X}$, (s.t.) $\hbar \in [0, 1)$. Then, $\mathcal{T}$ has a unique fixed-point.

Next, we present a suitable example to demonstrate Theorem (2).

Example 3. Let $\mathcal{X} = \mathbb{C}$ and $D_c^* : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$ be defined via

$$D_c^*(z_1, z_2, z_3) = (|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|) + i(|\mathcal{Y}_1 - \mathcal{Y}_2| + |\mathcal{Y}_2 - \mathcal{Y}_3| + |\mathcal{Y}_3 - 1|), \tag{24}$$

where $z_i = x_i + i y_i \in \mathbb{C} \forall i \in \{1, 2, 3\}$. In that case, $(\mathcal{X}, \mathcal{D}_c^*)$ is a complex-metric.

$\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ and $\Phi : [0,\infty) \longrightarrow [0, 1)$ are defined by $\mathcal{T}z = 1/10z$ for $z \in \mathcal{X}$,

$$\Phi(t) \coloneqq f(x) = \begin{cases} \dfrac{4}{5}, & \text{if } t = 0, \\ \dfrac{1}{3}, & \text{if } t > 0. \end{cases} \tag{25}$$

Then, Φ is $\mathfrak{R}$-mapping. For any $z_1, z_2, z_3 \in \mathbb{C}$, where $z_i = x_i + i y_i$, we have

$$\mathcal{T}z_i = \frac{z_i}{10} = \frac{x_i}{10} + i\frac{y_i}{10}, \forall i \in \{1, 2, 3\}. \tag{26}$$

Via common calculation, can verify that

$$\mathcal{D}_c^*(\mathcal{T}z_1, \mathcal{T}z_2, \mathcal{T}z_3) \preccurlyeq \Phi(|\mathcal{D}_c^*(z_1, z_2, z_3)|) \mathcal{D}_c^*(z_1, z_2, z_3). \tag{27}$$

As a result, each of the assumptions of Theorem 2 is fulfilled. It is consequently possible to apply Theorem 2.

Theorem 3. *Let $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping in a complete $(\mathcal{X}, \mathcal{D}^*)$, and assume there exists a $\mathfrak{R}$ – mapping $\Phi : [0,\infty) \longrightarrow [0, 1)$ (s.t.) for each $x, y, z \in \mathcal{X}$*

$$\mathcal{D}^*(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq \Phi(\mathcal{D}^*(x, y, z))\, \mathcal{D}^*(x, y, z). \tag{28}$$

In that case, $\mathcal{T}$ has a unique fixed-point.

Proof 6. The proof of Theorem 3 is a consequence of Theorem 2. □

Remark 5. Because any nondecreasing mapping or any nonincreasing mapping $\Phi : [0,\infty) \longrightarrow [0, 1)$ is $\mathfrak{R}$-mapping, by employing Theorem 2, we obtain the following outcomes.

Corollary 2. *Let $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping in a complete complex-valued $\mathcal{D}^*$-metric space $(\mathcal{X}, \mathcal{D}_c^*)$; assume there is nondecreasing mapping $\Phi : [0,\infty) \longrightarrow [0, 1)$, where*

$$\mathcal{D}_c^*(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \preccurlyeq \Phi(|D_c^*(x, y, z)|) \mathcal{D}_c^*(x, y, z), \forall x, y, z \in \mathcal{X}. \tag{29}$$

Therefore, $\mathcal{T}$ has a unique fixed-point on $\mathcal{X}$.

Corollary 3. *Let $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ be a map in a complete $\mathcal{D}^*$-metric space $(\mathcal{X}, \mathcal{D}^*)$; assume there is nondecreasing mapping $\Phi : [0,\infty) \longrightarrow [0, 1)$ such that for each $x, y, z \in \mathcal{X}$*

$$\mathcal{D}^*(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq \Phi(\mathcal{D}^*(x, y, z))\, \mathcal{D}^*(x, y, z) \tag{30}$$

So, $\mathcal{T}$ has unique fixed-point.

Corollary 4. *Let $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping in a complete complex-valued $\mathcal{D}^*$-metric $(\mathcal{X}, \mathcal{D}_c^*)$; assume there is nonincreasing mapping $\Phi : [0,\infty) \longrightarrow [0, 1)$, where*

$$\mathcal{D}_c^*(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \preccurlyeq \Phi(|D_c^*(x, y, z)|) \mathcal{D}_c^*(x, y, z), \text{for each } x, y, z \in \mathcal{X}. \tag{31}$$

Consequently, $\mathcal{T}$ has a unique fixed-point on $\mathcal{X}$.

Corollary 5. *Let $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping in a complete $(\mathcal{X}, \mathcal{D}^*)$, and assume there is nonincreasing mapping $\Phi : [0, \infty) \longrightarrow [0, 1)$, where*

$$\mathcal{D}^*(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq \Phi(\mathcal{D}^*(x, y, z))\, \mathcal{D}^*(x, y, z), \text{for each } x, y, z \in \mathcal{X}. \tag{32}$$

Consequently, $\mathcal{T}$ has a unique fixed-point on $\mathcal{X}$.

Corollary 6. *Let $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ be a contraction map in the complete complex-valued $\mathcal{D}^*$-metric space $(\mathcal{X}, \mathcal{D}_c^*)$; this means*

$$\mathcal{D}_c^*(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \precsim \hbar \mathcal{D}_c^*(x, y, z), \text{for each } x, y, z \in \mathcal{X}, \tag{33}$$

such that $\hbar \in [0, 1)$. In that case, $\mathcal{T}$ has a unique fixed-point.

4. Some Applications of Common Fixed-Point Theorems in Complex-Valued $\mathcal{D}_c^*$-Metric Spaces

Several implementations of common fixed-point theory are introduced in this segment with respect to closed balls on full complex-valued $\mathcal{D}_c^*$-metric spaces.

Theorem 4. *Let* $(\mathscr{X},\mathscr{D}_c^*)$ *be a complete complex-valued* $\mathscr{D}_c^*$*-metric space and let* $x_0\in\mathscr{X}, 0\prec m\in\mathbb{C}$, *and* e_1, e_2, e_3, e_4, e_5 *be real numbers such that* $e_1, e_2, e_3, e_4, e_5\geq 0$ *and* $e_1+e_2+e_3+3e_4+3e_5<1$. *Assume that* $F, T:(\mathscr{X},\mathscr{D}_c^*)\longrightarrow(\mathscr{X},\mathscr{D}_c^*)$ *are two mappings satisfying*

$$\begin{aligned}\mathscr{D}_c^*(Fx,Fx,Ty)&\precsim e_1\mathscr{D}_c^*(x,x,y)\\&+e_2\frac{\mathscr{D}_c^*(Fx,Fx,x)\mathscr{D}_c^*(Ty,Ty,y)}{1+\mathscr{D}_c^*(x,x,y)}\\&+e_3\frac{\mathscr{D}_c^*(Fx,Fx,x)\mathscr{D}_c^*(Ty,Ty,y)}{1+\mathscr{D}_c^*(x,x,y)}\\&+e_4\frac{\mathscr{D}_c^*(Fx,Fx,x)\mathscr{D}_c^*(Ty,Ty,y)}{1+\mathscr{D}_c^*(x,x,y)}\\&+e_5\frac{\mathscr{D}_c^*(Fx,Fx,x)\,\mathscr{D}_c^*(Ty,Ty,y)}{1+\mathscr{D}_c^*(x,x,y)},\end{aligned}\tag{34}$$

for every $x,y\in\mathscr{B}^c_{\mathscr{D}^*}[x_0,m]$. *If*

$$|\mathscr{D}_c^*(Fx_0,Fx_0,x_0)|\leq\frac{1-h}{2}|m|,\tag{35}$$

where $h=\max\{(e_1+e_4/1-e_2-e_4),(e_1+e_5/1-e_2-e_5)\}$; *so, there exists a unique common fixed-point* $p\in\mathscr{B}^c_{\mathscr{D}^*}[x_0,m]$ *of the self-maps* F *and* T.

Proof 7. Suppose that $x_0\in\mathscr{X}$ and that the sequence $\{x_s\}$ is defined as

$x_{2k+1}=Fx_{2k}$ and $x_{2k+2}=Tx_{2k+1}$,

(s.t.) $k=0,1,2,\cdots$. We explain $\{x_s\}\in\mathscr{B}^c_{\mathscr{D}^*}[x_0,m]\forall s\in\mathbb{N}$ by mathematical induction. Utilizing inequalities 7 and $h<1$, we obtain

$$|\mathscr{D}_c^*(Fx_0,Fx_0,x_0)|\leq|m|,$$

which implies $x_1\in\mathscr{B}^c_{\mathscr{D}^*}[x_0,m]$.

Assume that $x_2,\cdots,x_i\in B^c_{\mathscr{D}^*}[x_0,m]$ for some $i\in\mathbb{N}$. If $i=2k+1$ or $i=2k+2$ (s.t.) $k=0,1,2,\cdots,i-1/2$, by inequality (34), get

$$\begin{aligned}&\mathscr{D}_c^*(x_{2k+1},x_{2k+1},x_{2k+2})\\&=\mathscr{D}_c^*(Fx_{2k},Fx_{2k},Tx_{2k+1})\precsim e_1\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})\\&\quad+e_2\frac{\mathscr{D}_c^*(Fx_{2k},Fx_{2k},x_{2k})\mathscr{D}_c^*(Tx_{2k+1},Tx_{2k+1},x_{2k+1})}{1+\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})}\\&\quad+e_3\frac{\mathscr{D}_c^*(Fx_{2k},Fx_{2k},x_{2k+1})\mathscr{D}_c^*(Tx_{2k+1},Tx_{2k+1},x_{2k})}{1+\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})}\\&\quad+e_4\frac{\mathscr{D}_c^*(Fx_{2k},Fx_{2k},x_{2k})\mathscr{D}_c^*(Tx_{2k+1},Tx_{2k+1},x_{2k})}{1+\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})}\\&\quad+e_5\frac{\mathscr{D}_c^*(Fx_{2k},Fx_{2k},x_{2k+1})\mathscr{D}_c^*(Tx_{2k+1},Tx_{2k+1},x_{2k+1})}{1+\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})},\end{aligned}$$

and consequently,

$$\begin{aligned}\mathscr{D}_c^*(x_{2k+1},x_{2k+1},x_{2k+2})&\precsim e_1\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})\\&+e_2\frac{\mathscr{D}_c^*(Fx_{2k},Fx_{2k},x_{2k})\mathscr{D}_c^*(Tx_{2k+1},Tx_{2k+1},x_{2k+1})}{1+\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})}\\&+e_4\frac{\mathscr{D}_c^*(Fx_{2k},Fx_{2k},x_{2k})\mathscr{D}_c^*(Tx_{2k+1},Tx_{2k+1},x_{2k})}{1+\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})}\end{aligned}$$

implies

$$\begin{aligned}&\mathscr{D}_c^*(x_{2k+1},x_{2k+1},x_{2k+2})|\leq e_1|\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})|\\&\quad+e_2\frac{|\mathscr{D}_c^*(Fx_{2k},Fx_{2k},x_{2k})||\mathscr{D}_c^*(Tx_{2k+1},Tx_{2k+1},x_{2k+1})|}{|1+\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})|}\\&\quad+e_4\frac{|\mathscr{D}_c^*(Fx_{2k},Fx_{2k},x_{2k})||\mathscr{D}_c^*(Tx_{2k+1},Tx_{2k+1},x_{2k})|}{|1+\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})|}\\&\leq e_1|\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})|\\&\quad+e_2\frac{|\mathscr{D}_c^*(Fx_{2k+1},Fx_{2k+1},x_{2k})||\mathscr{D}_c^*(x_{2k+2},x_{2k+2},x_{2k+1})|}{|\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})|}\\&\quad+e_4\frac{|\mathscr{D}_c^*(x_{2k+1},x_{2k+1},x_{2k})||\mathscr{D}_c^*(x_{2k+2},x_{2k+2},x_{2k})|}{|\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})|}.\end{aligned}$$

Utilizing (3) of Lemma 1, we get

$$\begin{aligned}|\mathscr{D}_c^*(x_{2k+1},x_{2k+1},x_{2k+2})|\leq e_1|\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})|\\+e_2|\mathscr{D}_c^*(x_{2k+2},x_{2k+2},x_{2k+1})|\\+e_4|\mathscr{D}_c^*(x_{2k+2},x_{2k+2},x_{2k})|.\end{aligned}$$

By utilizing the condition $(C\mathscr{D}_4^*)$ and (3) of Lemma 1, we obtain

$$\begin{aligned}&\mathscr{D}_c^*(x_{2k+2},x_{2k+2},x_{2k+1})\precsim\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})\\&\quad+\mathscr{D}_c^*(x_{2k+2},x_{2k+2},x_{2k+1}),|\mathscr{D}_c^*(x_{2k+1},x_{2k+1},x_{2k+2})|\\&\leq\frac{e_1+e_4}{1-e_2-e_4}|\mathscr{D}_c^*(x_{2k},x_{2k},x_{2k+1})|.\end{aligned}\tag{36}$$

Via a similar method as above, we obtain

$$|\mathscr{D}_c^*(x_{2k+2},x_{2k+2},x_{2k+3})|\leq\frac{e_1+e_5}{1-e_2-e_5}|\mathscr{D}_c^*(x_{2k+1},x_{2k+1},x_{2k+2})|.\tag{37}$$

If we take $h=\max\{(e_1+e_4/1-e_2-e_4),(e_1+e_5/1-e_2-e_5)\}$, we obtain

$$|\mathscr{D}_c^*(x_i,x_i,x_{i+1})|\leq h^i|\mathscr{D}_c^*(x_0,x_0,x_1)|,$$

for all $i \in \mathbb{N}$. Let us consider

$$\begin{aligned}|\mathscr{D}_c^*(x_0, x_0, x_{i+1})| &\leq |\mathscr{D}_c^*(x_0, x_0, x_1)| + |\mathscr{D}_c^*(x_1, x_1, x_2)| \\ &\quad + \cdots + |\mathscr{D}_c^*(x_i, x_i, x_{i+1})| \\ &\leq |\mathscr{D}_c^*(x_0, x_0, x_1)|\left(1 + h + \cdots + h^{i-1}\right) \\ &\quad + h^i|\mathscr{D}_c^*(x_0, x_0, x_1)| \\ &\leq \frac{1-h}{2}|m|\left(1 + h + \cdots + h^{i-1}\right) \\ &\quad + h^i\frac{1-h}{2} \leq |m|(1-h)\left(1 + h + \cdots + h^i\right) \leq |m|,\end{aligned}$$

$\Longrightarrow x_{i+1} \in \mathscr{B}^c_{\mathscr{D}^*}[x_0, m]$. Therefore, $x_s \in \mathscr{B}^c_{\mathscr{D}^*}[x_0, m]$ and

$$|\mathscr{D}_c^*(x_s, x_s, x_{s+1})| \leq h^s|\mathscr{D}_c^*(x_0, x_0, x_1)|,$$

for each $s \in \mathbb{N}$. If take $r > s$, so we get

$$\begin{aligned}|\mathscr{D}_c^*(x_s, x_s, x_r)| &\leq |\mathscr{D}_c^*(x_s, x_s, x_{s+1})| + |\mathscr{D}_c^*(x_{s+1}, x_{s+1}, x_{s+2})| \\ &\quad + \cdots + |\mathscr{D}_c^*(x_{r-1}, x_{r-1}, x_r)| \longrightarrow 0,\end{aligned}$$

As $r, s \longrightarrow \infty$, that suggests $\{x_s\}$ is a Cauchy sequence in $\mathscr{B}^c_{\mathscr{D}^*}[x_0, m]$. Thus, there exists a point $p \in \mathscr{B}^c_{\mathscr{D}^*}[x_0, m]$ with $\lim_{s\longrightarrow\infty} x_s = p$.

Now, we show that $Fp = p$. Utilizing inequality 4, we obtain

$$\begin{aligned}|\mathscr{D}_c^*(Fp, Fp, p)| &\leq |\mathscr{D}_c^*(p, p, x_{2k+2})| + |\mathscr{D}_c^*(x_{2k+2}, x_{2k+2}, Fp)| \\ &= |\mathscr{D}_c^*(p, p, x_{2k+2})| + |\mathscr{D}_c^*(Fp, Fp, Tx_{2k+1})| \\ &\precsim |\mathscr{D}_c^*(p, p, x_{2k+2})| + e_1|\mathscr{D}_c^*(p, p, x_{2k+1})| \\ &\quad + e_2\frac{|\mathscr{D}_c^*(Fp, Fp, p)||\mathscr{D}_c^*(Tx_{2k+1}, Tx_{2k+1}, x_{2k+1})|}{|1 + \mathscr{D}_c^*(p, p, x_{2k+1})|} \\ &\quad + e_3\frac{|\mathscr{D}_c^*(Fp, Fp, x_{2k+1})||\mathscr{D}_c^*(Tx_{2k+1}, Tx_{2k+1}, p)|}{|1 + \mathscr{D}_c^*(p, p, x_{2k+1})|} \\ &\quad + e_4\frac{|\mathscr{D}_c^*(Fp, Fp, p)||\mathscr{D}_c^*(Tx_{2k+1}, Tx_{2k+1}, p)|}{|1 + \mathscr{D}_c^*(p, p, x_{2k+1})|} \\ &\quad + e_5\frac{|\mathscr{D}_c^*(Fp, Fp, x_{2k+1})||\mathscr{D}_c^*(Tx_{2k+1}, Tx_{2k+1}, x_{2k+1})|}{|1 + \mathscr{D}_c^*(p, p, x_{2k+1})|},\end{aligned}$$

which implies that this inequality converges to 0 as $s \longrightarrow \infty$. Consequently, $|\mathscr{D}_c^*(Fp, Fp, p)| = 0$, that is, $Fp = p$. Via a similar method as above, we illustrate that $Tp = p$.

Now, establish that a fixed-point p is unique. Presume $p^* \in \mathscr{B}^c_{\mathscr{D}^*}[x_0, m]$ also common fixed-point of F and T. In that case, we obtain

$$\begin{aligned}|\mathscr{D}_c^*(p, p, p^*)| &= |\mathscr{D}_c^*(Fp, Fp, Tp^*)| \leq e_1|\mathscr{D}_c^*(p, p, p^*)| \\ &\quad + e_2\frac{|\mathscr{D}_c^*(Fp, Fp, p)||\mathscr{D}_c^*(Tp^*, Tp^*, p^*)|}{|1 + \mathscr{D}_c^*(p, p, p^*)|} \\ &\quad + e_3\frac{|\mathscr{D}_c^*(Fp, Fp, p^*)||\mathscr{D}_c^*(Tp^*, Tp^*, p)|}{|1 + \mathscr{D}_c^*(p, p, p^*)|} \\ &\quad + e_4\frac{|\mathscr{D}_c^*(Fp, Fp, p)||\mathscr{D}_c^*(Tp^*, Tp^*, p)|}{|1 + \mathscr{D}_c^*(p, p, p^*)|} \\ &\quad + e_5\frac{|\mathscr{D}_c^*(Fp, Fp, p^*)||\mathscr{D}_c^*(Tp^*, Tp^*, p^*)|}{|1 + \mathscr{D}_c^*(p, p, p^*)|}.\end{aligned}$$

Therefore, we obtain

$$|\mathscr{D}_c^*(p, p, p^*)| \leq (e_1 + e_3)|\mathscr{D}_c^*(p, p, p^*)|,$$

since $|1 + \mathscr{D}_c^*(p, p, p^*)| > |\mathscr{D}_c^*(p, p, p^*)|$. Consequently, $p = p^*$ as $e_1 + e_3 < 1$. Therefore, p is unique common fixed-point of F and T.

Observe if we set $F = T$ in Theorem 4, so we obtain the next result. □

Corollary 7. *Let $(\mathscr{X}, \mathscr{D}_c^*)$ be a complete complex-valued $\mathscr{D}_c^*$-metric, $x_0 \in \mathscr{X}$, $0 \prec m \in \mathbb{C}$ and e_1, e_2, e_3, e_4, e_5 be real numbers such that $e_1, e_2, e_3, e_4, e_5 \geq 0$ and $e_1 + e_2 + e_3 + 3e_4 + 3e_5 < 1$. Assume that $F : (\mathscr{X}, \mathscr{D}_c^*) \longrightarrow (\mathscr{X}, \mathscr{D}_c^*)$ is a mapping satisfying*

$$\begin{aligned}\mathscr{D}_c^*(Fx, Fx, Fy) &\precsim e_1\mathscr{D}_c^*(x, x, y) \\ &\quad + e_2\frac{\mathscr{D}_c^*(Fx, Fx, x)\mathscr{D}_c^*(Fy, Fy, y)}{1 + \mathscr{D}_c^*(x, x, y)} \\ &\quad + e_3\frac{\mathscr{D}_c^*(Fx, Fx, y)\mathscr{D}_c^*(Fy, Fy, x)}{1 + \mathscr{D}_c^*(x, x, y)} \\ &\quad + e_4\frac{\mathscr{D}_c^*(Fx, Fx, x)\mathscr{D}_c^*(Fy, Fy, x)}{1 + \mathscr{D}_c^*(x, x, y)} \\ &\quad + e_5\frac{\mathscr{D}_c^*(Fx, Fx, y)\mathscr{D}_c^*(Fy, Fy, y)}{1 + \mathscr{D}_c^*(x, x, y)},\end{aligned} \tag{38}$$

$\forall x, y \in \mathscr{B}^c_{\mathscr{D}^*}[x_0, m]$. If $|\mathscr{D}_c^*(Fx_0, Fx_0, x_0)| \leq (1 - h/2)|m|$,

where $h = \max\{(e_1 + e_4/1 - e_2 - e_4), (e_1 + e_5/1 - e_2 - e_5)\}$; so, there exists a unique fixed-point $p \in \mathscr{B}^c_{\mathscr{D}^*}[x_0, m]$ of the self-map F.

Example 4. Suppose $\mathscr{X} = \mathbb{C}$ with the complex $\mathscr{D}^*$-metric on $\mathbb{C}$ is put as follows:

$$\mathscr{D}_c^*(z_1, z_2, z_3) = \left(\frac{(x_1 - x_3)^2}{9} + 4(y_1 - y_3)^2\right)^{1/2} + \left(\frac{(x_2 - x_3)^2}{9} + 4(y_2 - y_3)^2\right)^{1/2},$$

$\forall z_1, z_2, z_3 \in \mathbb{C}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, and $z_3 = (x_3, y_3)$.

Assume $F : \mathbb{C} \longrightarrow \mathbb{C}$ given by $Fz = z_0, \forall z \in \mathbb{C}$, wherein z_0 is the closed ball's center $\mathscr{B}^c_{\mathscr{D}^*}[z_0, m]$. If we set $e_1 = 1/2$, $e_2 = e_3 = e_4 = e_5 = 0$, we get

$$\mathscr{D}_c^*(Fz_1, Fz_1, Fz_2) = \mathscr{D}_c^*(z_0, z_0, z_0) = 0 \precsim \frac{1}{2}\mathscr{D}_c^*(z_1, z_1, z_2),$$

$\forall z_1, z_2 \in \mathscr{B}^c_{\mathscr{D}^*}[z_0, m]$. Then, the inequality (38) is satisfied. Thus, we get

$$h = \max\left\{\frac{e_1 + e_4}{1 - e_2 - e_4}, \frac{e_1 + e_5}{1 - e_2 - e_5}\right\} = e_1 = \frac{1}{2},$$

and $|\mathscr{D}_c^*(F\varkappa_0, F\varkappa_0, \varkappa_0)| = 0 \le 1/4|m|$.

Therefore, Corollary 7 is verified and there is a unique fixed-point $\varkappa_0 \in \mathscr{B}_{\mathscr{D}^*}^c[\varkappa_0, m]$ of the self-mapping F.

5. Conclusion

Banach's contraction principle plays a significant role in various fields of pure and applied mathematical analysis and scientific implementations. Therefore, the main aim of the present manuscript is to present a complex $\mathscr{D}_c^*$-metric space and verify the contraction principle in new spaces. Additionally, several novel fixed-point outcomes in complete complex $\mathscr{D}_c^*$-metric spaces have been proven which are extended and generalized to Banach's contraction principle and various distinguished outcomes in the previous studies. Additionally, some common fixed-point theory related implementations, with respect to closed balls on complete $(\mathscr{X}, \mathscr{D}_c^*)$, have been presented.

Data Availability Statement

The authors have nothing to report.

Ethics Statement

The authors have nothing to report.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

The first writer prepares the manuscript that enables to achieve the idea of the second writer. While the third writer corrects any errors, the fourth one takes a final look at the research and approves it.

Funding

This study has no fund to be taken.

References

[1] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[2] J. J. Nieto and R. Rodriguez-Lopez, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[3] A. Aghanians and K. Nourouzi, "Fixed points of integral type contractions in uniform spaces," *Univerzitet u Nišu*, vol. 29, no. 7, pp. 1613–1621, 2015.

[4] R. Akbar Zada and T. Li, "Integral type contraction and coupled coincidence fixed point theorems for two pairs in G-metric spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 5, no. 45, pp. 1475–1484, 2016.

[5] A. Azam, B. Fisher, and M. Khan, "Common fixed point theorems in complex valued metric spaces," *Numerical Functional Analysis and Optimization*, vol. 32, no. 3, pp. 243–253, 2011.

[6] K. Sitthikul and S. Saejung, "Some fixed point theorems in complex valued metric spaces," *Fixed Point Theory and Applications*, vol. 2012, no. 1, 2012.

[7] S. M. Kang, M. Kumar, P. Kumar, and S. Kumar, "Coupled fixed point theorems in complex valued metric spaces," *International Journal of Mathematical Analysis*, vol. 7, no. 46, pp. 2269–2277, 2013.

[8] S. Manro, "Some common fixed point theorems in complex valued metric space using implicit function," *International Journal of Analysis and Applications*, vol. 2, no. 1, pp. 62–70, 2013.

[9] M. A. Kutbi, A. Azam, A. Jamshaid, and C. Bari, "Some common coupled fixed point results for generalized contraction in complex-valued metric spaces," *Journal of Applied Mathematics*, vol. 2013, Article ID 352927, 10 pages, 2013.

[10] H. K. Nashine, M. Imdad, and M. Hassan, "Common fixed point theorems under rational contractions in complex valued metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 7, no. 1, pp. 42–50, 2015.

[11] M. Younisa, D. Singhb, L. Chenc, and M. Metwalid, "A study on the solutions of notable engineering models," *Mathematical Modelling and Analysis*, vol. 27, no. 3, pp. 492–509, 2022.

[12] M. Younis and D. Bahuguna, "A unique approach to graph-based metric spaces with an application to rocket ascension," *Computational and Applied Mathematics*, vol. 42, no. 1, p. 44, 2023.

[13] M. Oztürk and N. Kaplan, "Some common coupled fixed points of mappings satisfying contractive conditions with rational expressions in complex valued G_b-metric spaces," *Bangmod International Journal of Mathematical and Computational Science*, vol. 1, no. 1, pp. 190–204, 2015.

[14] A. K. Dubey, R. Shukla, and R. P. Dubey, "Some fixed point theorems in complex valued b-metric spaces," *Journal of Complex Systems*, vol. 2015, Article ID 832467, 7 pages, 2015.

[15] N. Singh, D. Singh, A. Badal, and V. Joshi, "Fixed point theorems in complex valued metric spaces," *Journal of the Egyptian Mathematical Society*, vol. 24, no. 3, pp. 402–409, 2016.

[16] O. Ege and I. Karaca, "Common fixed point results on complex valued G_b-metric spaces," *Thai Journal of Mathematics*, vol. 16, no. 3, pp. 775–787, 2018.

[17] R. A. Rashwan, H. A. Hammad, and M. G. Mahmoud, "Common fixed point results for weakly compatible mappings under implicit relations in complex valued G-metric spaces," *Information Sciences Letters*, vol. 8, no. 3, pp. 111–119, 2019.

[18] S. Shaban, S. Nabi, and Z. Haiyun, "A common fixed point theorem in D^*-metric spaces," *Fixed Point Theory and Applications*, vol. 2007, Article ID 27906, 13 pages, 2007.

[19] W. S. Du, "On coincidence point and fixed point theorems for nonlinear multivalued maps," *Topology and Its Applications*, vol. 159, no. 1, pp. 49–56, 2012.

[20] W. S. Du, "On Caristi type maps and generalized distances with applications," *Abstract and Applied Analysis*, vol. 2013, Article ID 407219, 8 pages, 2013.

[21] W. S. Du, E. Karapinar, and N. Shahzad, "The study of fixed point theory for various multivalued non-self-maps," *Abstract and Applied Analysis*, vol. 2013, no. 1, Article ID 938724, 9 pages, 2013.

[22] W. S. Du and E. Karapinar, "A note on Caristi-type cyclic maps: related results and applications," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 13 pages, 2013.